Nonlinear Functional Analysis and Applications
Vol. 17, No. 4 (2012), pp. 443-458

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A UNIFYING SEMILOCAL CONVERGENCE ANALYSIS FOR NEWTON-LIKE METHODS UNDER WEAK AND GATEAUX DIFFERENTIABILITY CONDITIONS

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Abstract. We present sufficient convergence conditions for the semilocal convergence of Newton-like methods in order to approximate a locally unique solution of a nonlinear equation containing a nondifferentiable term in a Banach space setting. The operators involved are Fréchet or Gateaux differentiable. Our results unify, improve the error bounds and also extend the applicability of earlier results. Numerical examples are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

where F is Fréchet or Gateaux differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y and $G : D \to Y$ is a continuous operator.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations. For example, dynamic

[0]Received August 24, 2012. Revised October 25, 2012.
[0]2000 Mathematics Subject Classification: 65H10, 65G99, 65J15, 47H17, 49M15.
[0]Keywords: Newton-like method, Banach space, Fréchet-Gateaux derivative, semilocal convergence, majorizing sequence, recurrent function.

systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, we assume that a time-invariant system is driven by the equation $\dot{x} = F(x) + G(x)$, for some suitable operators F and G, where x is the state. Then the equilibrium states are determined by solving equation $F(x) + G(x) = 0$. Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative. In fact, starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use Newton-like method

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)) \quad \text{for each} \quad n = 0, 1, 2, \dots, \quad x_0 \in D, \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating x^*. Here, $A_n \in \mathcal{L}(X, Y)$, for $n \geq 0$, is the space of bounded linear operators from X into Y.

For:

(*a*) $A_n = A(x_n) = F'(x_n)$, we obtain the Zabrejko-Zincenko iteration [1, 2, 5, 8, 9]. If $G(x) = 0$, we obtain Newton's method.

(*b*) $A_n = F'(x_n) + [x_{n-1}, x_n; G]$, we obtain Catinas' iteration [8], where $[x, y; G]$ is the divided difference of order one for G at the points $x, y \in D$ with $x \neq y$ satisfying

$$[x, y; G](x - y) = G(x) - G(y). \tag{1.3}$$

Several other choices of linear operators A_n are possible [1–13]. In particular we shall use Newton-like method (1.2) in the form

$$x_{n+1} = x_n - L_n^{-1}(F(x_n) + G(x_n)), \quad \text{for all} \quad n \geq 0, x_0 \in D, \tag{1.4}$$

where, for predetermined sequences $\{\gamma_n\}$, $\{\delta_n\}$ in $[0, 1]$ and $\{z_n\}$ in D,

$$L_n = B_n + C_n,$$

$$B_n = \gamma_n F'(x_n) + (1 - \gamma_n) F'(z_n)$$

and

$$C_n = \delta_n [x_{n-1}, x_n; G] + (1 - \delta_n)[z_{n-1}, z_n; G].$$

The convergence of Newton-like method (1.2) has been studied by many authors under various Lipschitz-type conditions. A survey of such results can be found in [5] and the references there (see also [9, 12]).

Here, we provide new semilocal convergence results that unify, extend the applicability and improve the error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$, for $n \geq 0$, of earlier ones under the same information as before.

Numerical examples validating the theoretical results and justifying the claims made above are also provided in this study.

The paper is organized as follows. In Section 2 we provide the semilocal convergence analysis of Newton-like method (1.2), when F' is Fréchet differentiable, whereas in Section 3, F' is only Gateaux differentiable. Special cases, applications and numerical examples are provided in the concluding Section 4 of this study.

2. Semilocal Convergence Analysis I

A semilocal convergence analysis for Newton-like method (1.2) is provided in this section. We need a result on majorizing sequences for Newton-like method (1.2).

Lemma 2.1. *Let $\eta, K > 0$, $L, M, N, l, \mu \geq 0$ be given constants. Define polynomial $h : [0, +\infty) \to (-\infty, +\infty)$ by*

$$h(s) = (1-l)s^2 - (\mu + N + 1 - l - L\eta)s + \mu + N + (2K + M)\eta. \tag{2.1}$$

Assume polynomial h has a minimal root $\alpha \in (0,1)$ such that

$$0 \leq \alpha_0 = \frac{K\eta + 2(\mu + N)}{2(1 - l - L\eta)} \leq \alpha. \tag{2.2}$$

Then, scalar sequence $\{t_n\}$ given by

$$\begin{cases} t_0 = 0, \quad t_1 = \eta, \\ t_{n+2} = t_{n+1} + \dfrac{K(2t_n + \frac{1}{2}(t_{n+1} - t_n)) + Mt_n + \mu + N}{1 - l - Lt_{n+1}}(t_{n+1} - t_n) \end{cases} \tag{2.3}$$

is non-decreasing, bounded from above by

$$t^{**} = \frac{\eta}{1-\alpha} + (\alpha_0 - \alpha)\eta \tag{2.4}$$

and converges to its unique least upper bound t^* satisfying

$$0 \leq t^* \leq t^{**}. \tag{2.5}$$

Moreover, the following estimates hold

$$t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n \eta \tag{2.6}$$

and

$$t^* - t_n \leq \frac{\alpha^n \eta}{1-\alpha}\cdot \tag{2.7}$$

Proof. We shall show using induction

$$0 \leq \frac{2(2K+M)t_n + K(t_{n+1}-t_n) + 2(\mu+N)}{2(1-l-Lt_{n+1})} \leq \alpha\cdot \tag{2.8}$$

Estimate (2.8) is true for $n = 0$ by hypothesis (2.2). We then have $0 \leq t_2 - t_1 \leq \alpha(t_1 - t_0) = \alpha\eta$.

Let us assume (2.8) holds for all $k \leq \eta$. We get

$$t_{k+1} - t_k \leq \alpha(t_k - t_{k-1}) \leq \alpha^k \eta, \tag{2.9}$$

so

$$t_{k+1} \leq \eta + \alpha_0\eta + \alpha^2\eta + \cdots + \alpha^k\eta \leq \frac{1-\alpha^{k+1}}{1-\alpha}\eta + (\alpha_0 - \alpha)\eta \leq t^{**}. \tag{2.10}$$

In view of (2.9) and (2.10) estimate (2.8) holds if

$$2(2K+M)t_k + K(t_{k+1}-t_k) + 2(\mu+N) + 2L\alpha t_{k+1} - 2\alpha(1-l) \leq 0 \tag{2.11}$$

or

$$\begin{aligned} &2(2K+M)(1+\alpha+\cdots+\alpha^{k-1})\eta + K\alpha^k\eta \\ &\quad + 2\alpha L(1+\alpha+\cdots+\alpha^k)\eta + 2(\mu+N) - 2\alpha(1-l) \\ &\leq 0. \end{aligned} \tag{2.12}$$

Estimate (2.12) motivates us to define recurrent function $f_k : [0, +\infty) \to (-\infty, +\infty)$ by

$$\begin{aligned} f_k(s) = {} & 2(2K+M)(1+s+\cdots+s^{k-1})\eta + Ks^k\eta \\ & + 2sL(1+s+\cdots+s^k)\eta + 2(\mu+N) - 2s(1-l). \end{aligned} \tag{2.13}$$

That is we can show instead of (2.12):

$$f_k(\alpha) \leq 0. \tag{2.14}$$

We need a relationship between two consecutive function f_k:

$$f_{k+1} = f_k(s) + 2(2K+M)s^k\eta + Ks^{k+1}\eta - Ks^k\eta + 2Ls^{k+2}\eta$$

so,

$$f_{k+1}(s) = f_k(s) + g(s)s^k\eta, \tag{2.15}$$

where,

$$g(s) = 2Ls^2 + Ks + 3K + 2M \geq 0. \tag{2.16}$$

In view of (2.15) and (2.16) we have

$$f_k(s) \leq f_{k+1}(s). \tag{2.17}$$

Let us define function $f_\infty : [0,1) \to (-\infty, +\infty)$ by

$$f_\infty(s) = \lim_{k\to+\infty} f_k(s). \tag{2.18}$$

Then, we have

$$f_k(\alpha) \le f_\infty(\alpha). \tag{2.19}$$

It follows from (2.14) and (2.19) that we can show instead of (2.19)

$$f_\infty(\alpha) = 0. \tag{2.20}$$

Using (2.13) and (2.18) we get

$$f_\infty(\alpha) = 2\left[\frac{(2K+M)\eta}{1-\alpha} + \frac{L\alpha\eta}{1-\alpha} + \mu + N - \alpha(1-l)\right]. \tag{2.21}$$

By the choice of α, (2.1) and (2.21) we get $f_\infty(\alpha) \le 0$. That completes the induction for (2.6) and (2.8). Hence, sequence $\{t_n\}$ is non-decreasing, bounded from above by t^{**} and as such it converges to t^*. Finally, estimate (2.7) follows from (2.6) by using standard majorization techniques [5, 12]. That completes the proof of the Lemma. □

We shall use A_n for $A(x_n)$ until the end of the study.

We can show the following semilocal convergence result for Newton-like method (1.2).

Theorem 2.2. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator, $G : D \to Y$ be continuous, $A(x)$ in $\mathcal{L}(X,Y)$ and $x_0 \in D$ be such that $A(x_0)^{-1} \in \mathcal{L}(Y,X)$. Assume there exist constants $\eta, L, M, N, l, \mu \ge 0$ and $K > 0$, such that for all $x, y \in D$*

$$\|A_0^{-1}(F(x_0) + G(x_0))\| \le \eta, \tag{2.22}$$

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \le K\|x - x_0\|, \tag{2.23}$$

$$\|A_0^{-1}(F'(x) - A(x))\| \le M\|x - x_0\| + \mu, \tag{2.24}$$

$$\|A_0^{-1}(A(x) - A_0)\| \le L\|x - x_0\| + l, \tag{2.25}$$

$$\|A_0^{-1}(G(x) - G(y))\| \le N\|x - y\|, \tag{2.26}$$

hypothesis of Lemma 2.1 hold and

$$\overline{U}(x_0, t^*) = \{x \in X : \|x - x_0\| \le t^*\} \subseteq D, \tag{2.27}$$

where t^* is given in Lemma 2.1. Then, sequence $\{x_n\}$ generated by Newton-like method (1.2) is well defined, remains in $\overline{U}(x_0, t^*)$ for all $n \ge 0$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) + G(x) = 0$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{2.28}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.29}$$

where sequence $\{t_n\}$ *is given by* (2.3)*. Furthermore, if there exists* $R \geq t^*$ *such that*

$$\overline{U}(x_0, R) \subset D \tag{2.30}$$

and

$$\frac{3}{2}KR + (K + M + L)t^* + \mu + N + l < 1, \tag{2.31}$$

then, x^* *is the unique solution of* R *equation* $F(x) + G(x) = 0$ *in* $\overline{U}(x_0, R)$*.*

Proof. We shall show using induction:

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{2.32}$$

and

$$\overline{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \overline{U}(x_k, t^* - t_k). \tag{2.33}$$

For every $z \in \overline{U}(x_1, t^* - t_1)$

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 - t_0 = t^* - t_0$$

shows $z \in \overline{U}(x_0, t^* - t_0)$.

Since, also

$$\|x_1 - x_0\| = \|A_0^{-1}(F(x_0) + G(x_0))\| \leq \eta = t_1 - t_0,$$

estimate (2.32) holds for $k = 0$. Assume (2.32) and (2.33) hold for all $k \leq n$. Then, we have

$$\begin{aligned} \|x_{k+1} - x_0\| &= \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq (t_{k+1} - t_k) + (t_k - t_{k-1}) + \cdots + (t_1 - t_0) = t_{k+1} \leq t^* \end{aligned}$$

and

$$\|x_k + \theta(x_{k+1} - x_k - x_0)\| \leq t_k + \theta(t_{k+1} - t_k) \leq t^*,\ \theta \in [0, 1].$$

Using (2.25) and the induction hypothesis we get

$$\|A_0^{-1}(A(x_k) - A_0)\| \leq L\|x_k - x_0\| + l \leq Lt_k + l < 1 \tag{2.34}$$

(by Lemma 2.1).

It follows from (2.34) and the Banach Lemma on invertible operators [5,12] that $A(x_k)^{-1} \in \mathcal{L}(Y, X)$ and

$$\|A(x_k)^{-1}A_0\| \leq \frac{1}{1 - l - L\|x_{k+1} - x_0\|} \leq \frac{1}{1 - l - Lt_{k+1}}. \tag{2.35}$$

Using (1.2) we obtain in term

$$\begin{aligned}F(x_k)+G(x_k) &= F(x_k)+G(x_k)-F(x_{k-1})-G(x_{k-1})\\&\quad -A(x_{k-1})(x_k-x_{k-1})\\&= \int_0^1 \Big[(F'(x_{k-1}+\theta(x_k-x_{k-1}))-F'(x_0))\\&\quad +(F'(x_0)-F'(x_{k-1}))\Big](x_k-x_{k-1})\,d\theta\\&\quad +(F'(x_{k-1})-A(x_{k-1}))(x_k-x_{k-1})\\&\quad +(G(x_k)-G(x_{k-1})).\end{aligned} \tag{2.36}$$

Then, by (2.3), (2.23), (2.24), (2.26), (2.36) and the induction hypothesis we get

$$\begin{aligned}\|A_0^{-1}(F(x_k)+G(x_k))\| &\le \int_0^1 \|A_0^{-1}[F'(x_{k-1}+\theta(x_k-x_{k+1}))\\&\quad -F'(x_0)]\|\|x_k-x_{k-1}\|\\&\quad +\|A_0^{-1}(F'(x_0)-F'(x_{k-1}))\|\|x_k-x_{k-1}\|\\&\quad +\|A_0^{-1}(F'(x_{k-1})-A(x_{k-1}))\|\|x_k-x_{k-1}\|\\&\quad +\|A_0^{-1}(G(x_k)-G(x_{k-1}))\|\\&\le [K(2\|x_{k-1}-x_0\|+\frac{1}{2}\|x_k-x_{k-1}\|)\\&\quad +M\|x_{k-1}-x_0\|+\mu+M]\|x_k-x_{k-1}\|\\&\le [K(2t_{k-1}+\frac{1}{2}(t_k-t_{k-1}))\\&\quad +Mt_{k-1}+\mu+N](t_k-t_{k-1}).\end{aligned} \tag{2.37}$$

It follows from (1.2), (2.35) and (2.37)

$$\begin{aligned}&\|x_{k+1}-x_k\|\\&=\|[A(x_k)^{-1}A_0][A_0^{-1}(F(x_k)+G(x_k))]\|\\&\le \|A(x_k)^{-1}A_0\|\|A_0^{-1}(F(x_k)+G(x_k))\|\\&\le \frac{K(2t_{k-1}+\frac{1}{2}(t_k-t_{k-1}))+Mt_{k-1}+\mu+N}{1-l-Lt_k}(t_k-t_{k-1})\\&=t_{k+1}-t_k,\end{aligned} \tag{2.38}$$

with completes the induction for (2.32).

For every $w\in\overline{U}(x_{k+1},t^*-t_{k+1})$ we have

$$\|w-x_k\|\le\|w-x_{k+1}\|+\|x_{k+1}-x_k\|\le t^*-t_{k+1}+t_{k+1}-t_k=t^*-t_k,$$

showing (2.33). Lemma 2.1 implies $\{t_n\}$ is a Cauchy sequence. It then follows from (2.32) and (2.33) that $\{x_n\}$ is a Cauchy sequence too in a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, t^*)$ (since $\overline{U}(x_0, t^*)$ is a closed set). By letting $k \to +\infty$ in (2.37) we get $F(x^*)+G(x^*) = 0$. Estimate (2.29) follows from (2.28) by using standard majorization techniques [5,12]. Finally, to show uniqueness, let $y^* \in \overline{U}(x_0, R)$ be a solution of equation $F(x) + G(x) = 0$. As in (2.36) we obtain the approximation

$$\begin{aligned} x_{k+1} - y^* &\\ = -A(x_k)^{-1}\Big[\int_0^1 & F'((y^* + \theta(x_k - y^*)) - F'(x_0))(x_k - y^*)\,d\theta \\ &+ (F'(x_0) - F'(x_k))(x_k - y^*) \\ &+ (F'(x_k) - A(x_k))(x_k - y^*) + (G(x_k) - G(y^*))\Big]. \end{aligned} \tag{2.39}$$

Then, as in (2.35) and (2.37) we get

$$\begin{aligned} &\|x_{k+1} - y^*\| \\ &\le \frac{[K(\|y^* - x_0\| + \frac{1}{2}\|x_k - y^*\| + \|x_k - x_0\|) + M\|x_k - x_0\|}{1 - l - L\|x_k - x_0\|} \\ &\quad + \frac{\eta + N]\|x_k - y^*\|}{1 - l - L\|x_k - x_0\|} \\ &\le \frac{\frac{3}{2}KR + (k + M)z^* + \mu + N}{1 - l - Lt^*}\|x_k - y^*\| \\ &< \|x_k - y^*\| \end{aligned} \tag{2.40}$$

(by (2.30) and (2.31)), which implies $\lim_{k\to+\infty} x_k = y^*$. But we showed $\lim_{k\to+\infty} x_k = x^*$. Hence, we deduce $x^* = y^*$. That completes the proof of the Theorem. □

(*a*) Note that t^{**} given in closed form by (2.4) can replace t^* in Theorem 2.2.

(*b*) We have already provided convergence results under conditions (2.22)–(2.27) together with the Lipschitz condition

$$\|A_0^{-1}(F'(x) - F'(y))\| \le \overline{K}\|x - y\| \tag{2.41}$$

for all $x, y \in D$. [5, 12]. However, there are operators F' satisfying (2.23) but not (2.41) (see Section 4). Hence, Theorem 2.2 is weaker than in earlier results using (2.22), (2.24)-(2.27) and (2.41) can be found in [1–12].

The importance of introducing center-Lipschitz condition (2.23) and using it (in combination with (2.41) or not) has been shown in [13], where weaker than before sufficient convergence conditions and tighter error bounds for Newton-like methods have been found.

3. Semilocal Convergence Analysis II

In Section 3 we study the semilocal convergence analysis of Newton-like method (1.4). As in Lemma 2.1 we need a result on majorizing sequences for Newton-like method (1.4).

Lemma 3.1. *Let $\eta > 0$, $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\mu_0 \in (0,3)$ be given. Set*

$$b_0 = \frac{\frac{\varepsilon}{4}(3-\mu_0)}{1-\frac{\varepsilon}{4}(3-\mu_0)} \quad \text{and} \quad b = \frac{\varepsilon}{1-\varepsilon}\cdot \tag{3.1}$$

Define scalar sequence $\{s_n\}$ by

$$\begin{cases} s_0 = 0, \quad s_1 = \eta, \quad s_2 = s_1 + b_0(s_1 - s_0), \\ s_{n+2} = s_{n+1} + b(s_{n+1} - s_n). \end{cases} \tag{3.2}$$

Then, sequence $\{s_n\}$ is non-decreasing bounded from above by

$$s^{**} = \frac{b_n}{1-b} + (b_0 - b)\eta \tag{3.3}$$

and converges to its unique least upper bound s^* satisfying

$$s^* \in [0, s^{**}]. \tag{3.4}$$

Moreover the following estimates hold

$$s_{n+1} - s_n \leq b(s_n - s_{n-1}) \leq b^n \eta \tag{3.5}$$

and

$$s^* - s_n \leq \frac{b^n \eta}{1-b}\cdot \tag{3.6}$$

As in Theorem 2.2 we can show the following semilocal convergence result for Newton-like method (1.4).

Theorem 3.2. *Let $F : D \subset X \to Y$ be Gateaux-differentiable, $G : X \to Y$ continuous with divided difference of order one denoted by $[x, y; G]$. Assume:*

(i) *there exist $x_0, z_{-1}, z_0 \in D$ such that $L_0^{-1} \in \mathcal{L}(Y, X)$, $\{\gamma_n\}$, $\{s_n\}$ in $[0,1]$, $\{z_n\} \in \overline{U}(x_0, s^*) \subseteq D$, $z_{-1} = x_0$, $z_0 = x_0$,*

(ii) *$F'(x)$, $[x, y; G]$ are precewise-hemicontinuous on $\overline{U}(x_0, s^*)$, $\overline{U}^2(x_0, s^*)$, respectively,*

(iii) *for* $\varepsilon \in \left(0, \frac{1}{2}\right)$

$$\|L_0^{-1}(F'(x) - F'(x_0))\| \leq \frac{\varepsilon}{4} \tag{3.7}$$

and

$$\|L_0^{-1}([x, y; G] - [x_{-1}, x_0; G])\| \leq \frac{\varepsilon}{4} \tag{3.8}$$

for all $x, y \in \overline{U}(x_0, s^*)$.

Then, sequence $\{x_n\}$ *generated by Newton-like method (1.4) is well defined, remains in* $\overline{U}(x_0, s^*)$ *for all* $n \geq 0$ *and converges to a unique solution* x^* *in* $\overline{U}(x_0, s^*)$ *of equation* $F(x) + G(x) = 0$*. Moreover, the following estimates hold*

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{3.9}$$

and

$$\|x_n - x^*\| \leq s^* - s_n. \tag{3.10}$$

Proof. We use the proof of Theorem 2.2 but instead of (2.34) and (2.36), we need, respectively

$$\begin{aligned}
&L_0^{-1}(L_k - L_0) \\
&= \gamma_k L_0^{-1}(F'(x_k) - F'(x_0)) + (1 - \gamma_k) L_0^{-1}(F'(z_k) - F'(x_0)) \\
&\quad + \delta_k L_0^{-1}([x_{k+1}, x_k; G] - [x_{-1}, x_0; G]) \\
&\quad + (1 - \delta_k) L_0^{-1}([z_{k-1}, z_k; G] - [x_{-1}, x_0; G]) \\
&\quad + (1 - \delta_k) L_0^{-1}([x_{-1}, x_0; G] - [z_{-1}, z_0; G])
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
F(x_k) + G(x_k) &= [F(x_k) - F(x_{k-1}) - B_{k-1}(x_k - x_{k-1})] \\
&\quad + [G(x_k) - G(x_{k-1}) - C_{k-1}(x_k - x_{k-1})] \\
&= P_k + Q_k,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
P_k = \gamma_k &\int_0^1 [F'(x_{k-1} + \theta(x_k - x_{k-1})) - F'(x_0)](x_k - x_{k-1})\, d\theta \\
&+ \gamma_k (F'(x_0) - F'(x_{k-1}))(x_k - x_{k-1}) \\
&+ (1 - \gamma_k) \int_0^1 [F'(x_{k-1} + \theta(x_k - x_{k-1})) - F'(x_0)](x_k - x_{k-1})\, d\theta \\
&+ (1 - \gamma_k)(F'(x_0) - F'(z_n))(x_k - x_{k-1}),
\end{aligned} \tag{3.13}$$

$$\begin{aligned} Q_k = \delta_k([x_k, x_{k-1}; G] - [x_{-1}, x_0; G])(x_k - x_{k-1}) \\ + \delta_k([x_{-1}, x_0; G] - [x_{k-2}, x_{k-1}; G])(x_k - x_{k-1}) \\ + (1 - \delta_k)([x_k, x_{k-1}; G] - [x_{-1}, x_0; G])(x_k - x_{k-1}) \\ + (1 - \delta_k)([x_{-1}, x_0; G] - [z_{k-2}, z_{k-1}; G])(x_k - x_{k-1}). \end{aligned} \tag{3.14}$$

Then, we have by (3.7), (3.8), (3.11) and the choices of $\{z_k\}$, ε:

$$\begin{aligned} \|L_0^{-1}(L_k - L_0)\| &\leq \gamma_k \frac{\varepsilon}{4} + (1 - \gamma_k)\frac{\varepsilon}{4} + \delta_k \frac{\varepsilon}{4} + (1 - \delta_k)\frac{\varepsilon}{4} \\ &= \varepsilon < 1. \end{aligned} \tag{3.15}$$

It follows from (3.15) and the Banach Lemma on invertible operators that $L_k^{-1} \in \mathcal{L}(Y, X)$ so that

$$\|L_k^{-1} L_0\| \leq \frac{1}{1 - \varepsilon}\cdot \tag{3.16}$$

Using (3.2), (3.7), (3.8) and (3.12), we get

$$\begin{aligned} \|L_0^{-1}(F(x_k) + G(x_k))\| &\leq [2\frac{\varepsilon}{4}\gamma_k + 2\frac{\varepsilon}{4}(1 - \gamma_k) + 2\frac{\varepsilon}{4}\delta_k \\ &\quad + 2\frac{\varepsilon}{4}(1 - \delta_k)]\|x_k - x_{k-1}\| \\ &\leq \varepsilon\|x_k - x_{k-1}\| \leq \varepsilon(s_k - s_{k-1}). \end{aligned} \tag{3.17}$$

In view of (1.4), (3.1), (3.2), (3.16) and (3.17) we get

$$\|x_2 - x_1\| \leq \|L_1^{-1} L_0\|\|L_0^{-1}(F(x_1) + G(x_1))\| \leq b_0\|x_1 - x_0\| \leq b_0(t_1 - t_0) = t_2 - t_1,$$

$$\|x_{k+1} - x_k\| \leq \|L_k L_0\|\|L_0^{-1}(F(x_k) + G(x_k))\| \leq b(t_k - t_{k-1}) = t_{k+1} - t_k,$$

which show estimate (3.5). Then, (3.6) follows from (3.5). Moreover, by (1.4) we have

$$\begin{aligned} \|L_0^{-1}(F(x_k) + G(x_k))\| &\leq \|L_0^{-1} L_k(x_{k+1} - x_k)\| \\ &\leq \|L_0^{-1} L_k\|\|x_{k+1} - x_k\| \\ &\leq \|L_0^{-1}(L_k - L_0 + L_0)\|(t_{k+1} - t_k) \\ &\leq (1 + \|L_0^{-1}(L_k - L_0)\|)(t_{k-1} - t_k) \\ &= (1 + \varepsilon)(t_{k+1} - t_k). \end{aligned} \tag{3.18}$$

That is $F(x^*) + G(x^*) = 0$. Finally, to show uniqueness let x^* and y^* be two distinct solutions of equation

$$F(x) + G(x) = 0 \quad \text{in} \quad \overline{U}(x_0, s^*).$$

Then, we have

$$
\begin{aligned}
\|x^*-y^*\| &= \|x^*-y^*-L_0^{-1}(F(x^*)+G(x^*)-F(y^*)-G(y^*))\| \\
&= \|L_0^{-1}\Big\{\gamma_0\int_0^1[F'(y^*+\theta(x^*-y^*))-F'(x_0)]\,d\theta \\
&\quad + (1-\gamma_0)\int_0^1[F'(y^*+\theta(x^*-y^*))-F'(x_0)]\,d\theta \\
&\quad + (1-\gamma_0)(F'(x_0)-F'(z_0)) \\
&\quad + \delta_0([x^*,y^*;G]-[x_{-1},x_0;G]) \\
&\quad + (1-\delta_0)([x^*,y^*;G]-[z_{-1},z_0;G])\Big\}(x^*-y^*)\| \\
&\le \left[\gamma_0\frac{\varepsilon}{4}+2(1-\gamma_0)\frac{\varepsilon}{4}+\delta_0\frac{\varepsilon}{4}+2(1-\delta_0)\frac{\varepsilon}{4}\right]\|x^*-y^*\| \\
&= \frac{\varepsilon}{4}(4-(\gamma_0+\delta_0))\|x^*-y^*\| \le \varepsilon\|x^*-y^*\| \\
&< \|x^*-y^*\|,
\end{aligned}
\tag{3.19}
$$

which is a contradiction. Hence, we deduce $x^* = y^*$. That completes the proof of the Theorem. □

(a) The point s^{**} given in closed form by (3.3) can replace s^* in Theorem 3.2.

(b) Theorem 3.2 generalizes the results of Vijesh and Subrahmanyan, Weerakoon and Fernando, Sahuand Singh. Theorem 2.2 also extends the results of Ozban from real line to Banach spaces [14].

(c) For $G(x) = 0$, $\gamma_n = 0$ and $z_n = x_0$, Theorem 3.2 reduces to the modified Newton method [3, 5].

Set $\gamma_n = \gamma$ and $\delta_n = \delta$ in Theorem 3.2 to obtain:

Corollary 3.3. *Suppose that F and G are continuous and satisfy all conditions of Theorem 3.2. Then, Newton-like method (1.4) converges strongly to the unique solution x^* of equation $F(x) + G(x) = 0$ in $\overline{U}(x_0, s^*)$, where $z_n \in \overline{U}(x_0, s^*)$.*

(a) If $\delta_n = \delta = 0$, then corollary 3.3 reduces to corollary 2.1 in [13]. Moreover, if $G(x) = 0$, corollary 3.3 reduces to a result by Vijeshand Subrahmanyam [14].

(b) If $z_n = \dfrac{x_n + y_n}{2}$ and $\delta_n = \delta = 0$, we obtain Theorem 2.2 in [5]. In this case, theorem 3.2 is more general than Argyros [5].

4. Special Cases and Numerical Examples

We first provide an application of Lemma 2.1 and theorem 2.2 in the special case when $A(x) = F'(x)$ and $G(x) = 0$. Then, Lemma 2.1 reduces to:

Proposition 4.1. *Let $\eta, k > 0$ be given constants. Assume:*

$$k\eta \leq 5 - 2\sqrt{6}. \tag{4.1}$$

Then, scalar sequence $\{t_n\}$ given by

$$\begin{cases} t_0 = 0, \quad t_1 = \eta, \\ t_{n+2} = t_{n+1} + \dfrac{k(2t_n + \frac{1}{2}(t_{n+1} - t_n))}{1 - kt_{n+1}}(t_{n+1} - t_n) \end{cases} \tag{4.2}$$

is non-decreasing, bounded from above by

$$t^{**} = \frac{\eta}{1-\alpha} + (\alpha_0 - \alpha)\eta \tag{4.3}$$

and converges to its unique least upper bound t^* satisfying

$$0 \leq t^* \leq t^{**}, \tag{4.4}$$

where

$$\alpha = \frac{1 - k\eta - \sqrt{(1-k\eta)^2 - 8k\eta}}{2} \quad \text{and} \quad \alpha_0 = \frac{k\eta}{2(1-k\eta)}\cdot \tag{4.5}$$

Moreover, the following estimates hold

$$t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n \eta \tag{4.6}$$

and

$$t^* - t_n \leq \frac{\alpha^n \eta}{1-\alpha} \tag{4.7}$$

Proof. Set $M = N = l = \mu = 0$ and $L = k$ in Lemma 2.1. □

Next we provide Numerical examples where (2.25) holds but not (2.41).

Example 1. Let $X = \mathbb{R}$, $D = (0, 1.9)$ and $F : D \to \mathbb{R}$ a function defined by

$$F(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) - 1.6, & x \neq 0, \\ -1.6, & x = 0. \end{cases}$$

Then F is Gateaux differentiable function on D and its derivative at point $x \in D$ is

$$F'_x = \begin{cases} 2x \cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let initial point $x_0 = 1.5 \in D$ and radius of closed ball $\overline{U(x_0, r)}$ is 0.3 then, we have

$$\|{F'}^{-1}_{x_0} F(x_0)\| \le 0.056553\ldots = \eta,$$
$$\|{F'}^{-1}_{x_0} (F'_x - F'_{x_0})\| < 0.25 = K.$$

For this equation, condition (4.1) in Proposition (4.1) is verified and we can construct the scalar sequence $\{t_n\}$ corresponding to (4.2) and obtain the iterations and a priori error bounds that are in Table 1. Besides, we have that the sequence converges to $t^* = 0.0569510105\ldots$.

Iteration	t_{n+2}	$\lvert t_{n+2} - t_{n+1}\rvert$
1	$0.0569390233\ldots$	$0.000405\ldots$
2	$0.0569506643\ldots$	$0.000011\ldots$
3	$0.0569510005\ldots$	$3.3621\ldots \times 10^{-7}$
4	$0.0569510102\ldots$	$9.7122\ldots \times 10^{-9}$

Table 1. Scalar sequence $\{t_n\}$ and a priori error bounds

Example 2. Let $X = \mathcal{L}^2([0,1])$ and consider el operator equation (1.1) with $G(x) = 0$, where

$$F(x)(t) = x(t) + \lambda \cos\left(\int_0^t x(s)\,ds\right), \quad \lambda \in \mathbb{R}.$$

Then F is nowhere Fréchet differentiable but everywhere Gateaux differentiable and $x \to F'_x$ is hemicontinuous. The Gateaux derivative of F is given by

$$F'_x h(t) = h(t) + \lambda \int_0^t h(s)\,ds \sin\left(\int_0^1 h(s)\,ds\right), \quad \text{for all} \quad h \in \mathcal{L}^2([0,1]).$$

For the choice $x_0 = 0$, take $r = 0.02$ and we have

$$F'_{x_0} = I.$$

Clearly, the operator F'_{x_0} is invertible. Again, we have

$$\|F'^{-1}_{x_0} F(x_0)\| = \lambda.$$

For $x \in U(x_0, r)$ and using definition of norm, we can write

$$\begin{aligned}
\|F'^{-1}_{x_0}(F'_x - F'_{x_0})h(t)\| &= \lambda \left\| \int_0^t h(s)\,ds \sin\left(\int_0^t x(s)\,ds\right)\right\| \\
&= \lambda \left[\int_0^1 \left|\int_0^t h(s)\,ds \sin\left(\int_0^t x(s)\,ds\right)\right|^2\right]^{\frac{1}{2}} \\
&\le \lambda \|h\|.
\end{aligned}$$

Hence we get

$$\|F_{x_0}'^{-1}(F_x' - F_{x_0}')\| \leq \lambda.$$

In particular, if $\lambda = 0.01$ then $\eta = 0.01$ and $K = 0.01$. The condition (4.1) of Proposition 4.1 is verified and the scalar sequence $\{t_n\}$ in (4.2) converges to its unique least upper bound $t^* = 0.010000500150046264\ldots$. We can see the results in Table 2.

Iteration	t_{n+2}	$\lvert t_{n+2} - t_{n+1}\rvert$
1	$0.010000500005000500\ldots$	$5.0005\ldots \times 10^{-7}$
2	$0.010000500150002625\ldots$	$1.0002\ldots \times 10^{-10}$
3	$0.010000500150004626\ldots$	$2.0006\ldots \times 10^{-14}$
4	$0.010000500150004626\ldots$	$3.4694\ldots \times 10^{-18}$

Table 2. Scalar sequence $\{t_n\}$ and a priori error bounds

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