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PARTIALLY ORDERED METRIC SPACES, RATIONAL CONTRACTIVE EXPRESSIONS AND COUPLED FIXED POINTS

Hemant Kumar Nashine[1] and Zoran Kadelburg[2]

[1]Department of Mathematics, Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud,
Raipur-492101(Chhattisgarh), India
e-mail: drhknashine@gmail.com, nashine_09@rediffmail.com

[2]University of Belgrade, Faculty of Mathematics
Studentski trg 16, 11000 Beograd, Serbia
e-mail: kadelbur@matf.bg.ac.rs

Abstract. We prove some coupled fixed point theorems for maps satisfying contractive conditions involving a rational expression in the setting of partially ordered metric spaces. We also present a result on the existence and uniqueness of coupled fixed points. An example is given to support the usability of our results, and to distinguish them from the known ones.

1. Introduction

The well-known Banach contraction theorem plays a major role in solving problems in many branches in pure and applied mathematics. A great number of generalizations of the Banach contraction principle were obtained in various directions. Many authors generalized this theorem to ordered metric spaces. The first such result was given by Ran and Reurings [19] who presented its applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [17] extended this result for non-decreasing mappings and applied it to obtain a unique solution for a periodic boundary value problem.

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[0]Corresponding Author: Zoran Kadelburg

Guo and Lakshmikantham [3] introduced the notion of a coupled fixed point for two mappings. Bhaskar and Lakshmikantham [2] proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property. Subsequently, several authors obtained many results of this kind (see, e.g., [5, 8, 11, 15, 16, 18, 21, 22, 24, 25]). These results have a lot of applications, e.g., in proving existence of solutions of periodic boundary value problems (e.g., [1, 2]) as well as particular integral equations (e.g., [7, 12, 13]).

Dass and Gupta [4] and Jaggi [10] proved fixed point theorems in metric spaces using contractive conditions involving rational expressions. Recently, Harjani et al. [6] and Luong and Thuan [14] derived results with such expressions in ordered metric spaces. In [23], Samet and Yazidi derived some coupled fixed point theorems of this kind.

In this paper we establish coupled fixed point results for mappings satisfying contractive condition involving a rational expression, more general than in [23], in the frame of partially ordered complete metric spaces. An example is given to support the usability of our results, and to distinguish them from the known ones.

2. Main results

Recall the following definitions.

Definition 2.1. ([3]) Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have *mixed monotone property* if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in X)\ x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y),$$
$$(\forall x, y_1, y_2 \in X)\ y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2).$$

Definition 2.2. ([3]) Let X be a nonempty set and let $F : X \times X \to X$. A point $(x, y) \in X \times X$ is said to be a *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

We will prove now coupled fixed point results which generalize the results of Samet and Yazidi [23].

Theorem 2.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property* *and satisfying*

$$\begin{aligned} & d(F(x,y),F(u,v)) \\ & \leq \frac{\alpha}{2}[d(x,u)+d(y,v)]+\beta M((x,y),(u,v)) \\ & \quad +\frac{\gamma}{2}[d(x,F(x,y))+d(u,F(u,v))+d(y,F(y,x))+d(v,F(v,u))] \\ & \quad +\frac{\delta}{2}[d(x,F(u,v))+d(y,F(v,u))+d(u,F(x,y))+d(v,F(y,x))], \end{aligned} \tag{2.1}$$

for all $(x,y),(u,v)\in X\times X$ *with* $x\succeq u$ *and* $y\preceq v$*, where*

$$M((x,y),(u,v))=\min\left\{d(x,F(x,y))\,\frac{2+d(u,F(u,v))+d(v,F(v,u))}{2+d(x,u)+d(y,v)},\right. \tag{2.2}$$
$$\left. d(u,F(u,v))\,\frac{2+d(x,F(x,y))+d(y,F(y,x))}{2+d(x,u)+d(y,v)}\right\}$$

and $\alpha,\beta,\gamma,\delta\geq 0$ *with* $\alpha+\beta+2\gamma+2\delta<1$*. We assume that there exists* $x_0,y_0\in X$ *such that*

$$x_0\preceq F(x_0,y_0) \quad \text{and} \quad y_0\succeq F(y_0,x_0). \tag{2.3}$$

Then, F *has a coupled fixed point* $(\bar{x},\bar{y})\in X\times X$*.*

Proof. Denote $x_1=F(x_0,y_0)$ and $y_1=F(y_0,x_0)$. Then $x_0\preceq x_1$ and $y_0\succeq y_1$, by (2.3). Further denote

$$x_2=F(x_1,y_1)=F(F(x_0,y_0),F(y_0,x_0))=F^2(x_0,y_0)$$

and

$$y_2=F(y_1,x_1)=F(F(y_0,x_0),F(x_0,y_0))=F^2(y_0,x_0).$$

Due to the mixed monotone property of F, we have

$$x_2=F(x_1,y_1)\succeq F(x_0,y_1)\succeq F(x_0,y_0)=x_1$$

and

$$y_2=F(y_1,x_1)\preceq F(y_0,x_1)\preceq F(y_0,x_0)=y_1.$$

Further, for $n=1,2,\ldots,$ we let

$$x_{n+1}=F^{n+1}(x_0,y_0)=F(F^n(x_0,y_0),F^n(y_0,x_0))$$

and

$$y_{n+1}=F^{n+1}(y_0,x_0)=F(F^n(y_0,x_0),F^n(x_0,y_0)).$$

We check easily that

$$x_0\preceq x_1\preceq x_2\preceq\cdots\leq x_n\preceq\cdots \tag{2.4}$$

and

$$y_0\succeq y_1\succeq y_2\succeq\cdots\succeq y_n\succeq\cdots. \tag{2.5}$$

If $x_{n+1} = x_n$ and $y_{n+1} = y_n$ for some n, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$, hence (x_n, y_n) is a coupled fixed point of F. Suppose, further, that

$$x_n \neq x_{n+1} \text{ or } y_n \neq y_{n+1} \text{ for each } n \in \mathbb{N}_0.$$

Now, we claim that, for $n \in \mathbb{N}_0$,

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)^n [d(x_1, x_0) + d(y_1, y_0)]. \quad (2.6)$$

Indeed, for $n = 1$, using $x_1 \succeq x_0$, $y_1 \preceq y_0$ and (2.1), we get:

$$\begin{aligned}
&d(x_2, x_1) = d(F(x_1, y_1), F(x_0, y_0)) && (2.7)\\
&\leq \frac{\alpha}{2}[d(x_1, x_0) + d(y_1, y_0)] + \beta M((x_1, y_1), (x_0, y_0))\\
&\quad + \frac{\gamma}{2}[d(x_1, F(x_1, y_1)) + d(x_0, F(x_0, y_0)) + d(y_1, F(y_1, x_1)) + d(y_0, F(y_0, x_0))]\\
&\quad + \frac{\delta}{2}[d(x_1, F(x_0, y_0)) + d(y_1, F(y_0, x_0)) + d(x_0, F(x_1, y_1)) + d(y_0, F(y_1, x_1))]\\
&\leq \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)]\\
&\quad + \beta d(x_1, F(x_1, y_1)) \frac{2 + d(x_0, F(x_0, y_0)) + d(y_0, F(y_0, x_0))}{2 + d(x_0, x_1) + d(y_0, y_1)}\\
&\quad + \frac{\gamma}{2}[d(x_1, x_2) + d(x_0, x_1) + d(y_1, y_2) + d(y_0, y_1)]\\
&\quad + \frac{\delta}{2}[d(x_1, x_1) + d(y_1, y_1) + d(x_0, x_2) + d(y_0, y_2)]\\
&\leq \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)] + \beta d(x_1, x_2)\\
&\quad + \frac{\gamma+\delta}{2}[d(x_0, x_1) + d(y_0, y_1) + d(x_1, x_2) + d(y_1, y_2)].
\end{aligned}$$

Similarly, using that $d(y_2, y_1) = d(F(y_1, x_1), F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$ and

$$\begin{aligned}
M((x_1, y_1), (x_0, y_0)) &\leq d(y_1, F(y_1, x_1)) \frac{2 + d(y_0, F(y_0, x_0)) + d(x_0, F(x_0, y_0))}{2 + d(y_0, y_1) + d(x_0, x_1)}\\
&= d(y_1, y_2),
\end{aligned}$$

we get

$$\begin{aligned}
d(y_2, y_1) \leq{}& \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)] + \beta d(y_1, y_2) && (2.8)\\
&+ \frac{\gamma+\delta}{2}[d(x_0, x_1) + d(y_0, y_1) + d(x_1, x_2) + d(y_1, y_2)].
\end{aligned}$$

Adding (2.7) and (2.8), we have

$$d(x_2, x_1) + d(y_2, y_1) \leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)[d(x_0, x_1) + d(y_0, y_1)].$$

In a similar way, proceeding by induction, if we assume that (2.6) holds, we get that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1}) &\leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] \\ &\leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)^{n+1}[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

Hence, by induction, (2.6) is proved.

Set

$$h_n := d(x_n, x_{n+1}) + d(y_n, y_{n+1}), \quad n \in \mathbb{N}$$

and $\Delta := \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Then, the sequence $\{h_n\}$ is decreasing and

$$h_n \leq \Delta^n h_0.$$

By assumption (2.4), $h_n > 0$ for $n \in \mathbb{N}_0$. Then, for each $n \geq m$ we have

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$

and

$$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m).$$

Therefore,

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\leq h_{n-1} + h_{n-2} + \cdots + h_m \\ &\leq (\Delta^{n-1} + \Delta^{n-2} + \cdots + \Delta^m)h_0 \\ &\leq \frac{\Delta^m}{1-\Delta}h_0 \end{aligned}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X since $0 \leq \Delta < 1$. Since (X, d) is a complete metric space, there exists $(\bar{x}, \bar{y}) \in X \times X$ such that

$$\lim_{n\to\infty} x_n = \bar{x} \quad \text{and} \quad \lim_{n\to\infty} y_n = \bar{y}. \tag{2.9}$$

Finally, we claim that $(\bar{x}, \bar{y})$ is a coupled fixed point of F. Indeed, from $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, using (2.9) and the continuity of F, it immediately follows that $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$. This completes the proof of the theorem. □

In the next theorem, we will substitute the continuity hypothesis on F by an additional property satisfied by the space $(X, d, \preceq)$.

Theorem 2.4. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property. Assume that there exist $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ such that*

$$\begin{aligned} &d(F(x,y), F(u,v)) \\ &\leq \frac{\alpha}{2}[d(x,u) + d(y,v)] + \beta M((x,y),(u,v)) \\ &\quad + \frac{\gamma}{2}[d(x, F(x,y)) + d(u, F(u,v)) + d(y, F(y,x)) + d(v, F(v,u))] \\ &\quad + \frac{\delta}{2}[d(x, F(u,v)) + d(y, F(v,u)) + d(u, F(x,y)) + d(v, F(y,x))] \end{aligned}$$

for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, where

$$\begin{aligned} M((x,y),(u,v)) = \min\Big\{ & d(x, F(x,y))\,\frac{2 + d(u, F(u,v)) + d(v, F(v,u))}{2 + d(x,u) + d(y,v)}, \\ & d(u, F(u,v))\,\frac{2 + d(x, F(x,y)) + d(y, F(y,x))}{2 + d(x,u) + d(y,v)} \Big\}. \end{aligned}$$

Suppose that there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0).$$

Finally, assume that X has the following properties:

(i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
(ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n.*

Then, F has a coupled fixed point $(\bar{x}, \bar{y}) \in X \times X$.

Proof. Following the proof of Theorem 2.3, we only have to show that $(\bar{x}, \bar{y})$ is a coupled fixed point of F. We have

$$\begin{aligned} d(F(\bar{x}, \bar{y}), \bar{x}) &\leq d(F(\bar{x}, \bar{y}), x_{n+1}) + d(x_{n+1}, \bar{x}) \\ &= d(F(\bar{x}, \bar{y}), F(x_n, y_n)) + d(x_{n+1}, \bar{x}). \end{aligned} \tag{2.10}$$

Since the nondecreasing sequence $\{x_n\}$ converges to $\bar{x}$ and the nonincreasing sequence $\{y_n\}$ converges to $\bar{y}$, by (i)–(ii), we have:

$$\bar{x} \succeq x_n \quad \text{and} \quad \bar{y} \preceq y_n, \quad \forall\, n.$$

Now, from the contractive condition (2.1), we have:

$$\begin{aligned}
& d(F(\bar{x},\bar{y}),F(x_n,y_n)) \\
& \leq \frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta M((\bar{x},\bar{y}),(x_n,y_n)) \\
& \quad +\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,F(x_n,y_n))+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,F(y_n,x_n))] \\
& \quad +\frac{\delta}{2}[d(\bar{x},F(x_n,y_n))+d(\bar{y},F(y_n,x_n))+d(x_n,F(\bar{x},\bar{y}))+d(y_n,F(\bar{y},\bar{x}))] \\
& \leq \frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)] \\
& \quad +\beta d(\bar{x},F(\bar{x},\bar{y}))\,\frac{2+d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2+d(\bar{x},x_n)+d(\bar{y},y_n)} \\
& \quad +\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,x_{n+1})+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,y_{n+1})] \\
& \quad +\frac{\delta}{2}[d(\bar{x},x_{n+1})+d(\bar{y},y_{n+1})+d(x_n,F(\bar{x},\bar{y}))+d(y_n,F(\bar{y},\bar{x}))].
\end{aligned}$$

Then, from (2.10), we get:

$$\begin{aligned}
& d(F(\bar{x},\bar{y}),\bar{x}) \\
& \leq d(x_{n+1},\bar{x}) \\
& \quad +\frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta d(\bar{x},F(\bar{x},\bar{y}))\,\frac{2+d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2+d(\bar{x},x_n)+d(\bar{y},y_n)} \\
& \quad +\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,x_{n+1})+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,y_{n+1})] \\
& \quad +\frac{\delta}{2}[d(\bar{x},x_{n+1})+d(\bar{y},y_{n+1})+d(x_n,F(\bar{x},\bar{y}))+d(y_n,F(\bar{y},\bar{x}))].
\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$d(F(\bar{x},\bar{y}),\bar{x}) \leq \beta d(\bar{x},F(\bar{x},\bar{y}))+\frac{\gamma+\delta}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))]. \quad (2.11)$$

Similarly,

$$d(\bar{y},F(\bar{y},\bar{x})) \leq \beta d(\bar{y},F(\bar{y},\bar{x}))+\frac{\gamma+\delta}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))]. \quad (2.12)$$

Adding (2.11) and (2.12), we have

$$\begin{aligned}
& d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x})) \\
& \leq (\beta+\gamma+\delta)[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))] \\
& \leq (\alpha+\beta+2\gamma+2\delta)[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))].
\end{aligned}$$

Since $0\leq\alpha+\beta+2\gamma+2\delta<1$, we obtain $d(F(\bar{x},\bar{y}),\bar{x})=0$ and $d(\bar{y},F(\bar{y},\bar{x}))$, i.e., $F(\bar{x},\bar{y})=\bar{x}$ and $F(\bar{y},\bar{x})=\bar{y}$. This completes the proof of the theorem. □

Now we shall prove a uniqueness theorem for the coupled fixed point. Note that, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x,y),(u,v) \in X \times X, \quad (u,v) \preceq (x,y) \Leftrightarrow x \succeq u,\ y \preceq v.$$

Theorem 2.5. *Assume that*

$$\begin{gathered} \forall (x,y),(x^*,y^*) \in X \times X,\ \exists\,(z_1,z_2) \in X \times X \\ \textit{that is comparable to } (x,y) \textit{ and } (x^*,y^*). \end{gathered} \tag{2.13}$$

Adding (2.13) *to the hypotheses of Theorem* 2.3*, we obtain the uniqueness of the coupled fixed point of* F*.*

Proof. From Theorem 2.3 we know that there exists a coupled fixed point $(\bar{x}, \bar{y})$ of F, which is obtained as $\bar{x} = \lim_{n\to\infty} F^n(x_0, y_0)$ and $\bar{y} = \lim_{n\to\infty} F^n(y_0, x_0)$. Suppose that (x^*, y^*) is another coupled fixed point, i.e.,

$$F(x^*, y^*) = x^* \quad \text{and} \quad F(y^*, x^*) = y^*.$$

Let us prove that

$$d(\bar{x}, x^*) + d(\bar{y}, y^*) = 0. \tag{2.14}$$

We distinguish two cases.

Case I: $(\bar{x}, \bar{y})$ is comparable with (x^*, y^*) with respect to the ordering in $X \times X$. Let, e.g., $\bar{x} \succeq x^*$ and $\bar{y} \preceq y^*$. Then, we can apply the contractive condition (2.1) to obtain

$$\begin{aligned} d(\bar{x}, x^*) &= d(F(\bar{x}, \bar{y}), F(x^*, y^*)) \\ &\le \frac{\alpha}{2}[d(\bar{x}, x^*) + d(\bar{y}, y^*)] + \delta[d(\bar{x}, x^*) + d(\bar{y}, y^*)], \end{aligned}$$

and

$$\begin{aligned} d(\bar{y}, y^*) &= d(F(\bar{y}, \bar{x}), F(y^*, x^*)) = d(F(y^*, x^*), F(\bar{y}, \bar{x})) \\ &\le \frac{\alpha}{2}[d(\bar{x}, x^*) + d(\bar{y}, y^*)] + \delta[d(\bar{x}, x^*) + d(\bar{y}, y^*)]. \end{aligned}$$

Adding up, we get that

$$d(\bar{x}, x^*) + d(\bar{y}, y^*) \le (\alpha + 2\delta)[d(\bar{x}, x^*) + d(\bar{y}, y^*)].$$

Since $0 \le \alpha + 2\delta < 1$, (2.14) holds.

Case II: $(\bar{x}, \bar{y})$ is not comparable with (x^*, y^*). In this case, there exists $(z_1, z_2) \in X \times X$ that is comparable both to $(\bar{x}, \bar{y})$ and (x^*, y^*). Then, for all $n \in \mathbb{N}$, $(F^n(z_1, z_2), F^n(z_2, z_1))$ is comparable both to $(F^n(\bar{x}, \bar{y}), F^n(\bar{y}, \bar{x})) =$

$(\bar{x},\bar{y})$ and $(F^n(x^*,y^*),F^n(y^*,x^*))=(x^*,y^*)$. We have

$$\begin{aligned} d(\bar{x},x^*)+d(\bar{y},y^*) &= d(F^n(\bar{x},\bar{y}),F^n(x^*,y^*))+d(F^n(\bar{y},\bar{x}),F^n(y^*,x^*)) \\ &\le d(F^n(\bar{x},\bar{y}),F^n(z_1,z_2))+d(F^n(z_1,z_2),F^n(x^*,y^*)) \\ &\quad + d(F^n(\bar{y},\bar{x}),F^n(z_2,z_1))+d(F^n(z_2,z_1),F^n(y^*,x^*)) \\ &\le (\alpha^n+2\delta^n)[d(\bar{x},z_1)+d(\bar{y},z_2)+d(x^*,z_1)+d(y^*,z_2)]. \end{aligned}$$

Since $0<\alpha,\delta<1$, (2.14) holds.

We deduce that in all cases (2.14) holds. This implies that $(\bar{x},\bar{y})=(x^*,y^*)$ and the uniqueness of the coupled fixed point of F is proved. □

If x_0,y_0 in X are comparable, we have the following result.

Theorem 2.6. *In addition to the hypotheses of Theorem* 2.3 *(resp. Theorem* 2.4*), suppose that x_0,y_0 in X are comparable. Then $\bar{x}=\bar{y}$.*

Proof. Suppose that $x_0\preceq y_0$. We claim that

$$x_n\preceq y_n,\ \forall\, n\in\mathbb{N}. \tag{2.15}$$

From the mixed monotone property of F, we have

$$x_1=F(x_0,y_0)\preceq F(y_0,y_0)\preceq F(y_0,x_0)=y_1.$$

Assume that $x_n\preceq y_n$ for some n. Now,

$$\begin{aligned} x_{n+1} &= F^{n+1}(x_0,y_0)=F(F^n(x_0,y_0),F^n(y_0,x_0)) \\ &= F(x_n,y_n) \\ &\preceq F(y_n,y_n)\preceq F(y_n,x_n) \\ &= y_{n+1}. \end{aligned}$$

Hence, (2.15) holds.

Now, using (2.15) and the contractive condition, we get

$$\begin{aligned}
&d(\bar{x},\bar{y})\\
&\le d(\bar{x},x_{n+1})+d(x_{n+1},y_{n+1})+d(y_{n+1},\bar{y})\\
&= d(\bar{x},x_{n+1})+d(F(y_n,x_n),F(x_n,y_n))+d(y_{n+1},\bar{y})\\
&\le d(\bar{x},x_{n+1})+d(y_{n+1},\bar{y})+\alpha d(x_n,y_n)+\beta M((y_n,x_n),(x_n,y_n))\\
&\quad+\frac{\gamma}{2}[d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))+d(y_n,F(y_n,x_n))+d(x_n,F(x_n,y_n)))]\\
&\quad+\frac{\delta}{2}[d(x_n,F(y_n,x_n))+d(y_n,F(x_n,y_n))+d(y_n,F(x_n,y_n))+d(x_n,F(y_n,x_n))]\\
&\le d(\bar{x},x_{n+1})+d(y_{n+1},\bar{y})+\alpha d(x_n,y_n)\\
&\quad+\beta d(y_n,y_{n+1})\,\frac{2+d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2+2d(y_n,x_n)}\\
&\quad+\gamma[d(x_n,x_{n+1})+d(y_n,y_{n+1})]+\delta[d(x_n,y_{n+1})+d(y_n,x_{n+1})]\\
&\le d(\bar{x},x_{n+1})+d(y_{n+1},\bar{y})+\alpha d(x_n,y_n)\\
&\quad+\beta d(y_n,y_{n+1})[2+d(x_n,x_{n+1})+d(y_n,y_{n+1})]\\
&\quad+\gamma[d(x_n,x_{n+1})+d(y_n,y_{n+1})]+\delta[d(x_n,y_{n+1})+d(y_n,x_{n+1})].
\end{aligned}$$

Passing to the limit as $n\to\infty$, we get that

$$d(\bar{x},\bar{y})\le(\alpha+2\delta)d(\bar{x},\bar{y}).$$

Since $0\le\alpha+2\delta<1$, this implies that $d(\bar{x},\bar{y})=0$, i.e., $\bar{x}=\bar{y}$. This completes the proof of the theorem. □

Remark 2.7. If we put

$$\mathcal{T}(x)=F(x,x),\quad\forall\, x\in X,$$

then for $x=y$ and $u=v$, the contractive condition (2.1) reduces to the condition for a single map (in the case without order) of Rhoades from [20, Corollary 15].

We illustrate our results by the following example which also distinguishes these result from the known ones.

Example 2.8. Let $X=[0,+\infty)$ be equipped with the standard metric and ordered by the relation $\preceq$ given by

$$x\preceq y\iff x=y\vee(x,y\in[0,1]\wedge x\le y).$$

Consider the (continuous) mapping $F:X\times X\to X$ given by

$$F(x,y)=\begin{cases}\dfrac{x^2-y^2}{8}, & x\ge y\\ 0, & x<y.\end{cases}$$

Let $\alpha, \beta, \gamma, \delta$ be nonnegative numbers satisfying $\frac{1}{2} \le \alpha < 1$ and $\alpha+\beta+2\gamma+2\delta < 1$, and denote by L and R, respectively, the left-hand and right-hand side of (2.1). Suppose that $x \succeq u$ and $y \preceq v$ and consider the following possible cases.

1) $x, u, y, v \in [0,1]$ and hence $x \ge u$, $y \le v$. Considering further six possibilities for the order of points x, y, u, v on the segment $[0,1]$ (and using that $x+y \le 2$ and $u+v \le 2$), we get that in each case

$$L \le \frac{x-y}{4} \le \frac{\alpha}{2}[d(x,u)+d(y,v)] \le R.$$

For example, if $0 \le y \le u \le x \le v \le 1$ then

$$L = d(F(x,y), F(u,v)) = d\left(\frac{x^2-y^2}{8}, 0\right) = \frac{x^2-y^2}{8} \le \frac{x-y}{4};$$

the other five cases are treated similarly.

2a) $x, u \in [0,1]$ and $y = v > 1$; then $L = 0$ and the condition is satisfied.

2b) $y, v \in [0,1]$ and $x = u > 1$; then

$$L = d\left(\frac{x^2-y^2}{8}, \frac{u^2-v^2}{8}\right) = \frac{x^2-y^2}{8} - \frac{x^2-v^2}{8} = \frac{v^2-y^2}{8} \le \frac{v-y}{4}$$
$$\le \frac{\alpha}{2}[d(x,u)+d(y,v)] \le R,$$

since $\frac{1}{2} \le \alpha < 1$.

3) $x = u > 1$ and $y = v > 1$; then obviously $L = 0$.

Thus, F satisfies all the assumptions of the given theorems and it has a unique coupled fixed point (which is $(0,0)$).

On the other hand, consider the same example in the case without order. Take $x = 4$ and $u = y = v = 0$. Then $F(x,y) = 2$, $F(u,v) = 0$, $L = 2$, but

$$R = \frac{\alpha}{2}[4+0] + \beta \cdot 0 + \frac{\gamma}{2}[2+0+0+0] + \frac{\delta}{2}[4+0+2+0] = 2\alpha + \gamma + 3\delta$$
$$\le 2(\alpha+\beta+2\gamma+2\delta) < 2,$$

whatever coefficients are taken satisfying the given condition.

The second main theorem uses contractive condition having a different type of rational expression.

Theorem 2.9. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property and satisfying*

$$\begin{aligned} &d(F(x,y), F(u,v)) \\ &\le \frac{\alpha}{2}[d(x,u)+d(y,v)] + \beta N((x,y),(u,v)) \\ &\quad + \frac{\gamma}{2}[d(x,F(x,y)) + d(u,F(u,v)) + d(y,F(y,x)) + d(v,F(v,u))], \end{aligned} \tag{2.16}$$

for all $(x,y),(u,v)\in X\times X$ *with* $x\succeq u$ *and* $y\preceq v$*, when* $D_1=d(x,F(u,v))+d(u,F(x,y))\neq 0$ *and* $D_2=d(y,F(v,u))+d(v,F(y,x))\neq 0$*, where*

$$N((x,y),(u,v)) = \min\left\{\frac{d^2(x,F(u,v))+d^2(u,F(x,y))}{d(x,F(u,v))+d(u,F(x,y))},\frac{d^2(y,F(v,u))+d^2(v,F(y,x))}{d(y,F(v,u))+d(v,F(y,x))}\right\} \tag{2.17}$$

and $\alpha,\beta,\gamma\geq 0$ *with* $\alpha+2\beta+2\gamma<1$*. Further,*

$$d(F(x,y),F(u,v))=0 \quad \text{if} \quad D_1=0 \text{ or } D_2=0. \tag{2.18}$$

We assume that there exist $x_0,y_0\in X$ *such that*

$$x_0\preceq F(x_0,y_0) \quad \text{and} \quad y_0\succeq F(y_0,x_0). \tag{2.19}$$

Then, F *has a coupled fixed point* $(\bar{x},\bar{y})\in X\times X$*.*

Proof. Following the proof of Theorem 2.3, we can construct sequences $\{x_n\}$ and $\{y_n\}$ satisfying conditions (2.4) and (2.5).

Now, we claim that, for $n\in\mathbb{N}$,

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)\leq\left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n[d(x_1,x_0)+d(y_1,y_0)]. \tag{2.20}$$

Indeed, for $n=1$, consider the following possibilities.

Case I: $x_0\neq x_2$ and $y_0\neq y_2$. Then $d(x_1,F(x_0,y_0))+d(x_0,F(x_1,y_1))\neq 0$ and $d(y_1,F(y_0,x_0))+d(y_0,F(y_1,x_1))\neq 0$. Hence, using $x_1\succeq x_0$, $y_1\preceq y_0$ and (2.16), we get:

$$\begin{aligned} d(x_2,x_1)&=d(F(x_1,y_1),F(x_0,y_0)) \qquad (2.21)\\ &\leq\frac{\alpha}{2}[d(x_1,x_0)+d(y_1,y_0)]+\beta N(x_1,y_1),(x_0,y_0))\\ &\quad+\frac{\gamma}{2}[d(x_1,F(x_1,y_1))+d(x_0,F(x_0,y_0))+d(y_1,F(y_1,x_1))+d(y_0,F(y_0,x_0))]\\ &\leq\frac{\alpha}{2}[d(x_0,x_1)+d(y_0,y_1)]+\beta\frac{d^2(x_1,F(x_0,y_0))+d^2(x_0,F(x_1,y_1))}{d(x_1,F(x_0,y_0))+d(x_0,F(x_1,y_1))}\\ &\quad+\frac{\gamma}{2}[d(x_1,x_2)+d(x_0,x_1)+d(y_1,y_2)+d(y_0,y_1)]\\ &\leq\frac{\alpha}{2}[d(x_0,x_1)+d(y_0,y_1)]+\beta[d(x_0,x_1)+d(x_1,x_2)]\\ &\quad+\frac{\gamma}{2}[d(x_0,x_1)+d(y_0,y_1)+d(x_1,x_2)+d(y_1,y_2)]. \end{aligned}$$

Similarly, using that $d(y_2,y_1)=d(F(y_1,x_1),F(y_0,x_0))=d(F(y_0,x_0),F(y_1,x_1))$ and

$$\begin{aligned} N((y_1,x_1),(y_0,x_0))&\leq\frac{d^2(y_1,F(y_0,x_0)+d^2(y_0,F(y_1,x_1))}{d(y_1,F(y_0,x_0))+d(y_0,F(y_1,x_1))}\\ &=d(y_0,y_2)\leq d(y_0,y_1)+d(y_1,y_2), \end{aligned}$$

we get

$$d(y_2, y_1) \leq \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)] + \beta[d(y_0, y_1) + d(y_1, y_2)] \\ + \frac{\gamma}{2}[d(x_0, x_1) + d(y_0, y_1) + d(x_1, x_2) + d(y_1, y_2)]. \tag{2.22}$$

Adding (2.21) and (2.22), we have

$$d(x_2, x_1) + d(y_2, y_1) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)[d(x_0, x_1) + d(y_0, y_1)]. \tag{2.23}$$

Case II: $x_0 = x_2$ and $y_0 \neq y_2$. The first equality implies that $d(x_1, F(x_0, y_0)) + d(x_0, F(x_1, y_1)) = 0$, and hence $d(x_1, x_2) = d(F(x_0, y_0), F(x_1, y_1)) = 0$, by (2.18). This means that $x_0 = x_1 = x_2$. From $y_0 \neq y_2$, as in the first case, we get that (2.22) holds true. As a consequence

$$d(y_1, y_2) \leq \frac{\frac{\alpha}{2} + \beta + \frac{\gamma}{2}}{1 - \beta - \frac{\gamma}{2}} d(y_0, y_1) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(y_0, y_1),$$

since $\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}} \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. But then $d(x_0, x_1) = d(x_1, x_2) = 0$ implies that (2.23) holds.

The case $x_0 \neq x_2$ and $y_0 = y_2$ is treated analogously.

Case III: If $x_0 = x_2$ and $y_0 = y_2$, then $d(x_1, F(x_0, y_0)) + d(x_0, F(x_1, y_1)) = 0$ and $d(y_1, F(y_0, x_0)) + d(y_0, F(y_1, x_1)) = 0$. Hence, (2.18) implies that $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, and so (2.23) holds trivially.

Thus, (2.20) holds for $n = 1$. In a similar way, proceeding by induction, if we assume that (2.20) holds, we get that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1}) &\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] \\ &\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^{n+1}[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

Hence, by induction, (2.20) is proved.

Using similar arguments as in the proof of Theorem 2.3, we have the desired result. This completes the proof of the theorem. □

In the next theorem, we will substitute the continuity hypothesis on F by an additional property satisfied by the space $(X, d, \preceq)$.

Theorem 2.10. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property.*

Assume that there exist $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + 2\beta + 2\gamma < 1$ *such that*

$$\begin{aligned} &d(F(x,y), F(u,v)) \\ &\leq \frac{\alpha}{2}[d(x,u) + d(y,v)] + \beta N((x,y),(u,v)) \\ &\quad + \frac{\gamma}{2}[d(x, F(x,y)) + d(u, F(u,v)) + d(y, F(y,x)) + d(v, F(v,u))], \end{aligned}$$

for all $(x,y),(u,v) \in X \times X$ *with* $x \succeq u$ *and* $y \preceq v$, *when*

$$D_1 = d(x, F(u,v)) + d(u, F(x,y)) \neq 0$$

and

$$D_2 = d(y, F(v,u)) + d(v, F(y,x)) \neq 0,$$

where

$$\begin{aligned} &N((x,y),(u,v)) \\ &= \min\left\{ \frac{d^2(x, F(u,v)) + d^2(u, F(x,y))}{d(x, F(u,v)) + d(u, F(x,y))}, \frac{d^2(y, F(v,u)) + d^2(v, F(y,x))}{d(y, F(v,u)) + d(v, F(y,x))} \right\}. \end{aligned}$$

Further, $d(F(x,y), F(u,v)) = 0$ *if* $D_1 = 0$ *or* $D_2 = 0$.

Suppose that there exist $x_0, y_0 \in X$ *such that*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0).$$

Finally, assume that X *has the following properties:*

(i) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to* $x \in X$, *then* $x_n \preceq x$ *for all* n,

(ii) *if a nonincreasing sequence* $\{y_n\}$ *in* X *converges to* $y \in X$, *then* $y_n \succeq y$ *for all* n.

Then, F *has a coupled fixed point* $(x,y) \in X \times X$.

Proof. Following the proof of Theorem 2.9, we only have to show that $(\bar{x}, \bar{y})$ is a coupled fixed point of F. Suppose this is not the case, i.e., $F(\bar{x}, \bar{y}) \neq \bar{x}$ or $F(\bar{y}, \bar{x}) \neq \bar{y}$ (e.g., let the first one of these holds). We have

$$\begin{aligned} d(F(\bar{x}, \bar{y}), \bar{x}) &\leq d(F(\bar{x}, \bar{y}), x_{n+1}) + d(x_{n+1}, \bar{x}) \\ &= d(F(\bar{x}, \bar{y}), F(x_n, y_n)) + d(x_{n+1}, \bar{x}). \end{aligned} \tag{2.24}$$

Since the nondecreasing sequence $\{x_n\}$ converges to $\bar{x}$ and the nonincreasing sequence $\{y_n\}$ converges to $\bar{y}$, by (i)–(ii), we have:

$$\bar{x} \succeq x_n \quad \text{and} \quad \bar{y} \preceq y_n, \quad \forall\, n.$$

Now, from the contractive condition, we have:

$$\begin{aligned}
&d(F(\bar{x},\bar{y}),F(x_n,y_n))\\
&\le \frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta N((\bar{x},\bar{y}),(x_n,y_n))\\
&\quad+\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,F(x_n,y_n))+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,F(y_n,x_n))]\\
&\le \frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta\frac{d^2(\bar{x},x_{n+1})+d^2(x_n,F(\bar{x},\bar{y}))}{d(\bar{x},x_{n+1})+d(x_n,F(\bar{x},\bar{y}))}\\
&\quad+\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,x_{n+1})+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,y_{n+1})].
\end{aligned}$$

We note that the case $d(\bar{x},x_{n+1})+d(x_n,F(\bar{x},\bar{y}))=0$ is impossible, since otherwise the condition (2.18) would imply $\bar{x}=F(\bar{x},\bar{y})$, which is excluded. Then, from (2.24), we get:

$$\begin{aligned}
&d(F(\bar{x},\bar{y}),\bar{x})\\
&\le d(x_{n+1},\bar{x})+\frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta\frac{d^2(\bar{x},x_{n+1})+d^2(x_n,F(\bar{x},\bar{y}))}{d(\bar{x},x_{n+1})+d(x_n,F(\bar{x},\bar{y}))}\\
&\quad+\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(x_n,x_{n+1})+d(\bar{y},F(\bar{y},\bar{x}))+d(y_n,y_{n+1})].
\end{aligned}$$

Taking limit as $n\to\infty$ (and again using that $F(\bar{x},\bar{y})\ne\bar{x}$), we have

$$d(F(\bar{x},\bar{y}),\bar{x})\le\beta d(\bar{x},F(\bar{x},\bar{y}))+\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))].\tag{2.25}$$

Now, if $\bar{y}=F(\bar{y},\bar{x})$, using that $\beta+\frac{\gamma}{2}<1$, it follows immediately that $\bar{x}=F(\bar{x},\bar{y})$, a contradiction. If this is not the case, we similarly get

$$d(\bar{y},F(\bar{y},\bar{x}))\le\beta d(\bar{y},F(\bar{y},\bar{x}))+\frac{\gamma}{2}[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))].\tag{2.26}$$

Adding (2.25) and (2.26), we have

$$\begin{aligned}
d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))&\le(\beta+\gamma)[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))]\\
&\le(\alpha+2\beta+2\gamma)[d(\bar{x},F(\bar{x},\bar{y}))+d(\bar{y},F(\bar{y},\bar{x}))].
\end{aligned}$$

Since $0\le\alpha+2\beta+2\gamma<1$, we obtain $d(F(\bar{x},\bar{y}),\bar{x})=0$ and $d(\bar{y},F(\bar{y},\bar{x}))=0$, i.e., $F(\bar{x},\bar{y})=\bar{x}$ and $F(\bar{y},\bar{x})=\bar{y}$, again a contradiction. This completes the proof of the theorem. □

Now we shall prove a uniqueness theorem for the coupled fixed point.

Theorem 2.11. *Assume that*

$$\begin{aligned}&\forall(x,y),(x^*,y^*)\in X\times X,\ \exists\,(z_1,z_2)\in X\times X\\ &\textit{that is comparable to } (x,y) \textit{ and } (x^*,y^*).\end{aligned}\tag{2.27}$$

Adding (2.27) *to the hypotheses of Theorem* 2.9*, we obtain the uniqueness of the coupled fixed point of* F*.*

Proof. From Theorem 2.9 we know that there exists a coupled fixed point $(\bar{x}, \bar{y})$ of F, which is obtained as $\bar{x} = \lim_{n\to\infty} F^n(x_0, y_0)$ and $\bar{y} = \lim_{n\to\infty} F^n(y_0, x_0)$. Suppose that (x^*, y^*) is another coupled fixed point, i.e.,

$$F(x^*, y^*) = x^* \quad \text{and} \quad F(y^*, x^*) = y^*.$$

Let us prove that

$$d(\bar{x}, x^*) + d(\bar{y}, y^*) = 0. \tag{2.28}$$

We distinguish two cases.

Case I: $(\bar{x}, \bar{y})$ is comparable with (x^*, y^*) with respect to the ordering in $X \times X$. Let, e.g., $\bar{x} \succeq x^*$ and $\bar{y} \preceq y^*$. Then, we can apply the contractive condition (2.16) to obtain

$$\begin{aligned} d(\bar{x}, x^*) &= d(F(\bar{x}, \bar{y}), F(x^*, y^*)) \\ &\le \frac{\alpha}{2}[d(\bar{x}, x^*) + d(\bar{y}, y^*)] + \beta d(\bar{x}, x^*), \end{aligned}$$

and

$$\begin{aligned} d(\bar{y}, y^*) &= d(F(\bar{y}, \bar{x}), F(y^*, x^*)) = d(F(y^*, x^*), F(\bar{y}, \bar{x})) \\ &\le \frac{\alpha}{2}[d(\bar{x}, x^*) + d(\bar{y}, y^*)] + \beta d(\bar{y}, y^*). \end{aligned}$$

Adding up, we get that

$$d(\bar{x}, x^*) + d(\bar{y}, y^*) \le (\alpha + \beta)[d(\bar{x}, x^*) + d(\bar{y}, y^*)].$$

Since $0 \le \alpha + \beta < 1$, (2.28) holds.

Case II: $(\bar{x}, \bar{y})$ is not comparable with (x^*, y^*). In this case, there exists $(z_1, z_2) \in X \times X$ that is comparable both to $(\bar{x}, \bar{y})$ and (x^*, y^*). Then, for all $n \in \mathbb{N}$, $(F^n(z_1, z_2), F^n(z_2, z_1))$ is comparable both to $(F^n(\bar{x}, \bar{y}), F^n(\bar{y}, \bar{x})) = (\bar{x}, \bar{y})$ and $(F^n(x^*, y^*), F^n(y^*, x^*)) = (x^*, y^*)$. We have

$$\begin{aligned} d(\bar{x}, x^*) + d(\bar{y}, y^*) &= d(F^n(\bar{x}, \bar{y}), F^n(x^*, y^*)) + d(F^n(\bar{y}, \bar{x}), F^n(y^*, x^*)) \\ &\le d(F^n(\bar{x}, \bar{y}), F^n(z_1, z_2)) + d(F^n(z_1, z_2), F^n(x^*, y^*)) \\ &\quad + d(F^n(\bar{y}, \bar{x}), F^n(z_2, z_1)) + d(F^n(z_2, z_1), F^n(y^*, x^*)) \\ &\le (\alpha^n + \beta^n)[d(\bar{x}, z_1) + d(\bar{y}, z_2) + d(x^*, z_1) + d(y^*, z_2)]. \end{aligned}$$

Since $0 < \alpha, \beta < 1$, (2.28) holds.

We deduce that in all cases (2.28) holds. This implies that $(\bar{x}, \bar{y}) = (x^*, y^*)$ and the uniqueness of the coupled fixed point of F is proved. □

Our next result is as follows:

Theorem 2.12. *In addition to the hypotheses of Theorem* 2.9 *(resp. Theorem* 2.10*), suppose that* x_0, y_0 *in* X *are comparable. Then* $\bar{x} = \bar{y}$.

Proof. Suppose that $x_0 \preceq y_0$. We claim that

$$x_n \preceq y_n, \ \forall\, n \in \mathbb{N}. \tag{2.29}$$

From the mixed monotone property of F, we have

$$x_1 = F(x_0, y_0) \preceq F(y_0, y_0) \preceq F(y_0, x_0) = y_1.$$

Assume that $x_n \preceq y_n$ for some n. Now,

$$\begin{aligned} x_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = y_{n+1}. \end{aligned}$$

Hence, (2.29) holds.

Now, using (2.29) and the contractive condition, we get

$$\begin{aligned} &d(\bar{x}, \bar{y}) \\ &\le d(\bar{x}, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, \bar{y}) \\ &= d(\bar{x}, x_{n+1}) + d(F(y_n, x_n), F(x_n, y_n)) + d(y_{n+1}, \bar{y}) \\ &\le d(\bar{x}, x_{n+1}) + d(y_{n+1}, \bar{y}) + \alpha d(x_n, y_n) + \beta N((y_n, x_n), (x_n, y_n)) \\ &\quad + \frac{\gamma}{2}[d(x_n,F(x_n, y_n))+d(y_n,F(y_n, x_n))+d(y_n,F(y_n, x_n))+d(x_n,F(x_n, y_n)))] \\ &\le d(\bar{x}, x_{n+1})+d(y_{n+1}, \bar{y})+\alpha d(x_n, y_n)+\beta\frac{d^2(x_n, F(y_n, x_n))+d^2(y_n, F(x_n, y_n))}{d(x_n, F(y_n, x_n))+d(y_n, F(x_n, y_n))} \\ &\quad + \gamma[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \\ &\le d(\bar{x}, x_{n+1}) + d(y_{n+1}, \bar{y}) + \alpha d(x_n, y_n) + \beta\frac{d^2(x_n, y_{n+1}) + d^2(y_n, x_{n+1})}{d(x_n, y_{n+1}) + d(y_n, x_{n+1})} \\ &\quad + \gamma[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \quad (\text{provided } d(x_n, y_{n+1}) + d(y_n, x_{n+1}) \ne 0). \end{aligned}$$

Passing to the limit as $n \to \infty$, we get that

$$d(\bar{x}, \bar{y}) \le (\alpha + \beta) d(\bar{x}, \bar{y}).$$

Since $0 \le \alpha + \beta < 1$, this implies that $d(\bar{x}, \bar{y}) = 0$, i.e., $\bar{x} = \bar{y}$.

In the case when $d(x_n, y_{n+1})+d(y_n, x_{n+1}) = 0$, the conditions of the theorem readily imply that $d(\bar{x}, \bar{y}) = 0$. This completes the proof of the theorem. □

Remark 2.13. If we put

$$\mathcal{T}(x) = F(x, x), \quad \forall\, x \in X,$$

then for $x = y$ and $u = v$, the contractive condition (2.16) reduces to the condition for a single map (in the case without order) of Imdad et al. from [9, Theorem 2.1].

Remark 2.14. The results of this paper can be easily modified in a way to obtain the existence of a coupled coincidence point of the mapping $F : X \times X \to X$ and an additional mapping $g : X \to X$, in the case when F has the g-mixed monotone property (see respective definitions in [11]).

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