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EXISTENCE AND EXPONENTIAL DECAY ESTIMATES FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. The paper is devoted to the study of a system of nonlinear equations with nonlinear boundary conditions. First, based on Faedo-Galerkin method and standard arguments of density corresponding to the regularity of initial conditions, we establish two global existence theorems of weak solutions. Next, the exponential decay property of the global solution via the construction of a suitable Lyapunov functional is presented.

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1. Introduction

In this paper, we consider the initial-boundary value problem for the following system of nonlinear wave equations

$$\begin{cases} u_{tt}-u_{xx}+\lambda_1 \left|u_t\right|^{r_1-2} u_t+f_1(u,v) = F_1(x,t), 0 < x < 1, 0 < t < T, \\ v_{tt}-v_{xx}+\lambda_2 \left|v_t\right|^{r_2-2} v_t+f_2(u,v) = F_2(x,t), 0 < x < 1, 0 < t < T, \end{cases} \tag{1.1}$$

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ v_x(0,t) = K \left|v(0,t)\right|^{p-2} v(0,t) + \mu \left|v_t(0,t)\right|^{q-2} v_t(0,t), v(1,t) = 0, \end{cases} \tag{1.2}$$

$$\begin{cases} u(x,0) = \tilde{u}_0(x), u_t(x,0) = \tilde{u}_1(x), \\ v(x,0) = \tilde{v}_0(x), v_t(x,0) = \tilde{v}_1(x), \end{cases} \tag{1.3}$$

where $p \geq 2$, $q \geq 2$, $K \geq 0$, $\mu > 0$, $\lambda_i > 0$, $r_i \geq 2$ $(i = 1,2)$ are given constants and f_1, f_2, F_1, F_2, $\tilde{u}_i$, $\tilde{v}_i$ $(i = 0,1)$ are given functions satisfying conditions specified later. Problems of this type arise in material science and physics, which have been studied by many authors. For example, we refer to [1]-[5], [7]-[16], and the references given therein. In these works, many interesting results about the existence and some properties of solutions were obtained.

In [12], Miao and Zhu proved the existence and regularity of global smooth solutions of a Cauchy problem for the non-linear system of wave equations with Hamilton structure. J. Wu and S. Li [14] considered a system of nonlinear wave equations with initial and Dirichlet boundary conditions, under some suitable conditions, the result on blow-up of solutions and upper bound of blow-up time were given.

M. M. Cavalcanti et al. [4] studied the existence of global solutions and the asymptotic behavior of the energy related to a degenerate system of wave equations with boundary conditions of memory type. By the construction of a suitable Lyapunov functional, the authors proved that the energy decay exponentially. In [11], [13], the existence, regularity and the decay properties of solutions for a wave equation (linear or nonlinear) associated with two-point boundary conditions have been established. The proofs are based on the Galerkin method associated to a priori estimates, weak convergence, compactness techniques and also by the construction of a suitable Lyapunov functional.

The main goal of this paper is to extend the results of [11], [13] to find a weak solution of problem (1.1)-(1.3) and prove exponential decay property of the global solution. The results are obtained by combination of the methods in [11], [13] with appropriate modifications. The paper consists of two main parts as follows.

Part 1 is devoted to the presentation of the existence results based on Faedo-Galerkin method and standard arguments of density corresponding to the regularity of initial conditions. In this part, problem (1.1)-(1.3) is dealt with two cases of $(p, q) : 2 \leq q \leq 4$ and $p \geq 2$; or $q > 4$ and $p \in \{2\} \cup [3, +\infty)$. In the cases $q = 2$ and $p \geq 2$; or $q > 2$ and $p \in \{2\} \cup [3, +\infty)$, the solution obtained here is unique.

In Part 2, we consider (1.1)-(1.3) with $p \geq 2$ and $q = r_1 = r_2 = 2$. Under some suitable conditions, there exists a unique solution $(u(t), v(t))$ defined on $\mathbb{R}_+$ and furthermore,

$$\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + |v(0,t)|^p + \int_0^1 F(u(x,t), v(x,t))dx$$

decays exponentially to 0 as $t \to +\infty$, with $\frac{\partial F}{\partial u} = f_1$, $\frac{\partial F}{\partial v} = f_2$.

2. The existence and the uniqueness of a weak solution

First, put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T), T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$.

Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $||\cdot||$ stands for the norm in L^2 and we denote by $||\cdot||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable such that $||u||_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1, we shall use the following norm

$$||v||_{H^1} = \left(||v||^2 + ||v_x||^2\right)^{1/2}. \tag{2.1}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}. \tag{2.2}$$

Then V is a closed subspace of H^1 and the following lemma is known as a standard one.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$||v||_{C^0(\overline{\Omega})} \leq \sqrt{2}||v||_{H^1} \text{ for all } v \in H^1. \tag{2.3}$$

Remark 2.2. On V, two norms $v \longmapsto ||v||_{H^1}$ and $v \longmapsto ||v_x||$ are equivalent. Furthermore, we have the following inequalities:

$$||v||_{C^0(\overline{\Omega})} \leq ||v_x|| \text{ for all } v \in V. \tag{2.4}$$

Remark 2.3. The weak formulation of the initial-boundary valued problem (1.1)-(1.3) can be given in the following manner: Find (u, v) belongs to $\widetilde{W} = \{(u, v) \in L^\infty(0, T; (H_0^1 \cap H^2) \times (V \cap H^2)) : (u_t, v_t) \in L^\infty(0, T; H_0^1 \times V),\ (u_{tt}, v_{tt}) \in L^\infty(0, T; L^2 \times L^2)\}$, such that (u, v) satisfies the following variational equation

$$\begin{cases} \langle u_{tt}(t), w\rangle + \langle u_x(t), w_x\rangle + \lambda_1 \langle \Psi_{r_1}(u_t(t)), w\rangle + \langle f_1(u, v), w\rangle \\ \quad = \langle F_1(t), w\rangle\,, \\ \langle v_{tt}(t), \phi\rangle + \langle v_x(t), \phi_x\rangle + \lambda_2 \langle \Psi_{r_2}(v_t(t)), \phi\rangle \\ \quad + \left[K\Psi_p(v(0, t)) + \mu\Psi_q(v_t(0, t))\right]\phi(0) + \langle f_2(u, v), \phi\rangle = \langle F_2(t), \phi\rangle\,, \end{cases} \tag{2.5}$$

for all $(w, \phi) \in H_0^1 \times V$, together with the initial conditions

$$(u(0), u_t(0)) = (\tilde{u}_0, \tilde{u}_1)\,,\ (v(0), v_t(0)) = (\tilde{v}_0, \tilde{v}_1)\,, \tag{2.6}$$

where $\Psi_r(z) = |z|^{r-2}z$, $r \in \{p, q, r_1, r_2\}$, with $p, q \geq 2, r_1 \geq 2$, $r_2 \geq 2$ are given constants.

Next, we consider (1.1)-(1.3) with $p \geq 2, q \geq 2, \lambda_1 > 0, \lambda_2 > 0, r_1 \geq 2$, $r_2 \geq 2, K \geq 0, \mu > 0$ and make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in (H_0^1 \cap H^2) \times H_0^1$, $(\tilde{v}_0, \tilde{v}_1) \in (V \cap H^2) \times V$;
(H_2) $F_1, F_2 \in L^1(0, T; L^2)$ such that $F_1', F_2' \in L^1(0, T; L^2)$;
(H_3) There exists $F \in C^2(\mathbb{R}^2; \mathbb{R})$ such that
 (i) $\frac{\partial F}{\partial u}(u, v) = f_1(u, v)$, $\frac{\partial F}{\partial v}(u, v) = f_2(u, v)$,
 (ii) there exists the constant $C_1 > 0$, such that
 $-F(u, v) \leq C_1\left(1 + u^2 + v^2\right), \forall u, v \in \mathbb{R}$;
(H_1') $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$, $(\tilde{v}_0, \tilde{v}_1) \in V \times L^2$;
(H_2') $F_1,\ F_2 \in L^2(Q_T)$.

Remark 2.4. We present below an example in which the functions f_1, f_2 satisfy the assumption (H_3). Consider the following functions

$$f_1(u, v) = 2\left(\frac{\alpha^2}{(\alpha^2+u^2)^2} + \frac{1}{\beta^2+v^2}\right)v^2u,$$

$$f_2(u, v) = 2\left(\frac{\beta^2}{(\beta^2+v^2)^2} + \frac{1}{\alpha^2+u^2}\right)u^2v,$$

where α, β are constants and α, β are not equal to 0. It is obvious that $(H_3, \text{(i)})$ holds, $(H_3, \text{(ii)})$ is also valid, since there exists a $F \in C^2(\mathbb{R}^2; \mathbb{R})$ defined by

$$F(u,v) = \left(\frac{1}{\alpha^2 + u^2} + \frac{1}{\beta^2 + v^2} \right) u^2 v^2$$

such that

$$\begin{aligned}
&\tfrac{\partial F}{\partial u}(u,v) = 2\left(\tfrac{\alpha^2}{(\alpha^2+u^2)^2} + \tfrac{1}{\beta^2+v^2} \right) v^2 u = f_1(u,v), \\
&\tfrac{\partial F}{\partial v}(u,v) = 2\left(\tfrac{\beta^2}{(\beta^2+v^2)^2} + \tfrac{1}{\alpha^2+u^2} \right) u^2 v = f_2(u,v), \\
&-F(u,v) = -\left(\tfrac{1}{\alpha^2+u^2} + \tfrac{1}{\beta^2+v^2} \right) u^2 v^2 \le u^2 + v^2, \text{ for all } (u,v) \in \mathbb{R}^2.
\end{aligned}$$

On the other hand, by

$$2F(u,v) \le u f_1(u,v) + v f_2(u,v) \le 4F(u,v), \text{ for all } (u,v) \in \mathbb{R}^2,$$

the functions f_1, f_2 also satisfy $(\widehat{H}_3)$ in next Section 3.

We have the following theorem about the existence of a "strong solution".

Theorem 2.5. *Let $T > 0$. Suppose that $(H_1) - (H_3)$ hold and the initial data satisfies the compatibility condition*

$$\tilde{v}_{0x}(0) = K\left|\tilde{v}_0(0)\right|^{p-2} \tilde{v}_0(0) + \mu \left|\tilde{v}_1(0)\right|^{q-2} \tilde{v}_1(0). \tag{2.7}$$

If either of the following cases is valid

$$2 \le q \le 4 \text{ and } p \ge 2; \text{ or } q > 4 \text{ and } p \in \{2\} \cup [3, +\infty),$$

then there exists a weak solution (u,v) of problem (1.1)-(1.3) *such that*

$$\begin{cases}
(u,v) \in L^\infty(0,T; (H_0^1 \cap H^2) \times V \cap H^2), \\
(u_t, v_t) \in L^\infty(0,T; H_0^1 \times V), \\
(u_{tt}, v_{tt}) \in L^\infty(0,T; L^2 \times L^2), \\
|u'|^{\frac{r_1}{2}-1} u', |v'|^{\frac{r_2}{2}-1} v' \in H^1(Q_T), |v'(0,\cdot)|^{\frac{q}{2}-1} v'(0,\cdot) \in H^1(0,T).
\end{cases} \tag{2.8}$$

Furthermore, in the cases $q = 2$ and $p \ge 2$; or $q > 2$ and $p \in \{2\} \cup [3, +\infty)$, the solution is unique.

Remark 2.6. (i) Theorem 2.5 gives no conclusion about the uniqueness of solution when $2 < p < 3$ and $q > 2$.
(ii) The regularity obtained by (2.8) shows that problem (1.1)-(1.3) has a strong solution

$$\begin{cases}
(u,v) \in L^\infty(0,T; (H_0^1 \cap H^2) \times (V \cap H^2)) \cap C^0\left(0,T; H_0^1 \times V\right) \\
\qquad\qquad \cap C^1\left(0,T; L^2 \times L^2\right), \\
(u_t, v_t) \in L^\infty(0,T; H_0^1 \times V) \cap C^0\left(0,T; L^2 \times L^2\right), \\
(u_{tt}, v_{tt}) \in L^\infty(0,T; L^2 \times L^2), \\
|u'|^{\frac{r_1}{2}-1} u', \, |v'|^{\frac{r_2}{2}-1} v' \in H^1(Q_T), \, |v'(0,\cdot)|^{\frac{q}{2}-1} v'(0,\cdot) \in H^1(0,T).
\end{cases} \tag{2.9}$$

With less regular initial data, if $q=2$, $p\geq 2$ or $q>2, p\in 2\cup[3,+\infty)$, we obtain the following theorem about the unique existence of a weak solution of problem (1.1)-(1.3).

Theorem 2.7. *Let $T>0$. Suppose that (H_1'), (H_2'), (H_3) hold. Then problem* (1.1)-(1.3) *has a unique solution*

$$\begin{cases}(u,v)\in C\left([0,T];H_0^1\times V\right)\cap C^1\left([0,T];L^2\times L^2\right),\\ (u',v')\in L^{r_1}(Q_T)\times L^{r_2}(Q_T), v(0,\cdot)\in H^1(0,T).\end{cases} \tag{2.10}$$

Proof of Theorem 2.5. The proof consists of four steps.
Step 1. *The Faedo-Galerkin approximation.* Let $\{(w_i,\phi_j)\}$ be a denumerable base of $(H_0^1\cap H^2)\times(V\cap H^2)$. We find the approximate solution of problem (1.1)- (1.3) in the form

$$u_m(t)=\sum\nolimits_{j=1}^m c_{mj}(t)w_j,\ v_m(t)=\sum\nolimits_{j=1}^m d_{mj}(t)\phi_j,$$

where the coefficient functions (c_{mj},d_{mj}) satisfy the system of ordinary differential equations

$$\begin{cases}\langle u_m''(t),w_j\rangle+\langle u_{mx}(t),w_{jx}\rangle+\lambda_1\left\langle\Psi_{r_1}(u_m'(t)),w_j\right\rangle\\ \qquad\qquad+\langle f_1(u_m,v_m),w_j\rangle=\langle F_1(t),w_j\rangle\\ \langle v_m''(t),\phi_j\rangle+\langle v_{mx}(t),\phi_{jx}\rangle+\lambda_2\left\langle\Psi_{r_2}(v_m'(t)),\phi_j\right\rangle\\ \qquad\qquad+[K\Psi_p(v_m(0,t))+\mu\Psi_q(v_m'(0,t))])\phi_j(0)\\ \qquad\qquad+\langle f_2(u_m,v_m),\phi_j\rangle=\langle F_2(t),\phi_j\rangle,\ \ 1\leq j\leq m,\\ (u_m(0),u_m'(0))=(\tilde u_0,\tilde u_1),\ \ (v_m(0),v_m'(0))=(\tilde v_0,\tilde v_1),\end{cases} \tag{2.11}$$

where $\Psi_r(z)=|z|^{r-2}z$, $r\in\{p,q,r_1,r_2\}$.

From the assumptions of Theorem 2.5, system (2.11) has a solution $(u_m(t), v_m(t))$ on an interval $[0,T_m]\subset[0,T]$. The following estimates allow one to take $T_m=T$ for all m.

Step 2. *A priori estimates I.* Multiplying the j^{th} equation of (2.11) by $(c_{mj}'(t), d_{mj}'(t))$ and summing with respect to j, and afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned}S_m(t)=S_m(0)-2\int_0^t&\left[\left\langle\frac{\partial F}{\partial u}(u_m(s),v_m(s)),u_m'(s)\right\rangle\right.\\ &\left.+\left\langle\frac{\partial F}{\partial v}(u_m(s),v_m(s)),v_m'(s)\right\rangle\right]ds\\ &+2\int_0^t\left[\left\langle F_1(s),u_m'(s)\right\rangle+\left\langle F_2(s),v_m'(s)\right\rangle\right]ds,\end{aligned} \tag{2.12}$$

where

$$\begin{aligned} S_m(t) &= ||u'_m(t)||^2 + ||v'_m(t)||^2 + ||u_{mx}(t)||^2 + ||v_{mx}(t)||^2 \\ &+ 2\lambda_1 \int_0^t \left\|u'_m(s)\right\|_{L^{r_1}}^{r_1} ds + 2\lambda_2 \int_0^t \left\|v'_m(s)\right\|_{L^{r_2}}^{r_2} ds \\ &+ \frac{2K}{p} \left|v_m(0,t)\right|^p + 2\mu \int_0^t \left|v'_m(0,s)\right|^q ds. \end{aligned} \tag{2.13}$$

By $(2.11)_3$, (2.13) and (H_1), there exists a positive constant C_0 depending only on $\tilde{u}_0$, $\tilde{u}_1, \tilde{v}_0$, $\tilde{v}_1$, p and K, such that

$$S_m(0) \le C_0, \text{ for all } m. \tag{2.14}$$

On the other hand

$$\left\|u_m(t)\right\|^2 = ||u_m(0) + \int_0^t u'_m(s)ds||^2 \le 2\left\|\tilde{u}_0\right\|^2 + 2t \int_0^t \left\|u'_m(s)\right\|^2 ds. \tag{2.15}$$

Hence, (2.13)-(2.15) lead to

$$\left\|u_m(t)\right\|^2 + \left\|v_m(t)\right\|^2 \le C_0 + 2T \int_0^t S_m(s)ds. \tag{2.16}$$

We shall estimate the following integrals in the right-hand side of (2.12).

First integral. Using the assumption (H_3) we deduce from (2.16) that

$$\begin{aligned} &-2\int_0^t \left[\left\langle \tfrac{\partial F}{\partial u}(u_m(s), v_m(s)), u'_m(s)\right\rangle + \left\langle \tfrac{\partial F}{\partial v}(u_m(s), v_m(s)), v'_m(s)\right\rangle\right] ds \\ &= -2\int_0^t \left[\tfrac{d}{ds}\int_0^1 F(u_m(x,s), v_m(x,s))dx\right] ds \\ &= 2\int_0^1 F(\tilde{u}_0(x), \tilde{v}_0(x))dx - 2\int_0^1 F(u_m(x,s), v_m(x,s))dx \\ &\le 2\sup_{|y|,|z|\le\sqrt{C_0}} |F(y,z)| + 2C_1\int_0^1 \left(1 + u_m^2(x,s) + v_m^2(x,s)\right) dx \\ &\le 2\sup_{|y|,|z|\le\sqrt{C_0}} |F(y,z)| + 2C_1\left[1 + C_0 + 2T\int_0^t S_m(s)ds\right] \\ &\le C_0 + C_T\int_0^t S_m(s)ds, \end{aligned} \tag{2.17}$$

where we remark that C_T always indicates a bound depending on T.

Second integral. Using the assumption (H_2), and deduce from the Cauchy-Schwartz inequality that

$$\begin{aligned} & 2\int_0^t \left[\left\langle F_1(s), u_m'(s)\right\rangle + \left\langle F_2(s), v_m'(s)\right\rangle\right] ds \\ & \leq 2\int_0^t \left[\|F_1(s)\| \left\|u_m'(s)\right\| + \|F_2(s)\| \left\|v_m'(s)\right\|\right] ds \\ & \leq \|F_1\|^2_{L^2(Q_T)} + \|F_2\|^2_{L^2(Q_T)} + \int_0^t S_m(s)ds \\ & \leq C_T + \int_0^t S_m(s)ds. \end{aligned} \tag{2.18}$$

Hence, it follows from (2.12), (2.14), (2.17), (2.18) that

$$S_m(t) \leq 2C_0 + C_T + (1 + C_T)\int_0^t S_m(s)ds. \tag{2.19}$$

By Gronwall's lemma, (2.19) yields

$$S_m(t) \leq (2C_0 + C_T)e^{(1+C_T)t} \leq C_T, \text{ for all } t \in [0,T]. \tag{2.20}$$

A priori estimates II. First of all, we are going to estimate $\|u_m''(0)\|^2 + \|v_m''(0)\|^2$.

Letting $t \to 0_+$ in Eq. $(2.11)_1$, multiplying the result by $c_{mj}''(0)$, we get

$$\begin{aligned} \left\|u_m''(0)\right\|^2 - \left\langle \tilde{u}_{0xx}, u_m''(0)\right\rangle + \lambda_1 \left\langle |\tilde{u}_1|^{r_1-2}\, \tilde{u}_1, u_m''(0)\right\rangle \\ + \left\langle f_1(\tilde{u}_0, \tilde{v}_0), u_m''(0)\right\rangle = \left\langle F_1(0), u_m''(0)\right\rangle. \end{aligned} \tag{2.21}$$

This implies that, for all m,

$$\left\|u_m''(0)\right\| \leq \|\tilde{u}_{0xx}\| + \lambda_1 \|\, |\tilde{u}_1|^{r_1-1}\, \| + \|f_1(\tilde{u}_0, \tilde{v}_0)\| + \|F_1(0)\| \equiv \overline{C}_{01}, \tag{2.22}$$

where $\overline{C}_{01}$ is a constant depending only on r_1, λ_1, $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{u}_1$, f_1, F_1.

Similarly, letting $t \to 0_+$ in Eq. $(2.11)_2$, multiplying the result by $d_{mj}''(0)$ and using the compatibility (2.7), we have

$$\begin{aligned} \left\|v_m''(0)\right\|^2 - \left\langle \tilde{v}_{0xx}, v_m''(0)\right\rangle + \lambda_2 \left\langle |\tilde{v}_1|^{r_1-2}\, \tilde{v}_1, v_m''(0)\right\rangle \\ + \left\langle f_2(\tilde{u}_0, \tilde{v}_0), v_m''(0)\right\rangle = \left\langle F_2(0), v_m''(0)\right\rangle. \end{aligned} \tag{2.23}$$

This implies that

$$\begin{aligned} \left\|v_m''(0)\right\| & \leq \|\tilde{v}_{0xx}\| + \lambda_2 \|\, |\tilde{v}_1|^{r_2-1}\, \| + \|f_2(\tilde{u}_0, \tilde{v}_0)\| + \|F_2(0)\| \\ & \equiv \overline{C}_{02} \quad \text{ for all } m, \end{aligned} \tag{2.24}$$

where $\overline{C}_{02}$ is a constant depending only on r_2, λ_2, $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{v}_1$, f_2, F_2.

Now differentiating (2.11) with respect to t, for all $1 \leq j \leq m$, we have

$$\begin{cases} \langle u_m'''(t), w_j\rangle + \langle u_{mx}'(t), w_{jx}\rangle + \lambda_1 \left\langle \Psi_{r_1}'(u_m'(t))u_m''(t), w_j\right\rangle \\ \quad + \left\langle \frac{\partial^2 F}{\partial u^2}(u_m, v_m)u_m' + \frac{\partial^2 F}{\partial u \partial v}(u_m, v_m)v_m', w_j \right\rangle = \langle F_1'(t), w_j\rangle, \\ \langle v_m'''(t), \phi_j\rangle + \langle v_{mx}'(t), \phi_{jx}\rangle + \lambda_2 \left\langle \Psi_{r_2}'(v_m'(t))v_m''(t), \phi_j\right\rangle \\ \quad + \left[K\Psi_p'(v_m(0,t))v_m'(0,t) + \mu\Psi_q'(v_m'(0,t))v_m''(0,t)\right]\phi_j(0) \\ \quad + \left\langle \frac{\partial^2 F}{\partial v \partial u}(u_m, v_m)u_m' + \frac{\partial^2 F}{\partial v^2}(u_m, v_m)v_m', \phi_j \right\rangle = \langle F_2'(t), \phi_j\rangle. \end{cases} \tag{2.25}$$

Multiplying the j^{th} equation of (2.25) by $\left(c_{mj}''(t),\ d_{mj}''(t)\right)$ and summing with respect to j, and afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} &X_m(t) \\ &= X_m(0) - 2\int_0^t \left\langle \frac{\partial^2 F}{\partial u^2}(u_m, v_m)u_m'(s) + \frac{\partial^2 F}{\partial u \partial v}(u_m, v_m)v_m'(s), u_m''(s)\right\rangle ds \\ &\quad -2\int_0^t \left\langle \frac{\partial^2 F}{\partial v \partial u}(u_m, v_m)u_m'(s) + \frac{\partial^2 F}{\partial v^2}(u_m, v_m)v_m'(s), v_m''(s)\right\rangle ds \\ &\quad +2\int_0^t \left[\langle F_1'(s), u_m''(s)\rangle + \langle F_2'(s), v_m''(s)\rangle\right] ds \\ &\quad -2K\int_0^t \Psi_p'(v_m(0,s))v_m'(0,s)v_m''(0,s)ds \\ &\equiv X_m(0) + \sum_{j=1}^4 J_j, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} X_m(t) &= \|u_m''(t)\|^2 + \|v_m''(t)\|^2 + \|u_{mx}'(t)\|^2 + \|v_{mx}'(t)\|^2 \\ &\quad + \frac{8\lambda_1(r_1-1)}{r_1^2}\int_0^t \left\| \frac{\partial}{\partial s}\left(|u_m'(x,s)|^{\frac{r_1}{2}-1} u_m'(x,s)\right)\right\|^2 ds \\ &\quad + \frac{8\lambda_2(r_2-1)}{r_2^2}\int_0^t \left\| \frac{\partial}{\partial s}\left(|v_m'(x,s)|^{\frac{r_2}{2}-1} v_m'(x,s)\right)\right\|^2 ds \\ &\quad + \frac{8\mu(q-1)}{q^2}\int_0^t \left| \frac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q}{2}-1} v_m'(0,s)\right)\right|^2 ds. \end{aligned} \tag{2.27}$$

By $(2.11)_3$, (2.22), (2.24), (2.27), we obtain

$$X_m(0) = \left\|u_m''(0)\right\|^2 + \left\|v_m''(0)\right\|^2 + \|\tilde{u}_{1x}\|^2 + \|\tilde{v}_{1x}\|^2 \leq C_0, \text{ for all } m, \tag{2.28}$$

where C_0 is a positive constant depending only on $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{u}_1$, $\tilde{v}_1$, f_1, f_2, F_1, F_2, r_1, r_2, λ_1, λ_2.

Put

$$K_2(T, F) = \sup_{|y|,\ |z| \leq \sqrt{C_T},\ |\alpha|=2} |D^\alpha F(y, z)|. \tag{2.29}$$

We shall estimate, respectively, the following integrals in the right-hand side of (2.26).

First integral. The Hölder inequality and the results in (2.13), (2.20), (2.27), (2.29) give

$$\begin{aligned} J_1 &= -2\int_0^t \left\langle \frac{\partial^2 F}{\partial u^2}(u_m, v_m)u_m'(s) + \frac{\partial^2 F}{\partial u \partial v}(u_m, v_m)v_m'(s), u_m''(s) \right\rangle ds \\ &\le 2\int_0^t \left[\left\| \frac{\partial^2 F}{\partial u^2}(u_m, v_m)u_m'(s) \right\| + \left\| \frac{\partial^2 F}{\partial u \partial v}(u_m, v_m)v_m'(s) \right\| \right] \|u_m''(s)\| \, ds \\ &\le 2K_2(T, F)\int_0^t \left[\|u_m'(s)\| + \|v_m'(s)\| \right] \|u_m''(s)\| \, ds \\ &\le 2K_2^2(T, F)\int_0^t S_m(s)ds + \int_0^t \|u_m''(s)\|^2 \, ds \le C_T + \int_0^t X_m(s)ds. \end{aligned} \tag{2.30}$$

Second integral. Similarly

$$\begin{aligned} J_2 &= -2\int_0^t \left\langle \frac{\partial^2 F}{\partial v \partial u}(u_m, v_m)u_m'(s) + \frac{\partial^2 F}{\partial v^2}(u_m, v_m)v_m'(s), v_m''(s) \right\rangle ds \\ &\le C_T + \int_0^t X_m(s)ds. \end{aligned} \tag{2.31}$$

Third integral. Using (H_2) and (2.27), we deduce from the Cauchy-Schwartz inequality that

$$\begin{aligned} J_3 &= 2\int_0^t \left[\langle F_1'(s), u_m''(s) \rangle + \langle F_2'(s), v_m''(s) \rangle \right] ds \\ &\le \|F_1'\|_{L^1(0,T;L^2)} + \|F_2'\|_{L^1(0,T;L^2)} + \int_0^t \|F_1'(s)\| \, \|u_m''(s)\| \, ds \\ &\quad + \int_0^t \|F_2'(s)\| \, \|v_m''(s)\| \, ds \\ &\le C_T + \int_0^t \left(\|F_1'(s)\| + \|F_2'(s)\| \right) X_m(s)ds. \end{aligned} \tag{2.32}$$

Fourth integral. We shall show that there exists C_T such that

$$J_4 \le C_T + \frac{1}{2}X_m(t) \tag{2.33}$$

in two cases for (p, q) as follows.

Case 1. $2 \le q \le 4$, $p \ge 2$: We have

$$\begin{aligned} J_4 &= -2K\int_0^t \Psi_p'(v_m(0,s))v_m'(0,s)v_m''(0,s)ds \\ &= -2K(p-1)\int_0^t |v_m(0,s)|^{p-2}\, v_m'(0,s)v_m''(0,s)ds \\ &= -2K(p-1)\int_0^t |v_m(0,s)|^{p-2}\, |v_m'(0,s)|^{1-\frac{q}{2}}\, v_m'(0,s) \\ &\quad \cdot |v_m'(0,s)|^{\frac{q}{2}-1}\, v_m''(0,s)ds \\ &= -2K(p-1)\tfrac{2}{q}\int_0^t |v_m(0,s)|^{p-2}\, |v_m'(0,s)|^{1-\frac{q}{2}}\, v_m'(0,s) \\ &\quad \cdot \tfrac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q}{2}-1}\, v_m'(0,s)\right) ds. \end{aligned} \tag{2.34}$$

Using the inequalities

$$\begin{aligned} &2ab \le \delta a^2 + \tfrac{1}{\delta}b^2, \text{ for all } a,\ b \ge 0,\ \delta > 0, \\ &x^{4-q} \le 1 + x^q, \text{ for all } x \ge 0,\ 2 \le q \le 4, \end{aligned} \tag{2.35}$$

it follows from (2.13), (2.20), (2.27), (2.34) that

$$\begin{aligned} J_4 &= -2K(p-1)\tfrac{2}{q}\int_0^t |v_m(0,s)|^{p-2}\, |v_m'(0,s)|^{1-\frac{q}{2}}\, v_m'(0,s) \\ &\quad \cdot \tfrac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q}{2}-1}\, v_m'(0,s)\right) ds \\ &\le K(p-1)\tfrac{2}{q}\int_0^t \Big[\tfrac{1}{\delta}\,|v_m(0,s)|^{2p-4}\, |v_m'(0,s)|^{4-q} \\ &\qquad + \delta\left|\tfrac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q}{2}-1}\, v_m'(0,s)\right)\right|^2\Big] ds \\ &\le K(p-1)\tfrac{2}{\delta q}\int_0^t |v_m(0,s)|^{2p-4}\, |v_m'(0,s)|^{4-q}\, ds \\ &\qquad + \delta K(p-1)\tfrac{2}{q}\int_0^t \left|\tfrac{\partial}{\partial s}\left(|v_m'(0,s)|^{\frac{q}{2}-1}\, v_m'(0,s)\right)\right|^2 ds \\ &\le K(p-1)\tfrac{2}{\delta q}\int_0^t (S_m(s))^{p-2}\left[1 + |v_m'(0,s)|^q\right] ds \\ &\qquad + \delta K(p-1)\tfrac{2}{q}\tfrac{q^2}{8\mu(q-1)}X_m(t) \\ &\le K(p-1)\tfrac{2}{\delta q}C_T^{p-2}\int_0^t \left[1 + |v_m'(0,s)|^q\right] ds \\ &\qquad + \delta K(p-1)\tfrac{q}{4\mu(q-1)}X_m(t) \\ &\le K(p-1)\tfrac{2}{\delta q}C_T^{p-2}\left[T + \tfrac{1}{2\mu}S_m(t)\right] + \delta K(p-1)\tfrac{q}{4\mu(q-1)}X_m(t) \\ &\le \tfrac{1}{\delta}C_T + \delta K(p-1)\tfrac{q}{4\mu(q-1)}X_m(t), \end{aligned} \tag{2.36}$$

for all $\delta > 0$ and C_T always indicates a bound depending on T.

Choosing $\delta > 0$ such that $\delta K(p-1)\frac{q}{4\mu(q-1)} \le \frac{1}{2}$, it follows from (2.36) that

$$J_4 \le C_T + \frac{1}{2}X_m(t).$$

Case 2. $q > 4,\ p \in \{2\} \cup [3, +\infty)$:

If $p \geq 3$: Using the inequality $x^3 \leq \frac{q-3}{q} + \frac{3}{q}x^q$ for all $x \geq 0$ and $q \geq 3$, it follows from (2.13), (2.20), (2.27), (2.28) that

$$\begin{aligned}
J_4 &= -2K \int_0^t \Psi_p'(v_m(0,s))v_m'(0,s)v_m''(0,s)ds \\
&= -K(p-1) \int_0^t |v_m(0,s)|^{p-2} \tfrac{d}{ds} |v_m'(0,s|^2 \, ds \\
&= -K(p-1) |v_m(0,t)|^{p-2} |v_m'(0,t)|^2 + K(p-1) |\tilde{v}_0(0)|^{p-2} \tilde{v}_1^2(0) \\
&\qquad + K(p-1)(p-2) \int_0^t |v_m(0,s)|^{p-4} v_m(0,s) \left(v_m'(0,s)\right)^3 ds \\
&= -K(p-1) |v_m(0,t)|^{p-2} |v_m'(0,t)|^2 + K(p-1) |\tilde{v}_0(0)|^{p-2} \tilde{v}_1^2(0) \\
&\qquad + K(p-1)(p-2) \int_0^t |v_m(0,s)|^{p-4} v_m(0,s) \left(v_m'(0,s)\right)^3 ds \\
&\leq C_T + K(p-1)(p-2) \int_0^t |v_m(0,s)|^{p-3} |v_m'(0,s)|^3 \, ds \\
&\leq C_T + K(p-1)(p-2) C_T^{p-3} \int_0^t \left[\tfrac{q-3}{q} + \tfrac{3}{q} |v_m'(0,r)|^q\right] ds \\
&\leq C_T \leq C_T + \tfrac{1}{2} X_m(t).
\end{aligned}$$

If $p = 2$:

$$\begin{aligned}
J_4 &= -2K \int_0^t \Psi_p'(v_m(0,s))v_m'(0,s)v_m''(0,s)ds \\
&= -K(p-1) \int_0^t \tfrac{d}{ds} |v_m'(0,s|^2 \, ds \\
&= -K(p-1) |v_m'(0,t)|^2 + K(p-1)\tilde{v}_1^2(0) \leq C_T.
\end{aligned}$$

Thus (2.33) holds. Combining (2.26) – (2.28), (2.30) – (2.32), and (2.33), we obtain

$$X_m(t) \leq C_T + 2\int_0^t \left(2 + \|F_1'(s)\| + \|F_2'(s)\|\right) X_m(s)ds. \tag{2.37}$$

By Gronwall's lemma, from (2.37) we obtain

$$X_m(t) \leq C_T \exp\left[2\int_0^t \left(2 + \|F_1'(s)\| + \|F_2'(s)\|\right) X_m(s)ds\right] \leq C_T, \tag{2.38}$$

$\forall m \in \mathbb{N}$, $\forall t \in [0,T]$, $\forall T > 0$.

Step 3. *Limiting process.* From (2.13), (2.20), (2.27), (2.38), we deduce the existence of a subsequence of $\{(u_m, v_m)\}$ still also so denoted such that

$$\begin{cases} (u_m, v_m) \to (u, v) & \text{in} \quad L^\infty(0, T; H_0^1 \times V) & \text{weak}^*, \\ (u'_m, v'_m) \to (u', v') & \text{in} \quad L^\infty(0, T; H_0^1 \times V) & \text{weak}^*, \\ (u''_m, v''_m) \to (u'', v'') & \text{in} \quad L^\infty(0, T; L^2 \times L^2) & \text{weak}^*, \\ (u'_m, v'_m) \to (u', v') & \text{in} \quad L^{r_1}(Q_T) \times L^{r_2}(Q_T) & \text{weakly}, \\ v_m(0, \cdot) \to v(0, \cdot) & \text{in} \quad W^{1,q}(0, T) & \text{weakly}, \\ v'_m(0, \cdot) \to v'(0, \cdot) & \text{in} \quad L^q(0, T) & \text{weakly}, \\ |v'_m(0, \cdot)|^{\frac{q}{2}-1} v'_m(0, \cdot) \to \chi_0 & \text{in} \quad H^1(0, T) & \text{weakly}, \\ \frac{\partial}{\partial t}\left(|u'_m|^{\frac{r_1}{2}-1} u'_m\right) \to \chi_1 & \text{in} \quad L^2(Q_T) & \text{weakly}, \\ \frac{\partial}{\partial t}\left(|v'_m|^{\frac{r_1}{2}-1} v'_m\right) \to \chi_2 & \text{in} \quad L^2(Q_T) & \text{weakly}. \end{cases} \tag{2.39}$$

By the compactness lemma of Lions ([6], p.57) and the imbeddings $H^2(0,T) \hookrightarrow C^1([0,T])$, $H^1(0,T) \hookrightarrow C^0([0,T])$, $W^{1,q}(0,T) \hookrightarrow C^0([0,T])$, we can deduce from (2.38) the existence of a subsequence also still denoted by $\{(u_m, v_m)\}$ such that

$$\begin{cases} (u_m, v_m) \to (u, v) & \text{strongly in} \quad L^2(Q_T) \times L^2(Q_T) \\ & \qquad \text{and a.e. in } Q_T, \\ (u'_m, v'_m) \to (u', v') & \text{strongly in} \quad L^2(Q_T) \times L^2(Q_T) \\ & \qquad \text{and a.e. in } Q_T, \\ v_m(0, \cdot) \to v(0, \cdot) & \text{strongly in} \quad C^0([0,T]), \\ |v'_m(0, \cdot)|^{\frac{q}{2}-1} v'_m(0, \cdot) \to \chi_0 & \text{strongly in} \quad C^0([0,T]). \end{cases} \tag{2.40}$$

By means of the continuity of f_1, we have

$$f_1(u_m, v_m) \to f_1(u, v) \text{ a.e. } (x, t) \text{ in } Q_T. \tag{2.41}$$

By $\|f_1(u_m, v_m)\|_{L^2(Q_T)} \leq \sqrt{T} \sup_{|y|,\, |z| \leq \sqrt{C_T}} |f_1(y, z)| < \infty$, from (2.41) and Lions's Lemma ([6], Lemma 1.3, p.12), the result is

$$f_1(u_m, v_m) \to f_1(u, v) \text{ in } L^2(Q_T) \text{ weakly}. \tag{2.42}$$

Similarly

$$f_2(u_m, v_m) \to f_1(u, v) \text{ in } L^2(Q_T) \text{ weakly}. \tag{2.43}$$

By means of the following inequality

$$|\Psi_r(x) - \Psi_r(y)| \leq (r-1) R^{r-2} |x - y|, \tag{2.44}$$

for all $x, y \in [-R, R], R > 0, r \geq 2$, it follows from (2.27), (2.38) and $(2.40)_2$, that

$$\begin{cases} \Psi_{r_1}(u'_m) \to \Psi_{r_1}(u') \text{ strongly in } L^2(Q_T), \\ \Psi_{r_2}(v'_m) \to \Psi_{r_2}(v') \text{ strongly in } L^2(Q_T). \end{cases} \tag{2.45}$$

By means of the continuity of Ψ_p, $(2.40)_3$ gives

$$\Psi_p(v_m(0,t)) \to \Psi_p(v(0,t)) \text{ strongly in } C^0([0,T]). \tag{2.46}$$

Put $\eta_m = |v'_m(0,\cdot)|^{\frac{q}{2}-1} v'_m(0,\cdot) \to \chi_0$, from $(2.40)_4$, we have

$$\eta_m \to \chi_0 \text{ strongly in } C^0([0,T]). \tag{2.47}$$

It follows from (2.47) that

$$v'_m(0,\cdot) = |\eta_m|^{\frac{2}{q}-1} \eta_m \to |\chi_0|^{\frac{2}{q}-1} \chi_0 \text{ strongly in } C^0([0,T]). \tag{2.48}$$

We deduce from $(2.39)_6$ and (2.48) that

$$|\chi_0|^{\frac{2}{q}-1} \chi_0 = v'(0,\cdot). \tag{2.49}$$

By means of the continuity of Ψ_q, it follows from (2.48) and (2.49) that

$$\Psi_q(v'_m(0,\cdot)) \to \Psi_q(v'(0,\cdot)) \text{ strongly in } C^0([0,T]). \tag{2.50}$$

Passing to the limit in (2.11) by $(2.39)_{1,2,3}$, (2.40), (2.42), (2.43), (2.45), (2.46), and (2.50), we have (u, v) satisfying the problem

$$\begin{cases} \langle u''(t), w\rangle + \langle u_x(t), w_x\rangle + \lambda_1 \langle \Psi_{r_1}(u'(t)), w\rangle \\ \quad + \langle f_1(u,v), w\rangle = \langle F_1(t), w\rangle, \\ \langle v''(t), \phi\rangle + \langle v_x(t), \phi_x\rangle + \lambda_2 \langle \Psi_{r_2}(v'(t)), \phi\rangle \\ \quad + [K\Psi_p(v(0,t)) + \mu\Psi_q(v'(0,t))]\, \phi(0) + \langle f_2(u,v), \phi\rangle = \langle F_2(t), \phi\rangle, \end{cases} \tag{2.51}$$

for all $(w, \phi) \in H_0^1 \times V$, together with the initial conditions

$$(u(0), u_t(0)) = (\tilde{u}_0, \tilde{u}_1), \; (v(0), v_t(0)) = (\tilde{v}_0, \tilde{v}_1). \tag{2.52}$$

On the other hand, we have from $(2.39)_{1,2,3}$, (2.51) and (H_2) that

$$\begin{cases} u_{xx} = u'' + \lambda_1\Psi_{r_1}(u') + f_1(u,v) - F_1 \in L^\infty(0,T;L^2), \\ v_{xx} = v'' + \lambda_2\Psi_{r_2}(v') + f_2(u,v) - F_2 \in L^\infty(0,T;L^2). \end{cases} \tag{2.53}$$

Thus $(u, v) \in L^\infty(0, T; (H_0^1 \cap H^2) \times (V \cap H^2))$.

On the other hand, by means of the continuity of function $z \longmapsto |z|^{\frac{r_1}{2}-1} z$, from $(2.40)_2$ and Lions's Lemma ([6], Lemma 1.3, p.12), we obtain

$$\left|u'_m\right|^{\frac{r_1}{2}-1} u'_m \to \left|u'\right|^{\frac{r_1}{2}-1} u' \text{ in } L^2(Q_T) \text{ weakly.}$$

So, from $(2.39)_8$, the result is

$$\frac{\partial}{\partial t}\left(\left|u'\right|^{\frac{r_1}{2}-1}u'\right)=\chi_1\in L^2(Q_T).$$

Thus, we obtain from $(2.39)_2$, that

$$\frac{\partial}{\partial x}\left(\left|u'\right|^{\frac{r_1}{2}-1}u'\right)=\frac{r_1}{2}\left|u'\right|^{\frac{r_1}{2}-1}u'_x\in L^2(Q_T).$$

Hence

$$\left|u'\right|^{\frac{r_1}{2}-1}u'\in H^1(Q_T).$$

Similarly

$$\left|v'\right|^{\frac{r_2}{2}-1}v'\in H^1(Q_T).$$

Consequently, the existence is proved.

Step 4. *Uniqueness of the solution.* Assume now that $q=2$ and $p\geq 2$; or $q>2$ and $p\in\{2\}\cup[3,+\infty)$. Let $(u_i,v_i), i=1,2$ be two weak solutions of problem (1.1) – (1.3), such that

$$\begin{cases}(u_i,v_i)\in L^\infty(0,T;(H_0^1\cap H^2)\times(V\cap H^2)),\\ (u_i',v_i')\in L^\infty(0,T;H_0^1\times V)\cap\left(L^{r_1}(Q_T)\times L^{r_2}(Q_T)\right),\\ (u_i'',v_i'')\in L^\infty(0,T;L^2\times L^2),\\ |u_i'|^{\frac{r_1}{2}-1}u_i',\ |v_i'|^{\frac{r_2}{2}-1}v_i'\in H^1(Q_T),\\ |v_i'(0,\cdot)|^{\frac{q}{2}-1}v_i'(0,\cdot)\in H^1\left(0,T\right),i=1,2.\end{cases}\tag{2.54}$$

Then $(u,v)=(u_1-u_2,\,v_1-v_2)$ satisfy the variational problem

$$\begin{cases}\langle u''(t),w\rangle+\langle u_x(t),w_x\rangle+\lambda_1\left\langle\Psi_{r_1}(u_1'(t))-\Psi_{r_1}(u_1'(t)),w\right\rangle\\ \qquad+\langle f_1(u_1,v_1)-f_1(u_2,v_2),w\rangle=0,\\ \langle v''(t),\phi\rangle+\langle v_x(t),\phi_x\rangle+\lambda_2\left\langle\Psi_{r_2}(v_1'(t))-\Psi_{r_2}(v_2'(t)),\phi\right\rangle\\ \qquad+K\left[\Psi_p(v_1(0,t))-\Psi_p(v_2(0,t))\right]\phi(0)\\ \qquad+\mu\left[\Psi_q(v_1'(0,t))-\Psi_q(v_2'(0,t))\right]\phi(0)\\ \qquad+\langle f_2(u_1,v_1)-f_2(u_2,v_2),\phi\rangle=0,\forall(w,\phi)\in H_0^1\times V,\\ u(0)=v(0)=u'(0)=v'(0)=0.\end{cases}\tag{2.55}$$

We take $(w,\phi)=(u',v')$ in $(2.55)_{1,2}$, and integrating with respect to t, we obtain

$$\begin{aligned}\sigma(t)=&-2\int_0^t\langle f_1(u_1,v_1)-f_1(u_2,v_2),u'\rangle\,ds\\ &-2\int_0^t\langle f_2(u_1,v_1)-f_2(u_2,v_2),v'\rangle\,ds\\ &-2K\int_0^t\left[\Psi_p(v_1(0,s))-\Psi_p(v_2(0,s))\right]v'(0,s)ds\\ \equiv&\ Z_1+Z_2+Z_3,\end{aligned}\tag{2.56}$$

where

$$\begin{aligned}\sigma(t) &= \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \\ &\quad +2\lambda_1 \int_0^t \langle \Psi_{r_1}(u_1'(s)) - \Psi_{r_1}(u_1'(s)), u'(s)\rangle\, ds \\ &\quad +2\lambda_2 \int_0^t \langle \Psi_{r_2}(v_1'(s)) - \Psi_{r_2}(v_2'(s)), v'(s)\rangle\, ds \\ &\quad +2\mu \int_0^t [\Psi_q(v_1'(0,s)) - \Psi_q(v_2'(0,s))]\, v'(0,s)ds.\end{aligned} \tag{2.57}$$

Using the following inequality

$$\forall q \geq 2, \exists C_q > 0 : (\Psi_q(x) - \Psi_q(y))\,(x-y) \geq C_q\,|x-y|^q\,, \forall x, y \in \mathbb{R}, \tag{2.58}$$

it follows from (2.57), that

$$\begin{aligned}\sigma(t) &\geq \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \\ &\quad +2C_{r_1}\lambda_1 \int_0^t \|u'(s)\|_{L^{r_1}}^{r_1}\, ds + 2C_{r_2}\lambda_2 \int_0^t \|v'(s)\|_{L^{r_2}}^{r_2}\, ds \\ &\quad +2C_q\mu \int_0^t |v'(0,s)|^q\, ds.\end{aligned} \tag{2.59}$$

Put

$$R = \max_{i=1,2}\left(\|u_{ix}\|_{L^\infty(0,T;H^1)} + \|v_{ix}\|_{L^\infty(0,T;H^1)}\right), \tag{2.60}$$

and

$$L_i(R) = \sup_{|y|,\ |z|\leq R}\left(\left|\frac{\partial f_i}{\partial y}(y,z)\right| + \left|\frac{\partial f_i}{\partial z}(y,z)\right|\right),\ i = 1, 2. \tag{2.61}$$

We shall estimate respectively the terms on the right-hand side of (2.56) as follows:

Integral Z_1. By (2.60), (2.61), and deduce from the Cauchy-Schwartz inequality that

$$\begin{aligned}|Z_1| &= \left|-2\int_0^t \langle f_1(u_1,v_1) - f_1(u_2,v_2), u'\rangle\, ds\right| \\ &\leq 2\int_0^t \|f_1(u_1,v_1) - f_1(u_2,v_2)\|\, \|u'(s)\|\, ds \\ &\leq 2L_1(R)\int_0^t [\|u(s)\| + \|v(s)\|]\, \|u'(s)\|\, ds \\ &\leq 2L_1(R)\int_0^t [\|u_x(s)\| + \|v_x(s)\|]\, \|u'(s)\|\, ds \leq 2L_1(R)\int_0^t \sigma(s)ds.\end{aligned} \tag{2.62}$$

Integral Z_2. Similarly,

$$|Z_2| = \left|-2\int_0^t \left\langle f_2(u_1,v_1) - f_2(u_2,v_2), v'\right\rangle ds\right| \leq 2L_2(R)\int_0^t \sigma(s)ds. \tag{2.63}$$

Integral Z_3. We consider

$$Z_3 = -2K\int_0^t [\Psi_p(v_1(0,s)) - \Psi_p(v_2(0,s))]\, v'(0,s)ds \tag{2.64}$$

in three cases for (p,q) as follows.

Case 1: $q = 2,\ p \geq 2$: Note that, by (2.59), we have

$$\sigma(t) \geq v^2(0,t) + 2\mu \int_0^t \left|v'(0,s)\right|^2 ds. \tag{2.65}$$

By (2.44), (2.60), (2.64), and (2.65), we have

$$\begin{aligned} Z_3 &\leq 2K(p-1)R^{p-2} \int_0^t |v(0,s)|\,|v'(0,s)|\,ds \\ &\leq \tfrac{K^2}{\mu}(p-1)^2 R^{2p-4} \int_0^t |v(0,s)|^2\,ds + \mu \int_0^t |v'(0,s)|^2\,ds \\ &\leq \tfrac{K^2}{\mu}(p-1)^2 R^{2p-4} \int_0^t \sigma(s)ds + \tfrac{1}{2}\sigma(t). \end{aligned} \tag{2.66}$$

Case 2: $q > 2,\ p = 2$:

$$Z_3 = -2K \int_0^t v(0,s)v'(0,s)ds = -Kv^2(0,t) \leq 0. \tag{2.67}$$

Case 3: $q > 2,\ p \geq 3$: By using integration by parts leads to

$$\begin{aligned} &Z_3 \\ &= -2K \int_0^t \left[\Psi_p(v_1(0,s)) - \Psi_p(v_2(0,s))\right] v'(0,s)ds \\ &= -2K \int_0^t \left[\int_0^1 \Psi_p'\left(v_2(0,s) + \theta v(0,s)\right) d\theta\right] v(0,s)v'(0,s)ds \\ &= -Kv^2(0,t) \int_0^1 \Psi_p'\left(v_2(0,t) + \theta v(0,t)\right) d\theta \\ &\quad + K\int_0^t v^2(0,s)ds \int_0^1 \Psi_p''\left(v_2(0,s) + \theta v(0,s)\right)\left(v_2'(0,s) + \theta v'(0,s)\right) d\theta \\ &\leq K(p-1)(p-2) \int_0^t v^2(0,s)ds \int_0^1 |v_2(0,s) + \theta v(0,s)|^{p-3} \\ &\qquad \cdot |v_2'(0,s) + \theta v'(0,s)|\,d\theta \\ &\leq \int_0^t \zeta_{p,q}(s)\sigma(s)ds, \end{aligned} \tag{2.68}$$

where

$$\begin{aligned} &\zeta_{p,q}(s) \\ &= K(p-1)(p-2)R^{p-3}\left(|v_2'(0,s)| + |v_1'(0,s)|\right),\ \ \zeta_{p,q} \in L^1(0,T). \end{aligned} \tag{2.69}$$

Put

$$\zeta_{p,q}^*(s) = \begin{cases} \frac{K^2}{\mu}(p-1)^2 R^{2p-4}, & q = 2,\ p \geq 2, \\ 0, & q > 2,\ p = 2, \\ \zeta_{p,q}(s), & q > 2,\ p \geq 3. \end{cases} \tag{2.70}$$

Hence, it follows from (2.66) - (2.70) that

$$Z_3(t) \leq \int_0^t \zeta_{p,q}^*(s)\sigma(s)ds + \frac{1}{2}\sigma(t). \tag{2.71}$$

Combining (2.56), (2.62), (2.63) and (2.71), we get

$$\sigma(t) \leq \int_0^t \zeta_*(s)\sigma(s)ds, \tag{2.72}$$

where

$$\zeta_*(s) = 4\left(L_1(R) + L_2(R)\right) + 2\zeta^*_{p,q}(s),\ \zeta_* \in L^1(0,T). \tag{2.73}$$

By Gronwall's lemma, (2.72) leads to $\sigma(t) \equiv 0$, i.e., $u = u_1 - u_2 \equiv 0$, $v = v_1 - v_2 \equiv 0$. Theorem 2.5 is proved completely. □

Proof of Theorem 2.7. Let $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$, $(\tilde{v}_0, \tilde{v}_1) \in V \times L^2$; $(F_1, F_2) \in L^2(Q_T)$; $q = 2$ and $p \geq 2$; or $q > 2$ and $p \in \{2\} \cup [3, +\infty)$.

In order to obtain the existence of a weak solution, we use standard arguments of density.

Let us consider the sequences $\{(u_{0m}, u_{1m})\} \subset C_0^\infty\left(\overline{\Omega}\right) \times C_0^\infty\left(\overline{\Omega}\right)$, $\{(v_{0m}, v_{1m})\} \subset C_0^\infty\left(\overline{\Omega}\right) \times C_0^\infty\left(\overline{\Omega}\right)$, and $\{(F_{1m}, F_{2m})\} \subset C_0^\infty\left(\overline{Q}_T\right) \times C_0^\infty\left(\overline{Q}_T\right)$, such that

$$\begin{cases} (u_{0m}, u_{1m}) \to (\tilde{u}_0, \tilde{u}_1) & \text{strongly in} \quad H_0^1 \times L^2, \\ (v_{0m}, v_{1m}) \to (\tilde{v}_0, \tilde{v}_1) & \text{strongly in} \quad V \times L^2, \\ (F_{1m}, F_{2m}) \to (F_1, F_2) & \text{strongly in} \quad L^2(Q_T) \times L^2(Q_T). \end{cases} \tag{2.74}$$

So $\{(v_{0m}, v_{1m})\}$ satisfy, for all $m \in \mathbb{N}$, the compatibility conditions

$$v_{0mx}(0) = K\left|v_{0m}(0)\right|^{p-2} v_{0m}(0) + \mu\left|v_{1m}(0)\right|^{q-2} v_{1m}(0). \tag{2.75}$$

Then, for each $m \in \mathbb{N}$ there exists a unique function (u_m, v_m) in the conditions of the Theorem 2.5. So we can verify

$$\begin{cases} \langle u''_m(t), w\rangle + \langle u_{mx}(t), w_x\rangle + \lambda_1\left\langle \Psi_{r_1}(u'_m(t)), w\right\rangle \\ \qquad + \langle f_1(u_m, v_m), w\rangle = \langle F_{1m}(t), w\rangle, \\ \langle v''_m(t), \phi\rangle + \langle v_{mx}(t), \phi_x\rangle + \lambda_2\left\langle \Psi_{r_2}(v'_m(t)), \phi\right\rangle + [K\Psi_p(v_m(0,t)) \\ \qquad + \mu\Psi_q(v'_m(0,t))]\,\phi(0) + \langle f_2(u_m, v_m), \phi\rangle = \langle F_{2m}(t), \phi\rangle, \\ \text{for all } (w, \phi) \in H_0^1 \times V, \\ (u_m(0), u'_m(0)) = (u_{0m}, u_{1m}),\ \ (v_m(0), v'_m(0)) = (v_{0m}, v_{1m}), \end{cases} \tag{2.76}$$

and

$$\begin{cases} (u_m, v_m) \in L^\infty(0,T; (H_0^1 \cap H^2) \times (V \cap H^2)) \\ \qquad \cap C^0\left(0,T; H_0^1 \times V\right) \cap C^1\left(0,T; L^2 \times L^2\right), \\ (u'_m, v'_m) \in L^\infty(0,T; H_0^1 \times V) \cap C^0\left(0,T; L^2 \times L^2\right), \\ (u''_m, v''_m) \in L^\infty(0,T; L^2 \times L^2), \\ \left|u'_m\right|^{\frac{r_1}{2}-1} u'_m,\ \ \left|v'_m\right|^{\frac{r_2}{2}-1} v'_m \in H^1(Q_T), \\ \left|v'_m(0,\cdot)\right|^{\frac{q}{2}-1} v'_m(0,\cdot) \in H^1(0,T). \end{cases} \tag{2.77}$$

By the same arguments used to obtain the above estimates, we get

$$\begin{aligned} S_m(t) = & ||u_m'(t)||^2 + ||v_m'(t)||^2 + ||u_{mx}(t)||^2 + ||v_{mx}(t)||^2 \\ & +2\lambda_1 \int_0^t \|u_m'(s)\|_{L^{r_1}}^{r_1} ds + 2\lambda_2 \int_0^t \|v_m'(s)\|_{L^{r_2}}^{r_2} ds \\ & +2\mu \int_0^t |v_m'(0,s)|^q ds \\ & \leq C_T, \end{aligned} \tag{2.78}$$

$\forall t \in [0, T_*]$, where C_T is a positive constant independent of m and t.

On the other hand, we put $U_{m,l} = u_m - u_l$, $V_{m,l} = v_m - v_l$, from (2.76), it follows that

$$\begin{cases} \langle U_{m,l}''(t), w\rangle + \langle U_{m,lx}(t), w_x\rangle + \lambda_1 \langle \Psi_{r_1}(u_m'(t)) - \Psi_{r_1}(u_l'(t)), w\rangle \\ \qquad + \langle f_1(u_m, v_m) - f_1(u_l, v_l), w\rangle = \langle F_{1m} - F_{1l}, w\rangle , \\ \langle V_{m,l}''(t), \phi\rangle + \langle V_{m,lx}(t), \phi_x\rangle + \lambda_2 \langle \Psi_{r_2}(v_m'(t)) - \Psi_{r_2}(v_l'(t)), \phi\rangle \\ \qquad +\mu \left[\Psi_q(v_m'(0,t)) - \Psi_q(v_l'(0,t))\right] \phi(0) + K \left[\Psi_p(v_m(0,t))\right. \\ \qquad \left. -\Psi_p(v_l(0,t))\right] \phi(0) + \langle f_2(u_m, v_m) - f_2(u_l, v_l), \phi\rangle \\ \quad = \langle F_{2m} - F_{2l}, \phi\rangle , \text{ for all } (w, \phi) \in H_0^1 \times V, \\ U_{m,l}(0) = u_{0m} - u_{0l},\, U_{m,l}'(0) = u_{1m} - u_{1l}, \\ V_{m,l}(0) = v_{0m} - v_{0l},\, V_{m,l}'(0) = v_{1m} - v_{1l}. \end{cases} \tag{2.79}$$

We take $w = U_{m,l}' = u_m' - u_l'$, $\phi = V_{m,l}' = v_m' - v_l'$, in $(2.79)_{1,2}$ and integrating with respect to t, we obtain

$$\begin{aligned} S_{m,l}(t) = & S_{m,l}(0) - 2 \int_0^t \left\langle f_1(u_m, v_m) - f_1(u_l, v_l), U_{m,l}'(s)\right\rangle ds \\ & -2 \int_0^t \left\langle f_2(u_m, v_m) - f_2(u_l, v_l), V_{m,l}'(s)\right\rangle ds \\ & +2 \int_0^t \left[\left\langle F_{1m}(s) - F_{1l}(s), U_{m,l}'(s)\right\rangle \right. \\ & \left. + \left\langle F_{2m}(s) - F_{2l}(s), V_{m,l}'(s)\right\rangle\right] ds \\ & -2K \int_0^t \left[\Psi_p(v_m(0,s)) - \Psi_p(v_l(0,s))\right] V_{m,l}'(0,s) ds \\ \equiv & S_{m,l}(0) + J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{2.80}$$

where

$$\begin{aligned} S_{m,l}(t) = & ||U_{m,l}'(t)||^2 + ||V_{m,l}'(t)||^2 + ||U_{m,l\,x}(t)||^2 + ||V_{m,l\,x}(t)||^2 \\ & +2\lambda_1 \int_0^t \left\langle \Psi_{r_1}(u_m'(s)) - \Psi_{r_1}(u_l'(s)), U_{m,l}'(s)\right\rangle ds \\ & +2\lambda_2 \int_0^t \left\langle \Psi_{r_2}(v_m'(s)) - \Psi_{r_2}(v_l'(s)), V_{m,l}'(s)\right\rangle ds \\ & +2\mu \int_0^t \left[\Psi_q(v_m'(0,s)) - \Psi_q(v_l'(0,s))\right] V_{m,l}'(0,s) ds, \end{aligned} \tag{2.81}$$

$$\begin{aligned} & S_{m,l}(0) \\ & = \|u_{1m} - u_{1l}\|^2 + \|v_{1m} - v_{1l}\|^2 + \|u_{0mx} - u_{0lx}\|^2 + \|v_{0mx} - v_{0lx}\|^2 \\ & \to 0, \end{aligned} \tag{2.82}$$

as $m,\, l \to \infty$. Using the following inequality (2.58), it follows from (2.81), that

$$\begin{aligned} S_{m,l}(t) \geq\ & ||U'_{m,l}(t)||^2 + ||V'_{m,l}(t)||^2 + ||U_{m,l\ x}(t)||^2 + ||V_{m,l\ x}(t)||^2 \\ & + 2\lambda_1 C_{r_1} \int_0^t ||U'_{m,l}(s)||_{L^{r_1}}^{r_1} ds + 2\lambda_2 C_{r_2} \int_0^t ||V'_{m,l}(s)||_{L^{r_2}}^{r_2} ds \\ & + 2\mu C_q \int_0^t |V'_{m,l}(0,s)|^q ds \equiv \bar{S}_{m,l}(t). \end{aligned} \tag{2.83}$$

We estimate the terms on the right-hand side of (2.80) as follows:

The term J_1. Put

$$L_T = \max_{i=1,2} \sup_{|y|,\ |z| \leq \sqrt{C_T}} \left(\left| \frac{\partial f_i}{\partial y}(y,z) \right| + \left| \frac{\partial f_i}{\partial z}(y,z) \right| \right). \tag{2.84}$$

we have

$$\begin{aligned} J_1 &= -2 \int_0^t \left\langle f_1(u_m, v_m) - f_1(u_l, v_l), U'_{m,l}(s) \right\rangle ds \\ &\leq 2L_T \int_0^t \left[\|u_m - u_l\| + \|v_m - v_l\| \right] ||U'_{m,l}(s)|| ds \\ &\leq 2L_T \int_0^t \left[||U_{m,l\ x}(s)|| + ||V_{m,l\ x}(s)|| \right] ||U'_{m,l}(s)|| ds \\ &\leq 4L_T \int_0^t \bar{S}_{m,l}(s) ds. \end{aligned} \tag{2.85}$$

The term J_2. Similarly

$$J_2 = -2 \int_0^t \left\langle f_2(u_m, v_m) - f_2(u_l, v_l), V'_{m,l}(s) \right\rangle ds \leq 4L_T \int_0^t \bar{S}_{m,l}(s) ds. \tag{2.86}$$

The term J_3.

$$\begin{aligned} J_3 &= 2 \int_0^t \left[\left\langle F_{1m}(s) - F_{1l}(s), U'_{m,l}(s) \right\rangle + \left\langle F_{2m}(s) - F_{2l}(s), V'_{m,l}(s) \right\rangle \right] ds \\ &\leq 2 \int_0^t \Big[\|F_{1m}(s) - F_{1l}(s)\| \, ||U'_{m,l}(s)|| \\ &\qquad + \|F_{2m}(s) - F_{2l}(s)\| \, ||V'_{m,l}(s)|| \ \Big] ds \\ &\leq \int_0^t \left[\|F_{1m}(s) - F_{1l}(s)\|^2 + \|F_{2m}(s) - F_{2l}(s)\|^2 \right] ds \\ &\qquad + \int_0^t \left[||U'_{m,l}(s)||^2 + ||V'_{m,l}(s)||^2 \right] ds \\ &\leq \|F_{1m} - F_{1l}\|_{L^2(Q_T)}^2 + \|F_{2m} - F_{2l}\|_{L^2(Q_T)}^2 + \int_0^t \bar{S}_{m,l}(s) ds. \end{aligned} \tag{2.87}$$

The term $J_4 = -2K \int_0^t \left[\Psi_p(v_m(0,s)) - \Psi_p(v_l(0,s))\right] V'_{m,l}(0,s)ds.$

Now, we need the following lemma.

Lemma 2.8. *Let $q = 2$ and $p \geq 2$; or $q > 2$ and $p \in \{2\} \cup [3,+\infty)$. Then there exist a constant $C_0 > 0$ depending only on $\tilde{v}_0$, p, K and a positive integrable functions $J_{m,l}$ such that $\|J_{m,l}\|_{L^1(0,T)} \leq C_T$, for all m, $l \in \mathbb{N}$, and*

$$J_4 \leq C_0 S_{m,l}(0) + \int_0^t J_{m,l}(s)\bar{S}_{m,l}(s)ds + \frac{1}{2}\bar{S}_{m,l}(t), \tag{2.88}$$

and C_T always indicates a bound depending on T.

Proof. We will consider the following cases for (p,q) :

Case 1: $q = 2$, $p \geq 2$;
Case 2: $q > 2$, $p = 2$;
Case 3: $q > 2$, $p \geq 3$.

Case 1: $q = 2$, $p \geq 2$: Note that, by (2.83), we have

$$\bar{S}_{m,l}(t) \geq V^2_{m,l}(0,t) + 2\mu \int_0^t |V'_{m,l}(0,s)|^2 ds. \tag{2.89}$$

By (2.44), (2.78), and (2.89), we have

$$\begin{aligned} J_4 &\leq 2K(p-1)\sqrt{C_T^{p-2}} \int_0^t |V_{m,l}(0,s)|\,|V'_{m,l}(0,s)|ds \\ &\leq \tfrac{K^2}{\mu}(p-1)^2 C_T^{p-2} \int_0^t V^2_{m,l}(0,s)ds + \mu \int_0^t |V'_{m,l}(0,s)|^2 ds \\ &\leq \tfrac{K^2}{\mu}(p-1)^2 C_T^{p-2} \int_0^t \bar{S}_{m,l}(s)ds + \tfrac{1}{2}\bar{S}_{m,l}(t), \end{aligned} \tag{2.90}$$

it is clear that (2.88) holds with

$$J_{m,l}(s) = \text{constant} \equiv \frac{K^2}{\mu}(p-1)^2 C_T^{p-2},\ J_{m,l} \in L^1(0,T). \tag{2.91}$$

Case 2: $q > 2$, $p = 2$: From (2.82), we obtain

$$\begin{aligned} J_4 &= -2K \int_0^t \left[\Psi_p(v_m(0,s)) - \Psi_p(v_l(0,s))\right] V'_{m,l}(0,s)ds \\ &= -2K \int_0^t V_{m,l}(0,s)V'_{m,l}(0,s)ds = -K\left[V^2_{m,l}(0,t) - V^2_{m,l}(0,0)\right] \\ &\leq KV^2_{m,l}(0,0) = K\,|v_{0m}(0) - v_{0l}(0)|^2 \\ &\leq K\,\|v_{0mx} - v_{0lx}\|^2 \leq KS_{m,l}(0). \end{aligned} \tag{2.92}$$

Then, (2.88) also holds with

$$C_0 = K, J_{m,l}(s) \equiv 0. \tag{2.93}$$

Case 3: $q > 2,\, p \geq 3$: Because

$$\Psi_p(v_m(0,s)) - \Psi_p(v_l(0,s)) = \left[\int_0^1 \Psi_p'\left(v_l(0,s) + \theta V_{m,l}(0,s)\right) d\theta\right] V_{m,l}(0,s),$$

we get

$$\begin{aligned}
J_4 &= -2K \int_0^t \left[\Psi_p(v_m(0,s)) - \Psi_p(v_l(0,s))\right] V_{m,l}'(0,s)ds \\
&= -2K \int_0^t \left[\int_0^1 \Psi_p'\left(v_l(0,s) + \theta V_{m,l}(0,s)\right) d\theta\right] V_{m,l}(0,s) V_{m,l}'(0,s)ds \\
&= -KV_{m,l}^2(0,t) \int_0^1 \Psi_p'\left(v_l(0,t) + \theta V_{m,l}(0,t)\right) d\theta \\
&\quad + K\left|v_{0m}(0) - v_{0l}(0)\right|^2 \int_0^1 \Psi_p'\left(v_{0l}(0) + \theta\left(v_{0m}(0) - v_{0l}(0)\right)\right) d\theta \\
&\quad + K \int_0^t V_{m,l}^2(0,s)ds \int_0^1 \Psi_p''\left(v_l(0,s) + \theta V_{m,l}(0,s)\right) \\
&\quad\cdot \left(v_l'(0,s) + \theta V_{m,l}'(0,s)\right) d\theta.
\end{aligned} \tag{2.94}$$

Note that $\Psi_p'(z) = (p-1)\left|z\right|^{p-2} \geq 0$, $\Psi_p''(z) = (p-1)(p-2)\left|z\right|^{p-4} z$, $\forall z \in \mathbb{R}$, combining (2.78), (2.82), (2.83) and (2.94), the result is

$$\begin{aligned}
J_4 &\leq K(p-1)\left|v_{0m}(0) - v_{0l}(0)\right|^2 \int_0^1 \left|v_{0l}(0) + \theta\left(v_{0m}(0) - v_{0l}(0)\right)\right|^{p-2} d\theta \\
&\quad + K(p-1)(p-2) \int_0^t V_{m,l}^2(0,s)ds \int_0^1 \left|v_l(0,s) + \theta V_{m,l}(0,s)\right|^{p-3} \\
&\quad\cdot \left|v_l'(0,s) + \theta V_{m,l}'(0,s)\right| d\theta \\
&\leq K(p-1)\left\|v_{0mx} - v_{0lx}\right\|^2 \left(\left\|v_{0mx}\right\| + \left\|v_{0lx}\right\|\right)^{p-2} \\
&\quad + \int_0^t J_{m,l}(s)\bar{S}_{m,l}(s)ds \\
&\leq C_0 S_{m,l}(0) + \int_0^t J_{m,l}(s)\bar{S}_{m,l}(s)ds.
\end{aligned} \tag{2.95}$$

This leads to (2.88) in which

$$\begin{aligned}
J_{m,l}(s) = K(p-1)(p-2) \int_0^1 \left|v_l(0,s) + \theta V_{m,l}(0,s)\right|^{p-3} \\
\cdot \left|v_l'(0,s) + \theta V_{m,l}'(0,s)\right| d\theta.
\end{aligned} \tag{2.96}$$

We have to prove that $\left\|J_{m,l}\right\|_{L^1(0,T)} \leq C_T$, for all $m,\, l \in \mathbb{N}$.

Note that, by (2.78), we have

$$\begin{aligned}
\left|v_l(0,s) + \theta V_{m,l}(0,s)\right|^{p-3} &\leq \left(\left\|v_{mx}(s)\right\| + \left\|v_{lx}(s)\right\|\right)^{p-3} \\
&\leq \left(2\sqrt{C_T}\right)^{p-3}, \text{ for all } m, l \in \mathbb{N}.
\end{aligned} \tag{2.97}$$

Hence, it follows from (2.96), (2.97) that

$$\begin{aligned}
\left|J_{m,l}(s)\right| &\leq K(p-1)(p-2)\left(2\sqrt{C_T}\right)^{p-3} \left(\left|v_m'(0,s)\right| + \left|v_l'(0,s)\right|\right) \\
&\leq Const.\left(\left|v_m'(0,s)\right| + \left|v_l'(0,s)\right|\right).
\end{aligned} \tag{2.98}$$

Thus, it follows from (2.78), (2.98), that

$$\begin{aligned} &\|J_{m,l}\|_{L^1(0,T)} \\ &\leq \text{Const}\,\|J_{m,l}\|_{L^q(0,T)} \\ &\leq \text{Const}\left[\left(\int_0^T \left|v_m'(0,s)\right|^q ds\right)^{1/q} + \left(\int_0^T \left|v_l'(0,s)\right|^q ds\right)^{1/q}\right] \\ &\leq C_T. \end{aligned} \tag{2.99}$$

Lemma 2.8 is proved completely. □

It follows from (2.80), (2.82), (2.83), (2.85)–(2.88), that

$$\bar{S}_{m,l}(t) \leq R_{m,l} + \int_0^t q_{m,l}(s)\bar{S}_{m,l}(s)ds, \tag{2.100}$$

where

$$\begin{aligned} q_{m,l}(s) &= 2\left(1 + 8L_T + J_{m,l}(s)\right), \quad \int_0^T q_{m,l}(s)ds \leq C_T, \\ R_{m,l} &= 2\left(1 + C_0\right) S_{m,l}(0) + 2\left\|F_{1m} - F_{1l}\right\|^2_{L^2(Q_T)} \\ &\quad + 2\left\|F_{2m} - F_{2l}\right\|^2_{L^2(Q_T)} \to 0, \end{aligned} \tag{2.101}$$

as $m, l \to \infty$. By Gronwall's lemma, it follows from (2.100) that

$$\bar{S}_{m,l}(t) \leq R_{m,l} \exp\left[\int_0^T q_{m,l}(s)ds\right] \leq C_T R_{m,l}, \ \forall t \in [0,T]. \tag{2.102}$$

Convergences of the sequences $\{(u_{0m}, u_{1m})\}$ and $\{(v_{0m}, v_{1m})\}$ imply the convergence to zero (when $m,\, l \to \infty$) of terms on the right hand side of (2.102). Therefore, we get

$$\begin{cases} (u_m, v_m) \to (u, v) & \text{strongly in} \\ & C^0([0,T]; H_0^1 \times V) \cap C^1([0,T]; L^2 \times L^2), \\ (u_m', v_m') \to (u', v') & \text{strongly in} \quad L^{r_1}(Q_T) \times L^{r_2}(Q_T), \\ v_m(0,\cdot) \to v(0,\cdot) & \text{strongly in} \quad W^{1,q}(0,T). \end{cases} \tag{2.103}$$

On the other hand, from (2.78), we deduce the existence of a subsequence of $\{(u_m, v_m)\}$ still also so denoted, such that

$$\begin{cases} (u_m, v_m) \to (u, v) & \text{in} \quad L^\infty(0,T; H_0^1 \times V) \quad \text{weakly*}, \\ (u_m', v_m') \to (u', v') & \text{in} \quad L^\infty(0,T; L^2 \times L^2) \quad \text{weakly*}. \end{cases} \tag{2.104}$$

Similarly, by (2.103), we deduce that

$$\begin{cases} f_1(u_m, v_m) \to f_1(u, v) & \text{strongly in } \ L^2(Q_T), \\ f_2(u_m, v_m) \to f_2(u, v) & \text{strongly in } \ L^2(Q_T), \\ \Psi_{r_1}(u'_m) \to \Psi_{r_1}(u') & \text{strongly in } \ L^2(Q_T), \\ \Psi_{r_2}(v'_m) \to \Psi_{r_2}(v') & \text{strongly in } \ L^2(Q_T). \end{cases} \tag{2.105}$$

Passing to the limit in (2.76) by (2.103) - (2.105), we have (u, v) satisfying the problem

$$\begin{cases} \frac{d}{dt}\langle u'(t), w\rangle + \langle u_x(t), w_x\rangle + \lambda_1 \langle \Psi_{r_1}(u'(t)), w\rangle \\ = \langle f_1(u, v), w\rangle + \langle F_1(t), w\rangle, \\ \frac{d}{dt}\langle v'(t), \phi\rangle + \langle v_x(t), \phi_x\rangle + \lambda_2 \langle \Psi_{r_2}(v'(t)), \phi\rangle \\ \qquad + [\mu v'(0, t) - K\Psi_p(v(0, t))\ ]\, \phi(0) \\ = \langle f_2(u, v), \phi\rangle + \langle F_2(t), \phi\rangle, \text{ for all } (w, \phi) \in H_0^1 \times V, \\ (u(0), u'(0)) = (\tilde{u}_0, \tilde{u}_1), \ (v(0), v'(0)) = (\tilde{v}_0, \tilde{v}_1). \end{cases} \tag{2.106}$$

Next, the uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions. Theorem 2.7 is proved completely. □

3. Exponential decay of solutions

In this section, we study the decay of the solution of problem (1.1) - (1.3) corresponding to $r_1 = r_2 = q = 2$. For this purpose, we also assume that the constants $K > 0$, $\mu > 0$, $p \geq 2$, $\lambda_1 > 0$, $\lambda_2 > 0$ and the functions ($\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$) satisfy the assumptions (H_1).

Henceforth, we strengthen the hypotheses as below.

$(\widehat{H}_3)$ There exist $F \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants d_1, $d_2 > 0$ such that

(i) $\frac{\partial F}{\partial u}(u, v) = f_1(u, v)$, $\frac{\partial F}{\partial v}(u, v) = f_2(u, v)$,

(ii) $F(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$,

(iii) $d_1 F(u, v) \leq u f_1(u, v) + v f_2(u, v) \leq d_2 F(u, v)$, for all $(u, v) \in \mathbb{R}^2$.

Let (u, v) be a weak solution of problem (1.1)-(1.3) satisfying (2.8). In order to obtain the decay result, we construct the functional

$$L(t) = E(t) + \delta\psi(t), \tag{3.1}$$

where δ is a positive constant and

$$\begin{aligned} E(t) = {} & \tfrac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) + \tfrac{K}{p}\,|v(0, t)|^p \\ & + \int_0^1 F(u(x, t), v(x, t))dx, \end{aligned} \tag{3.2}$$

$$\psi(t) = \langle u(t), u'(t)\rangle + \langle v(t), v'(t)\rangle + \frac{\lambda_1}{2}\,\|u(t)\|^2 + \frac{\lambda_2}{2}\,\|v(t)\|^2 + \frac{\mu}{2} v^2(0, t). \tag{3.3}$$

Put

$$E_1(t) = \left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + |v(0,t)|^p + \int_0^1 F(u(x,t), v(x,t))dx. \tag{3.4}$$

The following lemmas are necessary to ensure that the functional $L(t)$ is a Lyapunov function.

Lemma 3.1. *Let* $0 < \delta < 1$. *There exist the positive constants* β_1, β_2 *such that*

$$\beta_1 E_1(t) \le L(t) \le \beta_2 E_1(t). \tag{3.5}$$

Proof. By using the Cauchy inequality and (2.4), it follows from (3.1)-(3.4) that

$$\begin{aligned} L(t) \ge\ & \tfrac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) + \tfrac{K}{p}|v(0,t)|^p \\ & + \textstyle\int_0^1 F(u(x,t), v(x,t))dx \\ & - \tfrac{\delta}{2}\left[\|u'(t)\|^2 + \|u_x(t)\|^2 + \|v'(t)\|^2 + \|v_x(t)\|^2\right] \\ & + \tfrac{\delta}{2}\left[\lambda_1\|u(t)\|^2 + \lambda_2\|v(t)\|^2 + \mu v^2(0,t)\right] \\ \ge\ & \tfrac{1}{2}(1-\delta)\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ & + \tfrac{K}{p}|v(0,t)|^p + \textstyle\int_0^1 F(u(x,t), v(x,t))dx \ge \beta_1 E_1(t), \end{aligned} \tag{3.6}$$

which implies that

$$L(t) \ge \beta_1 E_1(t), \tag{3.7}$$

where

$$\beta_1 = \min\left\{1,\ \frac{1}{2}(1-\delta),\ \frac{K}{p}\right\}. \tag{3.8}$$

Similarly, we have

$$\begin{aligned} L(t) \le\ & \tfrac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) + \tfrac{K}{p}|v(0,t)|^p \\ & + \textstyle\int_0^1 F(u(x,t), v(x,t))dx + \tfrac{\delta}{2}\left[\|u'(t)\|^2 + \|u_x(t)\|^2 + \|v'(t)\|^2\right. \\ & \left. + \|v_x(t)\|^2\right] + \tfrac{\delta}{2}\left[\lambda_1\|u_x(t)\|^2 + \lambda_2\|v_x(t)\|^2 + \mu\|v_x(t)\|^2\right] \\ \le\ & \tfrac{1}{2}\left[1 + \delta(1 + \lambda_1 + \lambda_2 + \mu)\ \right]\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2\right. \\ & \left. + \|v_x(t)\|^2\right) + \tfrac{K}{p}|v(0,t)|^p + \textstyle\int_0^1 F(u(x,t), v(x,t))dx \\ \le\ & \beta_2 E_1(t), \end{aligned} \tag{3.9}$$

where

$$\beta_2 = \max\left\{\frac{1}{2}\left[1 + \delta(1 + \lambda_1 + \lambda_2 + \mu)\right], \frac{K}{p}, 1\right\}. \tag{3.10}$$

Lemma 3.1 is proved. □

Lemma 3.2. *The functional* $E(t)$ *defined by* (3.2) *satisfies*

$$E'(t) \le -\left(\lambda_1 - \tfrac{\varepsilon_1}{2}\right)\|u'(t)\|^2 - \left(\lambda_2 - \tfrac{\varepsilon_1}{2}\right)\|v'(t)\|^2 - \mu\,|v'(0,t)|^2 + \tfrac{1}{2\varepsilon_1}\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right), \tag{3.11}$$

for all $\varepsilon_1 > 0$.

Proof. Multiplying (1.1) by $(u'(x,t), v'(x,t))$ and integrating over $(0,1)$ leads to

$$E'(t) = -\lambda_1\left\|u'(t)\right\|^2 - \lambda_2\left\|v'(t)\right\|^2 - \mu\left|v'(0,t)\right|^2 + \langle F_1(t), u'(t)\rangle + \langle F_2(t), v'(t)\rangle. \tag{3.12}$$

On the other hand, for $\varepsilon_1 > 0$, we have

$$\begin{cases} \langle F_1(t), u'(t)\rangle \le \frac{\varepsilon_1}{2}\|u'(t)\|^2 + \frac{1}{2\varepsilon_1}\|F_1(t)\|^2, \\ \langle F_2(t), v'(t)\rangle \le \frac{\varepsilon_1}{2}\|v'(t)\|^2 + \frac{1}{2\varepsilon_1}\|F_2(t)\|^2. \end{cases} \tag{3.13}$$

Combining (3.12), (3.13), it follows that (3.11) holds. Lemma 3.2 is proved. □

Lemma 3.3. *The functional* $\psi(t)$ *defined by* (3.3) *satisfies*

$$\psi'(t) \le \left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2 - \left(1 - \frac{\varepsilon_2}{2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) - K\,|v(0,t)|^p - d_1\int_0^1 F(u(x,t), v(x,t))dx + \frac{1}{2\varepsilon_2}\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right), \tag{3.14}$$

for all $\varepsilon_2 > 0$.

Proof. Multiplying the equation (1.1) by $(u(x,t), v(x,t))$ and integrating over $(0,1)$, the result is

$$\psi'(t) = \left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2 - \|u_x(t)\|^2 - \|v_x(t)\|^2 - K\,|v(0,t)|^p - \langle f_1(u(t), v(t)), u(t)\rangle - \langle f_2(u(t), v(t)), v(t)\rangle + \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle. \tag{3.15}$$

On the other hand, for $\varepsilon_2 > 0$, by some estimations as in the proof of Lemma 3.2, we deduce the following inequalities

$$\begin{cases} \langle F_1(t), u(t)\rangle \le \frac{\varepsilon_2}{2}\|u_x(t)\|^2 + \frac{1}{2\varepsilon_2}\|F_1(t)\|^2, \\ \langle F_2(t), v(t)\rangle \le \frac{\varepsilon_2}{2}\|v_x(t)\|^2 + \frac{1}{2\varepsilon_2}\|F_2(t)\|^2. \end{cases} \tag{3.16}$$

In order to estimate the last term of (3.15), we use the hypotheses $\left(\widehat{H}_3, \text{(iii)}\right)$ and obtain that

$$\begin{aligned} & - \langle f_1(u(t), v(t)), u(t)\rangle - \langle f_2(u(t), v(t)), v(t)\rangle \\ & \leq -d_1 \int_0^1 F(u(x,t), v(x,t))\, dx. \end{aligned} \tag{3.17}$$

Combining (3.15) - (3.17), (3.14) follows. The Lemma 3.3 is proved. □

Now, we use the results obtained in Lemmas 3.1-3.3 to prove the decay of the solution of problem (1.1)-(1.3) given below.

Theorem 3.4. *Assume that*

$$\|F_1(t)\|^2 + \|F_2(t)\|^2 \leq \eta_1 \exp(-\eta_2 t) \text{ for all } t \geq 0, \tag{3.18}$$

where η_1, η_2 *are two positive constants. Then, there exist positive constants* γ, C_1 *such that*

$$E_1(t) \leq C_1 \exp(-\gamma t) \text{ for all } t \geq 0. \tag{3.19}$$

Proof. It follows from (3.1), (3.5), (3.11) and (3.14) that

$$\begin{aligned} L'(t) \leq & -\left(\lambda_1 - \frac{\varepsilon_1}{2} - \delta\right)\|u'(t)\|^2 - \left(\lambda_2 - \frac{\varepsilon_1}{2} - \delta\right)\|v'(t)\|^2 \\ & - \mu\left|v'(0,t)\right|^2 - \delta\left(1 - \frac{\varepsilon_2}{2}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ & - \delta K\left|v(0,t)\right|^p - \delta d_1 \int_0^1 F(u(x,t), v(x,t))dx \\ & + \left(\frac{1}{2\varepsilon_1} + \frac{\delta}{2\varepsilon_2}\right)\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right), \end{aligned} \tag{3.20}$$

for all ε_1, ε_2, $\delta > 0$.

Let δ, ε_2 be small enough such that

$$0 < \varepsilon_2 < 2,\ \ 0 < \delta < \min\{1, \lambda_1, \lambda_2\}. \tag{3.21}$$

Fix δ, we choose the positive constant ε_1 satisfies

$$0 < \frac{\varepsilon_1}{2} < \min\{\lambda_1 - \delta, \lambda_2 - \delta\}, \tag{3.22}$$

we deduce from (3.4), (3.5), (3.18), (3.20), (3.21) and (3.22) that there exist the positive constants γ_1, γ, with $\gamma < \eta_2$, such that

$$L'(t) \leq -\gamma L(t) + \gamma_1 \exp(-\eta_2 t) \text{ for all } t \geq 0. \tag{3.23}$$

Combining (3.4), (3.5) and (3.23), we get (3.19). The theorem 3.4 is completely proved. □

References

[1] M. Bergounioux, N. T. Long and A. P. N. Dinh, *Mathematical model for a shock problem involving a linear viscoelastic bar*, Nonlinear Anal. TMA. **43** (2001) 547–561.

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. S. Prates Filho and J. A. Soriano, *Existence and uniform decay of solutions of a degenerate equation with nonlinear boundary damping and boundary memory source term*, Nonlinear Anal. TMA. **38** (1999) 281–294.

[3] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, *On the existence and the uniform decay of a hyperbolic,* Southeast Asian Bulletin of Mathematics **24** (2000) 183–199.

[4] M. M. Cavalcanti, V. N. Domingos Cavalcanti and M. L. Santos, *Existence and uniform decay rates of solutions to a degenerate system with memory conditions at the boundary,* Applied Mathematics and Computation **150** (2004) 439–465.

[5] Liang Fei and Gao Hongjun, *Global nonexistence of positive initial-energy solutions for coupled nonlinear wave equations with damping and source terms*, Abstract and Applied Analysis, Vol. **2011**, Article ID 760209, 14 pages.

[6] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Dunod; Gauthier – Villars, Paris, 1969.

[7] N. T. Long and A. P. N. Dinh, *On the quasilinear wave equation*: $u_{tt} - \Delta u + f(u, u_t) = 0$ *associated with a mixed nonhomogeneous condition*, Nonlinear Anal. TMA. **19** (1992) 613–623.

[8] N. T. Long and A. P. N. Dinh, *A semilinear wave equation associated with a linear differential equation with Cauchy data*, Nonlinear Anal. TMA. **24** (1995) 1261–1279.

[9] N. T. Long and T. N. Diem, *On the nonlinear wave equation* $u_{tt} - u_{xx} = f(x, t, u, u_x, u_t)$ *associated with the mixed homogeneous conditions*, Nonlinear Anal. TMA. **29** (11) (1997) 1217–1230.

[10] N. T. Long, A. P. N. Dinh and T. N. Diem, *On a shock problem involving a nonlinear viscoelastic bar*, J. Boundary Value Problems, **2005** (3) (2005) 337–358.

[11] N. T. Long and L. T. P. Ngoc, *On a nonlinear wave equation with boundary conditions of two-point type*, J. Math. Anal. Appl. **385** (2) (2012) 1070–1093.

[12] Changxing Miao and Youbin Zhu, *Global smooth solutions for a non-linear system of wave equations*, Nonlinear Anal. TMA. **67** (2007) 3136–3151.

[13] L. X. Truong, L. T. P. Ngoc, A. P. N. Dinh and N. T. Long, *The regularity and exponential decay of solution for a linear wave equation associated with two-point boundary conditions*, Nonlinear Anal. RWA. **11** (2010) 1289–1303.

[14] Jieqiong Wu and Shengjia Li, *Blow-up for coupled nonlinear wave equations with damping and source*, Applied Mathematics Letters, **24** (2011) 1093–1098.

[15] Yaojun Ye, *Global existence and asymptotic behavior of solutions for some nonlinear hyperbolic equation*, Journal of Inequalities and Applications, Vol. **2010**, Article ID 895121, 10 pages.

[16] Zai-yun Zhang and Xiu-jin Miao, *Global existence and uniform decay for wave equation with dissipative term and boundary damping,* Computers and Math. with Appli. **59** (2010) 1003–1018.