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STRONG CONVERGENCE THEOREMS BY HALPERN-TYPE ITERATIONS FOR RELATIVELY NONEXPANSIVE MULTI-VALUED MAPPINGS IN BANACH SPACES

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Abstract. In this paper, an iterative sequence for relatively nonexpansive multi-valued mapping by modifying Halpern's iterations is introduced and the strong convergence theorems are proved. At the end of the paper some applications are given also.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space E. A single-valued mapping $T : D \to D$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\right\}, \tag{1.1}$$

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for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x-y\|, y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if $H(T(x), T(y)) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to N(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in E. \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.3}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1. The following basic properties for Banach space E and for the normalized duality mapping J can be found in Cioranescu [1].

(i) If E is an arbitrary Banach space, then J is monotone and bounded;
(ii) If E is a strictly convex Banach space, then J is strictly monotone;
(iii) If E is a a smooth Banach space, then J is single-valued, and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E;
(iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;
(v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*, JJ^* = I_E^*$ and $J^*J = I_E$;
(vi) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;
(vii) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Let E be a smooth Banach space. In the sequel, we always use $\phi : E \times E \to \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{1.5}$$

In addition, the function ϕ has the following property:

$$\phi(y,x) = \phi(z,x) + \phi(y,z) + 2\langle z-y, Jx-Jz\rangle, \quad \forall x,y,z \in E \tag{1.6}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \tag{1.7}$$

for all $\lambda \in [0,1]$ and $x,y,z \in E$.

Let C is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E. Following Alber [2], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = arg \inf_{y\in C} \phi(y,x), \quad \forall x \in E.$$

Let D be a nonempty subset of a smooth Banach space. A mapping $T : D \to E$ is relatively nonexpansive [3-5], if the following properties are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p,Tx) \le \phi(p,x)$ for all $p \in F(T)$ and $x \in D$;
(R3) $I-T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

If T satisfies (R1) and (R2), then T is called *quasi-ϕ-nonexpansive* [6].

Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors, see for instance [7-11].

Let D be a nonempty closed convex subset of a smooth Banach space E. A mapping $T : D \to N(D)$ is relatively nonexpansive multi-valued mapping [11], if the following properties are satisfied:

(S1) $F(T) \neq \emptyset$;
(S2) $\phi(p,z) \le \phi(p,x), \forall x \in D, z \in T(x), p \in F(T)$;
(S3) $I-T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D which weakly to p and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, it follows that $p \in F(T)$.

In this article, we introduce the following iterative sequence for finding a fixed point of strongly reatively nonexpansive multi-valued mapping $T : D \to N(D)$. Given $u \in E, x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n), \tag{1.8}$$

where $w_n \in Tx_n$ for all $n \in N$, D is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, Π_D is the generalized projection of E onto D and $\{\alpha_n\}$ is sequences in (0,1). We proved the strong convergence theorems in uniformly convex and uniformly smooth Banach space E.

2. Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

First, we recall some conclusions.

Lemma 2.1. (cf. [12, Proposition 2]) *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.2. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\phi(x_n, y_n) \to 0 \Longleftrightarrow x_n - y_n \to 0 \Longleftrightarrow Jx_n - Jy_n \to 0.$$

Lemma 2.3. (cf. [12, Propositions 4 and 5]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;*
(b) *If $x \in E$ and $z \in C$, then $z = \Pi_C x \Longleftrightarrow \langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$;*
(c) *For $x, y \in E, \phi(x, y) = 0$ if and only $x = y$.*

Remark 2.4. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Lemma 2.5. (cf. [11, Proposition 2.1]) *Let E be a strictly convex and smooth Banach space, and D a nonempty closed convex subset of E. Suppose $T : D \to N(D)$ is a relatively nonexpansive multi-valued mapping. Then, $F(T)$ is closed and convex.*

Lemma 2.6. (cf. [13, Lemma1]) *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \quad n = 1, 2, 3, \ldots$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, we use Halpern's idea [14] for finding fixed point of relatively nonexpansive multi-valued mappings in a uniformly convex and smooth Banach space.

Theorem 3.1. *Let D be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E and let $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping. Let $\{x_n\}$ be the sequence in D defined by (1.8), where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. The proof of Theorem 3.1 is divided into two steps:

Step 1. Firstly We show that $\{x_n\}$ converges strongly in D.

Let $y_n \equiv J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n)$. Then $x_{n+1} \equiv \Pi_D y_n$. By Lemma 2.5, $F(T)$ is nonempty, closed and convex, so, we can define the generalized projection $\Pi_{F(T)}$ onto $F(T)$. Let $z \in F(T)$, From the definition of relatively nonexpansive multi-valued mapping and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\phi(z, x_{n+1})\\
&= \phi(z, \Pi_D J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n))\\
&\le \phi(z, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n))\\
&= \|z\|^2 - 2\langle z, \alpha_n Ju + (1-\alpha_n)Jw_n\rangle + \|\alpha_n Ju + (1-\alpha_n)Jw_n\|^2 \qquad (3.1)\\
&= \|z\|^2 - 2\alpha_n\langle z, Ju\rangle - 2(1-\alpha_n)\langle z, Jw_n\rangle + \alpha_n\|u\|^2 + (1-\alpha_n)\|w_n\|^2\\
&= \alpha_n\phi(z,u) + (1-\alpha_n)\phi(z,w_n)\\
&= \alpha_n\phi(z,u) + \phi(z,x_n).
\end{aligned}$$

By $\sum_{n=1}^{\infty} \alpha_n < \infty$, from Lemma 2.6, we have $\lim_{n\to\infty} \phi(p, x_n)$ exists and in particular, $\{\phi(p, x_n)\}$ is bounded. This implies $\{x_n\}, \{Tx_n\}$ are bounded. Since the interior of $F(T)$ is nonempty, there exist $p \in F(T)$, $h \in E$ with $\|h\| \le 1$ and $r > 0$, such that $p + rh \in F(T)$. By (1.6), we have

$$\phi(z, x_n) = \phi(x_{n+1}, x_n) + \phi(z, x_{n+1}) + 2\langle x_{n+1} - z, Jx_n - Jx_{n+1}\rangle,$$

this implies

$$\langle x_{n+1} - z, Jx_n - Jx_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}(\phi(z, x_n) - \phi(z, x_{n+1})). \qquad (3.2)$$

Since $p + rh \in F(t)$, we obtain

$$\phi(p + rh, x_{n+1}) \le \phi(p + rh, x_n) + \alpha_n\phi(p + rh, u).$$

Notice (3.2), above inequality is equivalent to

$$0 \le \langle x_{n+1} - (p + rh), Jx_n - Jx_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\alpha_n\phi(p + rh, u).$$

Then by (3.2), we have

$$\begin{aligned} r\langle h, Jx_n - Jx_{n+1}\rangle &\le \langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle \\ &\quad + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\alpha_n \phi(p + rh, u) \\ &= \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})) + \frac{1}{2}\alpha_n\phi(p + rh, u), \end{aligned}$$

and hence

$$\langle h, Jx_n - Jx_{n+1}\rangle \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})) + \frac{1}{2r}\alpha_n\phi(p + rh, u).$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|Jx_n - Jx_{n+1}\| \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})) + \frac{1}{2r}\alpha_n\phi(p + rh, u). \tag{3.3}$$

So, if $n > m$, then

$$\begin{aligned} \|Jx_m - Jx_n\| &= \|Jx_m - Jx_{m+1} + Jx_{m+1} - \cdots - Jx_{n-1} + Jx_{n-1} - Jx_n\| \\ &\le \sum_{i=m}^{n-1} \|Jx_i - Jx_{i+1}\| \\ &\le \frac{1}{2r}\sum_{i=m}^{n-1}(\phi(p, x_i) - \phi(p, x_{i+1})) + \frac{\phi(p + rh, u)}{2r}\sum_{i=m}^{n-1}\alpha_i \\ &= \frac{1}{2r}(\phi(p, x_m) - \phi(p, x_n)) + \frac{\phi(p + rh, u)}{2r}\sum_{i=m}^{n-1}\alpha_i. \end{aligned}$$

We know that $\{\phi(p, x_n)\}$ converges. So, $\{Jx_n\}$ is a Cauchy sequence. Since E^* is complete, so $\{Jx_n\}$ converges strongly to some point in E^*. Since E^* has a Fréchet differentiable norm, then J^{-1} is continuous on E^*. Hence x_n converges strongly to some point q in D.

Step 2. Next we prove that $q \in F(T)$ where $q = \lim_{n\to\infty} \Pi_{F(T)}x_n$. By (3.3) and the convergence of $\{\phi(p, x_n)\}$, it follows that

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0. \tag{3.4}$$

Since $y_n \equiv J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n)$, we have

$$\|Jy_n - Jw_n\| = \|\alpha_n Ju - \alpha_n Jw_n\| = \alpha_n\|Ju - Jw_n\|,$$

from $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} \|Jy_n - Jw_n\| = 0. \tag{3.5}$$

By J^{-1} is uniformly norm-to-norm continuous on bounded sets, from (3.4) and (3.5) we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0, \tag{3.6}$$

and

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.7}$$

Since $x_{n+1} \equiv \Pi_D y_n$, from Lemma 2.3, we have

$$\begin{aligned}\phi(w_n, x_{n+1}) + \phi(x_{n+1}, y_n) &= \phi(w_n, \Pi_D y_n) + \phi(\Pi_D y_n, y_n)\\ &\leq \phi(w_n, y_n).\end{aligned}$$

Since

$$\begin{aligned}\phi(w_n, y_n) &= \phi(w_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n))\\ &= \|w_n\|^2 - 2\langle w_n, \alpha_n Ju + (1-\alpha_n)Jw_n\rangle\\ &\quad + \|\alpha_n Ju + (1-\alpha_n)Jw_n\|^2\\ &\leq \|w_n\|^2 - 2\alpha_n\langle w_n, Ju\rangle - 2(1-\alpha_n)\langle w_n, Jw_n\rangle\\ &\quad + \alpha_n\|u\|^2 + (1-\alpha_n)\|w_n\|^2\\ &= \alpha_n\phi(w_n, u) + (1-\alpha_n)\phi(w_n, w_n),\end{aligned}$$

and $\lim_{n\to\infty} \alpha_n = 0$, Then we have

$$\lim_{n\to\infty} \phi(w_n, x_{n+1}) = \lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

By Lemma2.1 we obtain

$$\lim_{n\to\infty} \|w_n - x_{n+1}\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.8}$$

From

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - w_n\|.$$

Therefore, by (3.6), (3.6), (3.8) we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0.$$

So, we have $q \in F(T)$, where $q = \lim_{n\to\infty} \Pi_{F(T)} x_n$. □

4. Application to zero point problem of maximal monotone mappings

Let E be a smooth, strictly convex and reflexive Banach space. An operator $A : E \to 2^{E^*}$ is said to be monotone, if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$. We denote the zero point set $\{x \in E : 0 \in Ax\}$ of A by $A^{-1}0$. A monotone operator A is said to be maximal, if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then $A^{-1}0$ is closed and convex. Let A be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $J(x) \in J(x_r) + rA(x_r)$ (see, for example, [2]). We define the *resolvent* of A by $J_r x = x_r$. In other words $J_r = (J + rA)^{-1}J$, $\forall r > 0$. We know that J_r is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(J_r)$, $\forall r > 0$, where $F(J_r)$ is the set of fixed points of J_r.

We have the following:

Theorem 4.1. *Let E, $\{\alpha_n\}$ be the same as in Theorem 3.1. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$ for all $r > 0$ such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_r x_n),$$

then $\{x_n\}$ converges strongly to some point of $A^{-1}0$.

Proof. In Theorem 3.1 taking $D = E$, $T = J_r$, $r > 0$, then $T : E \to E$ is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(T) = F(J_r)$, $\forall r > 0$ is a nonempty closed convex subset of E. Therefore all the conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 4.1 can be obtained from Theorem 3.1 immediately. □

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