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REVIEW OF RECENT STUDIES ON THE KKM THEORY

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Abstract. Since we introduced the KKM theory in 1992, there have appeared more than twelve hundred publications related to the theory. Many of them are concerned with KKM type theorems on particular spaces, their equivalent formulations, and their applications to various problems. Recently the KKM theory tends to the study on abstract convex spaces properly including such particular spaces. In this paper, we give a short history of the theory and review the current study through our previous comments or surveys.

1. Introduction

Since we introduced the KKM theory as an independent branch of Nonlinear Analysis in 1992 [11], there have appeared more than twelve hundred publications related to the theory. Many of them are concerned with KKM type theorems on particular spaces, their equivalent formulations, and their applications to various problems. Recently the KKM theory tends to the study on abstract convex spaces properly including generalized convex spaces (G-convex spaces for short) due to the author.

The concept of G-convex spaces has a number of modifications or imitations. In some of our previous works, we showed that such spaces are unified to the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ which can be made into G-convex spaces in several ways. Since then, in order to unify various types of relevant

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spaces and to upgrade the KKM theory, the present author has published a sequence of papers related to abstract convex spaces [16-19,21-24,26-40].

In this paper, we give a short history of the KKM theory and review the current study by recalling our previous comments or surveys in the sequence of papers.

In Section 2, the basic concepts on our abstract convex spaces are introduced. We recall some subclasses of abstract convex spaces; namely, convexity spaces, convex spaces, H-spaces, G-convex spaces, ϕ_A-spaces and their various particular forms. Section 3 deals with some equivalents of the KKM principle and their concise history. Finally, in Section 4, we review our previous works which were concerned with comments on particular researches on the KKM theory. Some of them are closely related to modifications of our G-convex space theory.

2. Abstract convex spaces

We follow our recent works [22, 23, 26, 32, 34, 35] and the references therein:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Example 2.3. The following are typical examples of abstract convex spaces. Others can be seen in [32] and the references therein.

(1) A *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a topological space E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection.
(2) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$ [10].
(3) An abstract convex space $(X, D; \Gamma)$ is called an H-*space* by Park if $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a *c*-space by Horvath [6, 7].
(4) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

    Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$.

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Example 2.5. We give examples of partial KKM spaces as in [32], where the references can be seen:

(1) Every G-convex space is a KKM space.
(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.
(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.
(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X \rangle$, the triple $(H \supset X; \Gamma)$ is a partial KKM space.
(5) Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.
(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.
(7) Kulpa and Szymanski [9] found some partial KKM spaces which are not KKM spaces.

Recently, we were concerned with a particular subclass of abstract convex spaces as follows [16, 19, 20, 25]:

Definition 2.6. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Example 2.7. We give examples of ϕ_A-spaces as follows; for the references, see [40]:

(1) An L-space, which is a G-convex space $(E; \Gamma)$.
(2) An MC-space, which is known to be a G-convex space.
(3) A topological space Y with property (H), that is, if, for each $N \in \langle Y\rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$ with $|N| = n + 1$.
(4) FC-spaces $(Y, \{\varphi_N\})$ whenever Y is a topological space and for each $N \in \langle Y\rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$ with $|N| = n + 1$.
(5) Any G-convex space is a ϕ_A-space. Conversely, a ϕ_A-space can be made into a G-convex space $(X, D; \Gamma)$ in several ways; see [16, 19, 20, 25, 40].
(6) Later ϕ_A-spaces are called GFC-spaces by Khanh et al. [8].

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. New basic theorems of the KKM theory

For an abstract convex space $(E, D; \Gamma)$, consider the following related four conditions for a multimap $G : D \multimap E$:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

For a multimap $G : D \multimap E$, consider the following related four conditions:

(a) $\bigcup_{z\in D} G(z) = E$ implies $\bigcup_{z\in D} \operatorname{Int} G(z) = E$.

(b) $\operatorname{Int} \bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *unionly open-valued*).

(c) $\bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *transfer open-valued*).

(d) G is open-valued.

From the partial KKM principle, we have the following KKM type theorem in [35, 39]:

Theorem 3.1. [KKM type theorem] *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map; and*
(2) *there exists a nonempty compact subset K of E such that either*
  (i) *$\bigcap_{z\in M} \overline{G(z)} \subset K$ for some $M \in \langle D\rangle$; or*
  (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z\in D'} \overline{G(z)} \subset K.$$

Then $K \cap \bigcap_{z\in D} \overline{G(z)} \neq \emptyset$. Furthermore, the following hold:
(α) *If G is transfer closed-valued, then $K \cap \bigcap_{z\in D} G(z) \neq \emptyset$.*
(β) *If G is intersectionally closed-valued, then $\bigcap_{z\in D} G(z) \neq \emptyset$.*

Theorem 3.1 can be reformulated to several equivalent forms. We give here only two of them:

Theorem 3.2. [Fan–Browder type fixed point theorem] *Let $(E, D; \Gamma)$ be a partial KKM space and $S : D \multimap E$, $T : E \multimap E$ maps. Suppose that*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S^-(x) \subset T^-(x)$;*
(2) *$E = S(D)$; and*
(3) *there exists a nonempty compact subset K of E such that either*
  (i) *$\bigcap_{z\in M} \overline{E \setminus S(z)} \subset K$ for some $M \in \langle D\rangle$; or*
  (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z\in D'} \overline{E \setminus S(z)} \subset K.$$

(α) *If S is transfer open-valued, then there exists an $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.*
(β) *If S is unionly open-valued, then there exists an $\bar{x} \in E$ such that $\bar{x} \in T(\bar{x})$.*

Theorem 3.3. [Fan type minimax inequality] *Let $(E, D; \Gamma)$ be a partial KKM space, $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ extended real-valued functions, and $\gamma \in \overline{\mathbb{R}}$ such that*

(0) *for each $x \in E$, $g(x, x) \leq \gamma$;*
(1) *for each $z \in D$, $G(z) := \{y \in E \mid f(z, y) \leq \gamma\}$ is intersectionally closed [resp., transfer closed];*
(2) *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in E \mid g(x, y) > \gamma\}$; and*
(3) *there exists a nonempty compact subset K of E such that the map $G : D \multimap E$ satisfies either*
  (i) *$K \supset \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D \rangle$; or*
  (ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K \supset \overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \neq \emptyset.$$

Then

(a) *there exists a $y_0 \in E$ [resp., $y_0 \in K$] such that $f(z, y_0) \leq \gamma$ for all $z \in D$; and*
(b) *if $\gamma := \sup_{x \in E} g(x, x)$, then we have*

$$\inf_{y \in E} \sup_{z \in D} f(z, y) \leq \sup_{x \in E} g(x, x).$$

In our previous work [35], we showed that these three theorems are mutually equivalent and basic in the KKM theory of abstract convex spaces. For the origins of Theorems 3.1-3.3, see [23] and references therein.

Here we recall briefly applications of particular forms of Theorems 3.1-3.3 on particular types of KKM spaces in the chronological order.

(1) At the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Ky Fan in 1961-84. Actually, Fan [2] obtained his well-known KKM lemma and its direct applications. A number of intersection theorems and applications to equilibrium problems followed; see [13].

(2) The origin of Theorem 3.2 is due to Browder in 1968 [1], and equivalent to Fan's geometric lemma in 1961 [2]. Browder gave applications to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. Later Browder's theorem was also applied to the existence of maximal elements in mathematical economics.

(3) Fan in 1972 [3] established the origin of the minimax inequality (Theorem 3.3) from his KKM lemma and applied it to variational inequalities (of

Hartman-Stampacchia and of Browder), geometric formulations, separation properties of u.d.c. multimaps, coincidence and fixed point theorems, properties of sets with convex sections, and potential theory.

Further applications of the inequality appeared in various fields in mathematical sciences, for example, nonlinear analysis, especially in fixed point theory, variational inequalities, various equilibrium theory, mathematical programming, partial differential equations, game theory, impulsive control, and mathematical economics; see [13, 23] and the references therein.

(4) In 1981, Gwinner [5] displayed relations and connections between some of the most fundamental results of modern nonlinear analysis in the form of a circular tour. His tour started in a traditional way, but also ended with the classical KKM theorem; thus eight results in [5] are in some wide sense equivalent to the KKM theorem. Nowadays there are nearly one hundred such equivalent results; see [13] and references therein. Especially, in [5], an infinite dimensional analogue of the Walras' excess demand theorem was given, which is equivalent to the Fan-Glicksberg fixed point theorem.

(5) Granas [4] studied generalizations and applications of early results in the KKM theory on convex subsets of topological vector spaces and unified classical results systematically. In fact, Granas expanded Fan's works systematically and established new topological methods in convex analysis and nonlinear analysis.

(6) Then, the KKM theory has been extended to convex spaces by Lassonde in 1983 [10], and to c-spaces (or H-spaces) by Horvath in 1984-93 [6, 7] and others. For other literature, see [13].

(7) Since 1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. In our previous work [14] on G-convex spaces, there exist more than 15 equivalent formulations of the KKM principle such as Alexandroff-Pasynkoff theorem, Fan type matching theorem, Tarafdar type intersection theorem, geometric or section properties, Fan-Browder type fixed point theorems, maximal element theorems, analytic alternatives, Fan type minimax inequalities, variational inequalities, and others. This is also true for KKM spaces.

(8) In 2001, Park and Jeong [41] showed that the Brouwer fixed point theorem is equivalent to a number of results closely related to the Euclidean spaces or n-simplexes or n-balls. Among them are the Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

(9) Since 2007 we introduced the concept of ϕ_A-spaces as a generalization of various variants of G-convex spaces and showed that they can be made into G-convex spaces in several ways; see [16, 19, 20, 25, 40]. Later ϕ_A-spaces are called GFC-spaces by Khanh et al. [8] in 2009.

(10) There are several more statements equivalent to the KKM principle for abstract convex spaces; see [22, 32]. In [22-24, 32], we applied Theorems 3.1-3.3 and other equivalents of KKM spaces to obtain fixed point, minimax, and equilibrium theorems including the Nash equilibrium theorem. Especially, in [24], we showed that some of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces. Examples of such results are theorems of Sperner and Alexandroff-Pasynkoff, Horvath type fixed point theorem, Fan-Browder type coincidence theorems, Fan type minimax inequalities, variational inequalities, von Neumann type minimax theorem, Nash type equilibrium theorem, and Himmelberg type fixed point theorem. Moreover, in our other works, we obtained generalizations of the Gale-Nikaido-Debreu theorem.

4. Comments on other works in KKM theory

In this section, we review our previous works which were concerned with comments on particular researches on the KKM theory. Some of them are closely related to modifications of our G-convex space theory.

(1) NFAA 5: 2000 [15] — In the KKM theory, by giving finer topologies on the underlying spaces, we can destroy many of artificial terminology. Examples of such terminology are compact closures (ccl) and compact interiors (cint), finitely metrically closed sets, closed transfer completeness, transfer compactly open values, compactly local intersection property, and others.

(2) PNIMS 2: 2007 [16] — We introduce some subclasses of the class of abstract convex spaces, namely, abstract convex minimal spaces, minimal KKM spaces, generalized convex minimal spaces, and others. Each of these subclasses is convenient to establish the KKM theory and contains G-convex spaces properly. The class of G-convex spaces contains the classes of L-spaces, spaces having property (H), FC-spaces, and others. Some related matters are also discussed.

(3) NAF 12: 2007 [17] — We introduce basic results in the KKM theory on abstract convex spaces and the KKM maps. These are applied to various modifications of the concepts of G-convex spaces and KKM type maps. We discuss the nature of those modifications and criticize recently appeared 'generalizations' of our previous works due to many other authors.

(4) NAF 13: 2008 [18] — Some modifications of the concept of G-convex spaces are actually particular types of them. As an example, we introduce ϕ_A-spaces which include L-spaces, FC-spaces, and others. In this paper, we show that contents of sixteen recent papers on FC-spaces are consequences of the abstract convex space theory or the G-convex space theory developed by the author.

(5) JGO 45: 2009 [25] — We show that FC-spaces can be made into L-spaces, and hence particular types of G-convex spaces. Some examples are given and related matters are also discussed.

(6) PAMJ 18: 2008 [19] — Basic results in the KKM theory on abstract convex spaces and the KKM maps are applied to ϕ_A-spaces which unify various imitations of G-convex spaces. We show that basic theorems on ϕ_A-spaces can be applied to correct and improve results on the so-called R-KKM maps on the so-called L-convex spaces.

(7) CUBO 10: 2008 [20] — Various types of ϕ_A-spaces $(X, D; \phi_A)$ are simply G-convex spaces. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces.

(8) TJM 12: 2008 [21] — In [21], KKM theorems or coincidence theorems on abstract convex spaces are applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtain generalizations or improvements of a number of known equilibria results, especially, on the so-called FC-spaces.

(9) NA 73: 2010 [32] — We clearly derive a sequence of a dozen statements which characterize the KKM spaces and several equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces. Therefore, the whole contents of [32] is applicable to any ϕ_A-space $(X, D; \phi_A)$.

(10) NAF 15: 2010 [33] — In the KKM theory, various types of ϕ_A-spaces $(X, D; \phi_A)$ due to other authors are simply G-convex spaces. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. In 2006-09, G-convex spaces are extended to KKM spaces. In

[33], we review the recent transition from G-convex spaces to partial KKM spaces, and introduce a basic KKM theorem on these spaces.

(11) CKMS 25: 2010 [28] — Recently, some authors adopted the concept of the so-called *generalized R-KKM maps* which are used to rewrite some known results in the KKM theory. In [28], we show that those maps are simply KKM maps on G-convex spaces. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

(12) NAF 15: 2010 [29] — Recently there have appeared a very large number of papers on the KKM theory. In [29], we select several of them and give comments.

(13) NA 74: 2011 [34] — In the KKM theory, some authors adopt the concepts of the compact closure (ccl), compact interior (cint), transfer compactly closed-valued multimap, transfer compactly l.s.c. multimap, and transfer compactly local intersection property, respectively, instead of the closure, interior, closed-valued multimap, l.s.c. multimap, and possession of a finite open cover property. In this paper, we show that such adoption is inappropriate and artificial. In fact, any theorem with a "transfer" attached term is equivalent to the corresponding one without "transfer". Moreover, we can invalidate terms with "compactly" attached by giving a finer topology on the underlying space. In such ways, we obtain simpler formulations of KKM type theorems, Fan-Browder type fixed point theorems, and other results in the KKM theory on abstract convex spaces.

(14) NAF 16: 2011 [36] — After our previous work [12] appeared, some new results on continuous selection problem on G-convex spaces were obtained by a number of other authors. Some of them claimed to obtain already known results. In [36], we show that selection theorems in [12] with a few generalized forms of them contain certain selection theorems in more than a dozen papers mainly appeared after [12].

(15) CKMS 27: 2012 [37] — Recently, at least four of Ding's papers give examples of his FC-spaces which are not L-spaces. We show that they can be made into L-spaces. We also clarify that all statements in one of Ding's paper can be stated in corrected and generalized forms for the class of abstract convex spaces beyond FC-spaces.

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