Nonlinear Functional Analysis and Applications
Vol. 17, No. 4 (2012), pp. 519-537

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON THE CONVERGENCE OF NEWTON'S METHOD UNDER UNIFORM CONTINUITY CONDITIONS

I. K. Argyros[1] and S. K. Khattri[2]

[1]Department of Mathematical Sciences
Cameron University, Lawton, Oklahoma 73505-6377, USA
e-mail: `iargyros@cameron.edu`

[2]Department of Engineering
Stord Haugesund University College, Norway
e-mail: `sanjay.khattri@hsh.no`

Abstract. A new semilocal convergence analysis for Newton's method is developed under uniformly continuity assumptions on the Fréchet-derivative of the operator. It turns out that our analysis has several advantages over earlier studies. For example, error bounds derived in this work are finer than the known results in scientific literature [1, 3, 18, 24, 25, 27, 28, 31, 38, 40, 41, 43, 46, 48, 53, 54, 55] and, under the same or weaker sufficient convergence conditions, our analysis provide at least as precise information on the location of the solution. Numerical examples are also presented which further validate the developed theoritical results.

1. Introduction

In this work, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is a Fréchet differentiable operator defined on an open ball $U(x_0, R)$ $(R > 0)$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Numerous problems in science and engineering can be reduced to solving the above equation [4, 9, 15, 28, 31, 33, 34, 41, 44, 47]. Consequently, solving these equations is an important scientific field of research. In many situations, finding a closed

[0]Received November 23, 2012. Revised December 7, 2012.
[0]2000 Mathematics Subject Classification: 65J15, 65G99, 47H17, 49M15.
[0]Keywords: Newton's method, Banach space, semilocal convergence, Kantorovich hypotheses, uniform continuity.

form solution for the non-linear equation (1.1) is not possible. Therefore, iterative solution techniques are employed for solving these equations.

One of the most important iterative method is the Newton's method (**NM**) which is given as

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \quad (n \geq 0), \quad (x_0 \in \mathcal{D}). \tag{1.2}$$

The Newton's method is one of the most popular, and may be the most used, iterative procedure for generating a sequence $\{x_n\}$ to approximate the solution $x^\star$ of equation (1.1). There exists an extensive literature on local as well as semilocal convergence for the Newton's method under various Lipschitz type conditions. A recent survey of such results can be found in [9, 12, 15], and the references therein (see also [1–5] and [15–50]).

We study convergence of (**NM**) assuming there exists $x_0 \in \mathbf{X}$, $R > 0$ and a non-decreasing function $w\colon [0, R) \to [0, R)$ such that

$$\mathcal{F}'(x_0)^{-1} \in L(\mathbf{Y}, \mathbf{X}), \tag{1.3}$$

$$\lim_{t\to\infty} w(t) = 0 \tag{1.4}$$

and

$$\left\| F'(x_0)^{-1} \left(F'(x) - F'(y) \right) \right\| \leq w(\|x - y\|) \tag{1.5}$$

for all $x, y \in U(x_0, R)$. More generally, we assume that

$$\left\| F'(x_0)^{-1} \left(F'(x) - F'(y) \right) \right\| \leq v(r, \|x - y\|) \tag{1.6}$$

for all $x, y \in U(x_0, r)$ and $0 < r < R$. For some non-decreasing (in both arguments) function $v\colon [0, R) \times [0, R) \to [0, +\infty)$ satisfying

$$\lim_{t\to\infty} v(r, t) = 0, \quad 0 \leq r \leq R. \tag{1.7}$$

Sufficient convergence conditions for the semilocal convergence of (**NM**) under the assumptions (1.3)-(1.6) were given in [1]. Later in [3], and under the same assumptions (1.3)-(1.6), we provided finer error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ $(n \geq 0)$ by introducing the center-Lipschitz type condition

$$\left\| F'(x_0)^{-1} \left(F'(x) - F'(x_0) \right) \right\| \leq w_0 \left(\|x - x_0\| \right) \tag{1.8}$$

for all $x \in U(x_0, R)$ (see also Remark 2.3). Note that (1.5) (or (1.6)) imply the existence of function $w_0\colon [0, R) \to [0, +\infty)$ which can be chosen to be non-decreasing and which satisfies

$$\lim_{t\to\infty} w_0(t) = 0. \tag{1.9}$$

Moreover, for all $t \in [0, R)$ the following

$$w_0(t) \leq w(t), \tag{1.10}$$

$$w_0(t) \leq v(t,t) \tag{1.11}$$

hold and w/w_0, v/w_0 can be arbitrarily large (see Example 4.3).

In this study, we develop a finer convergence analysis and also provide sufficient conditions which are weaker than before [1, 3]. The rest of the paper is organized as follows. In the Section 2, we present results on majorizing sequences for (**NM**). While the Section 3 develops a semilocal convergence analysis of (**NM**). Finally, numerical examples are presented in the concluding Section 4.

2. Majorizing sequences for (NM)

We need a result on majorizing sequences, involving functions (w_0, w) and constants (η, R), for (**NM**).

Lemma 2.1. *Let the constants $\eta \geq 0$, $R > 0$ and non-decreasing functions $w_0, w\colon [0,\infty) \to [0, +\infty)$ with $\lim\limits_{t\to 0} w_0(t) = \lim\limits_{t\to 0} w(t) = 0$ be given. Define scalar sequence $\{t_n\}$ by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\int_0^1 w(\theta(t_{n+1}-t_n))(t_{n+1}-t_n)\,d\theta}{1 - w_0(t_{n+1})}, \tag{2.1}$$

sequences of functions $\{f_n\}$, $\{g_n\}$ on $[0,1)$ by

$$f_n(t) = \int_0^1 w\left(t^n\theta\eta\right)d\theta + t w_0\left[(1+t+\cdots+t^n)\eta\right] - t, \tag{2.2}$$

$$\begin{aligned} g_n(t) &= f_{n+1}(t) - f_n(t) \\ &= \int_0^1 \left[w(t^{n+1}\theta\eta) - w(t^n\theta\eta)\right]d\theta \\ &\quad + t\left[w_0\left((1+t+\cdots+t^{n+1})\eta\right) - w_0\left((1+t+\cdots+t^n)\eta\right)\right] \end{aligned} \tag{2.3}$$

and function f_∞ on $[0,1)$ by

$$f_\infty(t) = t\left[w_0\left(\frac{\eta}{1-t}\right) - 1\right]. \tag{2.4}$$

Additionally, assume that either of the following set of conditions hold:

H1. *there exists* $\alpha \in [0,1)$ *such that*

$$0 \leq \frac{\int_0^1 w(\theta\eta)\,d\theta}{1 - w_0(\eta)} \leq \alpha, \tag{2.5}$$

$$w_0\left(\frac{\eta}{1-\alpha}\right) \leq 1 \tag{2.6}$$

and

$$g_n(\alpha) \geq 0 \quad \textit{for all } n. \tag{2.7}$$

H2. *there exists* $\alpha \in [0,1)$ *such that*

$$f_1(\alpha) \leq 0, \tag{2.8}$$

$$0 \leq \frac{\int_0^1 w(\theta\eta)\,d\theta}{1 - w_0(\eta)} \leq \alpha,$$

and

$$g_n(\alpha) \geq 0 \quad \textit{for all } n. \tag{2.9}$$

Then, the sequence $\{t_n\}$ *is well defined, non-decreasing, bounded from above by*

$$t^{\star\star} = \frac{\eta}{1-\alpha} \tag{2.10}$$

and converges to its unique least upper bound $t^\star$ *satisfying*

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.11}$$

Moreover, the following estimates hold

$$0 \leq t_{n+1} - t_n \leq \alpha^n \eta \tag{2.12}$$

and

$$0 \leq t^\star - t_n \leq \frac{\alpha^n \eta}{1-\alpha}. \tag{2.13}$$

Proof. We consider the following two parts.

Part I. We shall show using induction

$$0 < \frac{\int_0^1 w(\theta(t_{n+1} - t_n))\,d\theta}{1 - w_0(t_{n+1})} \leq \alpha. \tag{2.14}$$

Estimate (2.14) holds for $n = 0$ by the initial conditions and (2.5). It then follows from (2.1) that

$$0 \leq t_2 - t_1 \leq \alpha(t_1 - t_0) = \alpha\eta.$$

Let us assume that (2.14) holds for all $n \leq k$. Then, we have by the induction hypotheses

$$t_{k+1} - t_k \leq \alpha^k \eta \quad \text{and} \quad t_{k+1} \leq \frac{1-\alpha^{k+1}}{1-\alpha}\eta \leq t^{\star\star}.$$

Estimate (2.14) can be written as

$$\int_0^1 w(\theta(t_{n+1} - t_n))\, d\theta + \alpha w_0 \left(\frac{1-\alpha^{k+1}}{1-\alpha}\eta\right) - \alpha \leq 0,$$

or

$$\int_0^1 w(\alpha^k \theta \eta)\, d\theta + \alpha w_0 \left(\frac{1-\alpha^{k+1}}{1-\alpha}\eta\right) - \alpha \leq 0. \tag{2.15}$$

Estimate (2.15) motivates us to define recurrent functions f_k given by (2.2) and instead of the estimate (2.15) prove that

$$f_k(\alpha) \leq 0. \tag{2.16}$$

We need a relationship between two consecutive functions f_k. We obtain from (2.2) and (2.3)

$$f_{k+1}(\alpha) = f_k(\alpha) + g_k(\alpha). \tag{2.17}$$

Moreover by the hypotheses (2.7)

$$f_k(\alpha) \leq f_{k+1}(\alpha). \tag{2.18}$$

Furthermore, let us define function f_∞ on $[0, 1)$ as follows

$$f_\infty(\alpha) = \lim_{k\to\infty} f_k(\alpha). \tag{2.19}$$

Then, from (2.3) we get

$$f_\infty(\alpha) = \alpha \left[w_0 \left(\frac{\eta}{1-\alpha}\right) - 1 \right]. \tag{2.20}$$

In view of (2.16)-(2.20), instead of (2.16), we can show

$$f_\infty(\alpha) \leq 0, \tag{2.21}$$

which is true by (2.6) and (2.20). The induction for (2.14) is completed. It follows that the sequence $\{t_n\}$ is non-decreasing and bounded from above by $t^{\star\star}$ and as such it converges to $t^\star$. Estimate (2.13) follows from (2.12) (which is implied by (2.14) and (2.1)). That completes the proof of the **Part I**.

Part II. In this case by using (2.9), (2.16) and (2.17) we can show, instead of (2.16), $f_1(\alpha) \leq 0$ and which is true by (2.8). The rest of the proof follows as in **Part I**. That completes the proof of the Lemma. $\square$

The corresponding results of majorizing sequences for (**NM**) involving functions w_0, v and constants η, R are given in a similar way by:

Lemma 2.2. *Let constants $\eta \geq 0$, $R > 0$ and non-decreasing functions $w_0\colon [0,R) \to [0,\infty)$, $v\colon [0,R)\times[0,R) \to [0,\infty)$ with $\lim_{t\to 0} w_0(t) = 0$, $\lim_{t\to 0} v(r,t) = 0$ $(0 \leq r < R)$ be given. Define scalar sequence $\{r_n\}$*

$$r_0 = 0, \quad r_1 = \eta,$$
$$r_{n+2} = r_{n+1} + \frac{\int_0^1 v(r_{n+1}, \theta(r_{n+1} - r_n))(r_{n+1} - r_n)\, d\theta}{1 - w_0(r_{n+1})}, \tag{2.22}$$

sequences of functions $\{f_n^1\}$, $\{g_n^1\}$ on $(0,1)$ by

$$f_n^1(t) = \int_0^1 v\left((1 + t + \cdots + t^n)\eta, t^n\theta\eta\right) d\theta + t w_0\left[(1 + t + \cdots + t^n)\eta\right] - t, \tag{2.23}$$

$$g_n^1(t) = f_{n+1}^1(t) - f_n^1(t) \tag{2.24}$$

and function f_∞^1 on $(0,1)$ by

$$f_\infty^1(t) = t\left[w_0\left(\frac{\eta}{1-t}\right) - 1\right]. \tag{2.25}$$

Assume that either of the set of conditions hold:

H1. *there exists $\phi \in (0,1)$ such that*

$$0 \leq \frac{\int_0^1 v(\eta, \theta\eta)\, d\theta}{1 - w_0(\eta)} \leq \phi, \tag{2.26}$$

$$w_0\left(\frac{\eta}{1-\phi}\right) \leq 1 \tag{2.27}$$

and

$$g_n^1(\phi) \geq 0 \quad \textit{for all } n. \tag{2.28}$$

H2. *there exists $\phi \in [0,1)$ such that*

$$f_1^1(\alpha) \leq 0, \tag{2.29}$$

$$0 \leq \frac{\int_0^1 v(\eta, \theta\eta)\, d\theta}{1 - w_0(\eta)} \leq \phi,$$

and

$$g_n^1(\phi) \geq 0 \quad \textit{for all } n. \tag{2.30}$$

Then, the sequence $\{r_n\}$ *is well defined, non-decreasing, bounded from above by*

$$r^{\star\star} = \frac{\eta}{1-\phi} \tag{2.31}$$

and converges to its unique least upper bound $r^\star$ *satisfying*

$$0 \leq r^\star \leq r^{\star\star}. \tag{2.32}$$

Moreover, the following estimates hold

$$0 \leq r_{n+1} - r_n \leq \phi^n \eta \tag{2.33}$$

and

$$0 \leq r^\star - r_n \leq \frac{\phi^n \eta}{1-\phi}. \tag{2.34}$$

Remark 2.3. Let us define scalar sequences $\{\bar{t}_n\}$, $\{\bar{r}_n\}$ by

$$\begin{aligned} &\bar{t}_0 = 0, \quad \bar{t}_1 = \eta, \quad \bar{t}_2 = \bar{t}_1 + \frac{\int_0^1 w_0(\theta(\bar{t}_1 - \bar{t}_0))(\bar{t}_1 - \bar{t}_0)\, d\theta}{1 - w_0(\bar{t}_1)}, \\ &\bar{t}_{n+2} = \bar{t}_{n+1} + \frac{\int_0^1 w(\theta(\bar{t}_{n+1} - \bar{t}_n))(\bar{t}_{n+1} - \bar{t}_n)\, d\theta}{1 - w_0(\bar{t}_{n+1})}, \quad (n \geq 1) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} &\overline{r}_0 = 0, \quad \overline{r}_1 = \eta, \quad \overline{r}_2 = \overline{r}_1 + \frac{\int_0^1 w_0(\theta(\overline{r}_1 - \overline{r}_0))(\overline{r}_1 - \overline{r}_0)\, d\theta}{1 - w_0(\overline{r}_1)}, \\ &\overline{r}_{n+2} = \overline{r}_{n+1} + \frac{\int_0^1 v(\overline{r}_{n+1}, \theta(\overline{r}_{n+1} - \overline{r}_n))(\overline{r}_{n+1} - \overline{r}_n)\, d\theta}{1 - w_0(\overline{r}_{n+1})}, \quad (n \geq 1) \end{aligned} \tag{2.36}$$

A simple induction argument shows that under the hypotheses of Lemma 2.1 and 2.2 the sequences $\{\bar{t}_n\}$, $\{\overline{r}_n\}$ are finer than $\{t_n\}$, $\{r_n\}$. That is for $n > 1$

$$\bar{t}_n < t_n, \tag{2.37}$$

$$\bar{t}_{n+1} - \bar{t}_n < t_{n+1} - t_n, \tag{2.38}$$

$$\bar{t}^\star \leq t^\star, \tag{2.39}$$

$$\overline{r}_n < r_n, \tag{2.40}$$

$$\overline{r}_{n+1} - \overline{r}_n < r_{n+1} - r_n, \tag{2.41}$$

and

$$\overline{r}^\star \leq r^\star. \tag{2.42}$$

Later we shall show that $\{t_n\}$, $\{r_n\}$, $\{\bar{t}_n\}$, $\{\overline{r}_n\}$ are majorizing sequences for $\{x_n\}$. Before doing that let us show that these sequences are finer than the known results in the published literature [1, 3, 18, 24, 25, 27, 28, 31, 38, 40, 41, 43, 46, 48, 53, 54, 55].

Proof. Let us define functions $\overline{w}$, $\psi_0, \psi, \overline{\psi}$: $[0, R) \to [0, \infty)$ by

$$\overline{w} = \sup\{w(u) + w(v) \colon u + v = r\}, \tag{2.43}$$

$$\psi_0(r) = \eta + \int_0^r w_0(t)\, dt - r, \tag{2.44}$$

$$\psi(r) = \eta + \int_0^r w(t)\, dt - r, \tag{2.45}$$

$$\overline{\psi}(r) = \eta + \int_0^r \overline{w}(t)\, dt - r, \tag{2.46}$$

and sequences $\{s_n\}$, $\{\overline{s}_n\}$ by

$$s_{n+1} = s_n - \frac{\overline{\psi}(s_n)}{\psi'(s_n)}, \tag{2.47}$$

$$\overline{s}_{n+1} = \overline{s}_n - \frac{\overline{\psi}(\overline{s}_n)}{\psi_0'(\overline{s}_n)}. \tag{2.48}$$

If equation

$$\overline{\psi}(r) = 0 \tag{2.49}$$

has a unique solution $s^\star$ in $[0, R]$, then the sequence $\{x_n\}$ generated by the Newton's method for (1.1) is well defined and converges to a solution $x^\star \in U(x_0, s^\star)$ of equation $\mathcal{F}(x) = 0$ such that

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{2.50}$$

and

$$\|x_{n+1} - x^\star\| \leq s^\star - s_n. \tag{2.51}$$

Here, $s^\star = \lim_{n\to\infty} s_n$. We have shown, under the same assumptions, that

$$\|x_{n+1} - x_n\| \leq \overline{s}_{n+1} - \overline{s}_n \leq s_{n+1} - s_n \tag{2.52}$$

and

$$\|x_{n+1} - x^\star\| \leq \overline{s}^\star - \overline{s}_n \tag{2.53}$$

(see [3]). Here, $\overline{s}^\star = \lim_{n\to\infty} \overline{s}_n$. A simple inductive argument shows that

$$t_n < \overline{s}_n, \tag{2.54}$$

$$t_{n+1} - t_n < \overline{s}_{n+1} - \overline{s}_n, \tag{2.55}$$

and

$$t^\star \leq \overline{s}^\star. \tag{2.56}$$

Hence $\{t_n\}$ is a finer sequence than $\{\overline{s}_n\}$. Later we shall show that the sufficient convergence conditions of $\{t_n\}$ can be weaker than those of $\{\overline{s}_n\}$. Similar favorable comparisons follow for the case of sequence $\{r_n\}$ and the corresponding ones in [1, 3] (quasi majorant case). □

3. Semilocal convergence analysis of (**NM**)

This section develops semilocal convergence results for (**NM**) using functions w_0 and w.

Theorem 3.1. *Let $x_0 \in \mathbf{X}$ and $R > 0$ be such that $\mathcal{F} \colon U(x_0, R) \to \mathbf{Y}$ is Fréchet-differentiable. Assume conditions* (1.3)-(1.5)*, hypotheses of Lemma* 2.1*,*

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta \tag{3.1}$$

and

$$t^\star < R \tag{3.2}$$

*hold. Then, sequence $\{x_n\}$ generated by (**NM**) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, x^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{3.3}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{3.4}$$

Furthermore, if there exists $R_0 \in [t^\star, R)$ such that

$$\int_0^1 w_0\left[(1-\theta)t^\star + \theta\, R_0\right] d\theta \leq 1 \tag{3.5}$$

then, the solution $x^\star$ is unique in $U(x_0, R_0)$.

Proof. We shall show using induction

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{3.6}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k). \tag{3.7}$$

For every $z \in \overline{U}(x_1, t^\star - t_1)$,

$$\|z - x_0\| \le \|z - x_0\| + \|x_1 - x_0\| \le t^\star - t_1 + t_1 - t_0 = t^\star - t_0$$

implies that $z \in \overline{U}(x_0, t^\star - t_0)$. We also have from (2.1) and (3.1)

$$\|x_1 - x_0\| = \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \le \eta = t_1 - t_0.$$

That is estimates (3.6) and (3.7) hold for $k = 0$. Given they hold for $n \le k$, then

$$\|x_{k+1} - x_0\| \le \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1} \le t^{\star\star}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \le t_k + \theta(t_{k+1} - t_k) \le t^{\star\star}$$

for all $\theta \in [0, 1]$. Using (1.8), (2.19) and the induction hypotheses, we get

$$\left\|F'(x_0)^{-1}\left(F'(x_{k+1}) - F'(x_0)\right)\right\| \le w_0\left(\|x_{k+1} - x_0\|\right) \le w_0(t_{k+1}) < 1. \tag{3.8}$$

It follows from (3.8) and the Banach Lemma on invertible operators [28] that $F'(x_{k+1})^{-1} \in L(\mathbf{Y}, \mathbf{X})$ and

$$\left\|F'(x_{k+1})^{-1}F'(x_0)\right\| \le \frac{1}{1 - w_0(t_{k+1})}. \tag{3.9}$$

In view of (1.2), we have the approximation

$$\begin{aligned} \mathcal{F}(x_{k+1}) &= \mathcal{F}(x_{k+1}) - \mathcal{F}(x_k) - F'(x_k)(x_{k+1} - x_k) \\ &= \int_0^1 \left[F'(x_k + \theta(x_{k+1} - x_k)) - \mathcal{F}'(x_k)\right](x_{k+1} - x_k)\, d\theta\,. \end{aligned} \tag{3.10}$$

Then, by (1.5), (1.8), (2.1), (3.10) and the induction hypotheses, we obtain

$$\begin{aligned} \left\|F'(x_0)^{-1}F'(x_1)\right\| &\le \int_0^1 \left\|F'(x_0)^{-1}\left[F'(x_0 + \theta(x_1 - x_0)) - F'(x_0)\right](x_1 - x_0)\, d\theta\right\| \\ &\le \int_0^1 w_0\left(\|\theta(x_1 - x_0)\|\right)\|x_1 - x_0\|\, d\theta \\ &\le \int_0^1 w_0\left(\theta t_1\right) t_1\, d\theta \\ &\le \int_0^1 w\left(\theta t_1\right) t_1\, d\theta \end{aligned} \tag{3.11}$$

and for $k \geq 1$

$$\begin{aligned}
&\left\|F'(x_0)^{-1}F(x_{k+1})\right\| \\
&\leq \int_0^1 \left\|F'(x_0)^{-1}\left[F'(x_k+\theta(x_{k+1}-x_k))-F'(x_k)\right](x_{k+1}-x_k)\,d\theta\right\| \\
&\leq \int_0^1 w\left(\|\theta(x_{k+1}-x_k)\|\right)\|x_{k+1}-x_k\|\,d\theta \\
&\leq \int_0^1 w\left(\|\theta(t_{k+1}-t_k)\|\right)(t_{k+1}-t_k)\,d\theta\,. && (3.12)
\end{aligned}$$

Moreover, by (1.2), (2.1), (3.9), (3.12) we get

$$\begin{aligned}
\|x_2-x_1\| &\leq \left\|F'(x_1)^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}F(x_1)\right\| \\
&\leq \frac{1}{1-w_0(t_1)}\int_0^1 w_0(\theta\, t_1)t_1\,d\theta \\
&\leq \frac{1}{1-w_0(t_1)}\int_0^1 w(\theta\, t_1)t_1\,d\theta = t_2-t_1 && (3.13)
\end{aligned}$$

and for $k \geq 1$

$$\begin{aligned}
\|x_{k+2}-x_{k+1}\| &\leq \left\|F'(x_{k+1})^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}F(x_{k+1})\right\| \\
&\leq \frac{1}{1-w_0(t_{k+1})}\int_0^1 w_0(\theta(t_{k+1}-t_k))(t_{k+1}-t_k)\,d\theta \\
&= t_{k+2}-t_{k+1} && (3.14)
\end{aligned}$$

which completes the induction for (3.6). Furthermore, for every $w \in \overline{U}(x_{k+2}, t^\star - t_{k+2})$, we obtain

$$\begin{aligned}
\|w-x_{k+1}\| &\leq \|w-x_{k+2}\| + \|x_{k+2}-x_{k+1}\| \\
&\leq t^\star - t_{k+2} + t_{k+2} - t_{k+1} = t^\star - t_{k+1},
\end{aligned}$$

therefore $w \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$, which completes the induction for (3.7). Lemma (2.1) implies that sequence $\{t_n\}$ is Cauchy. It follows from (3.6) and (3.7) that $\{x_n\}$ is also Cauchy sequence in a Banach space and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \to \infty$ in (3.12) and using (1.4), we obtain $\mathcal{F}(x^\star) = 0$. Estimate (3.4) follows from (3.3) by using standard majorization techniques [4, 9, 31]. Finally to show the uniqueness let $y^\star \in U(x_0, R_0)$ such that $\mathcal{F}(y^\star) = 0$. Define operator $\mathcal{M}$ as follows

$$\mathcal{M} = \int_0^1 \mathcal{F}'(x^\star + \theta(y^\star - x^\star))\,d\theta\,.$$

Then, using (1.8) we obtain in turn

$$\begin{aligned}
&\left\| \mathcal{F}'(x_0)^{-1} \int_0^1 \left[\mathcal{F}'(x^\star + \theta(y^\star - x^\star)) - F'(x_0) \right] \right\| \\
&\leq \int_0^1 w_0 \left(\|x^\star + \theta(y^\star - x^\star) - x_0\| \right) d\theta \\
&\leq \int_0^1 w_0 \left((1-\theta)\|x^\star - x_0\| + \theta\|y^\star - x_0\| \right) d\theta \\
&< \int_0^1 w_0 \left((1-\theta)t^\star + \theta R_0 \right) d\theta \leq 1.
\end{aligned}$$

It follows that $\mathcal{M}^{-1} \in L(\mathbf{Y}, \mathbf{X})$. Using the identity

$$\mathcal{F}(y^\star) - \mathcal{F}(x^\star) = \mathcal{M}(y^\star - x^\star)$$

we deduce $x^\star = y^\star$. That completes the proof of the Theorem. □

Remark 3.2.

a. It follows from the proof of the Theorem 3.1 that $\{\bar{t}_n\}$ given by (2.35) is also a linear majorizing sequence for $\{x_n\}$. In particular, we have

$$\|x_{n+1} - x_n\| \leq \bar{t}_{n+1} - \bar{t}_n \tag{3.15}$$

and

$$\|x_{n+1} - x^\star\| \leq t^\star - \bar{t}_n. \tag{3.16}$$

See also (2.37)-(2.39).

b. If w, $\{t_n\}$ (or $\{\bar{t}_n\}$), (1.4), (1.5) are replaced by v, $\{r_n\}$ (or $\{\bar{r}_n\}$), (1.7), (1.6), respectively, then following verbatim the proof of Theorem 2.2, we arrive at:

Theorem 3.3. *Let $x_0 \in \mathbf{X}$ and $R > 0$ be such that $\mathcal{F}\colon U(x_0, R) \to \mathbf{Y}$ is Fréchet-differentiable. Assume conditions* (1.3)*,* (1.6)*,* (1.7)*,* (1.8)*, hypotheses of Lemma 2.2,* (3.1) *and*

$$r^\star < R \tag{3.17}$$

*hold. Then, sequence $\{x_n\}$ generated by (**NM**) is well-defined, remains in $\overline{U}(x_0, r^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, r^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates holds*

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n \tag{3.18}$$

and

$$\|x_n - x^\star\| \leq r^\star - r_n. \tag{3.19}$$

Furthermore, if there exists $R_1 \in [r^\star, R)$ *such that*

$$\int_0^1 w_0\,[(1-\theta)R_1 + \theta\, r^\star]\, d\theta \le 1 \tag{3.20}$$

then, the solution is unique in $U(x_0, R_1)$.

4. Applications

4.1. **Application - I.** We shall examine the interesting Lipschitz case for Fréchet-differentiable operator. Other cases and choices for functions w_0, w, v can be found in [1, 3, 4, 9, 41, 44]. We set

$$w_0(t) = L_0\, t, \quad w(t) = L\, t \quad \text{and} \quad v(r,t) = L\, t.$$

According to (2.1), (2.35), (2.47) and (2.48) we have corresponding sequences

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2}{2(1-L_0\, t_{n+1})}, \tag{4.1}$$

$$\bar{t}_0 = 0,\ \ \bar{t}_1 = \eta,\ \ \bar{t}_2 = \bar{t}_1 + \frac{L_0(\bar{t}_1-\bar{t}_0)^2}{2(1-L_0\bar{t}_1)},\ \ \bar{t}_{n+2} = \bar{t}_{n+1} + \frac{L(\bar{t}_{n+1}-\bar{t}_n)^2}{2(1-L_0\,\bar{t}_{n+1})}, \tag{4.2}$$

$$s_0 = 0, \quad s_{n+1} = s_n - \frac{1/2L\, s_n^2 - s_n + \eta}{L\, s_n - 1}. \tag{4.3}$$

The related functions are

$$f_\infty(t) = t\left[\frac{L_0\eta}{1-t} - 1\right], \tag{4.4}$$

$$g_n(t) = \frac{1}{2}p(t)t^n\eta, \qquad p(t) = 2L_0t^2 + Lt - L \tag{4.5}$$

and

$$\psi(t) = \frac{1}{2}L\, t^2 - t + \eta. \tag{4.6}$$

The sufficient convergence conditions for the sequences $\{t_n\}$, $\{\bar{t}_n\}$, $\{s_n\}$, respectively(see, [4, 12, 28]) are

$$h_1 = L_1\eta \le 1, \quad L_1 = \frac{1}{4}\left(L + 4L_0 + \sqrt{L^2 + 8LL_0}\right), \tag{4.7}$$

$$h_2 = L_2\eta \le 1, \quad L_2 = \frac{1}{4}\left(L_0 + \sqrt{L\, L_0 + 8L_0^2} + \sqrt{L_0L}\right), \tag{4.8}$$

and

$$h_k = L\,\eta \le \frac{1}{2}. \tag{4.9}$$

Condition (4.9) is the well-known Newton-Kantorovich hypotheses for solving equation (1.1) which is famous for its simplicity and clarity [28]. Note that

$$L_0 \leq L \tag{4.10}$$

holds in general and L/L_0 can be arbitrarily large (see Example 4.3). In case $L_0 = L$ conditions (4.7) and (4.8) reduce to (4.9). Otherwise (that is if $L_0 < L$) we have

$$h_k \leq 1 \Longrightarrow h_1 \leq 1 \Longrightarrow h_2 \leq 1 \tag{4.11}$$

but not necessarily vice versa. We also have as $L_0/L \to 0$

$$\frac{h_1}{h_k} \to \frac{1}{4}, \quad \frac{h_2}{h_k} \to 0 \quad \text{and} \quad \frac{h_2}{h_1} \to 0. \tag{4.12}$$

We shall complete this application by considering a numerical example.

Example 4.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^2$ be equipped with the max-norm, $x_0 = (1,1)^T$, $\mathcal{D} = \overline{U}(x_0, 1-q)$. Let us define $\mathcal{F}$ on $\mathcal{D}$ by

$$\mathcal{F}(x) = \left(\xi_1^3 - q, \xi_2^3 - q\right)^T, \quad x = (\xi_1, \xi_2)^T.$$

Through algebraic manipulations, we obtain

$$\eta = \frac{1-q}{2}, \quad L_0 = 3 - q \quad \text{and} \quad L = 2(2-q) > L_0.$$

From (4.7), (4.8) and (4.10) we get

$$\begin{aligned} L_1 &= 4 - \frac{3}{2}\,q + \frac{1}{2}\,\sqrt{(q-2)\,(5\,q-14)}, \\ L_2 &= 3 - q + \frac{\sqrt{2}}{4}\,\left[\sqrt{(5\,q-14)\,(q-3)} + \sqrt{(q-2)\,(q-3)}\right], \end{aligned}$$

$$\begin{aligned} h_1 &= \frac{1}{4}\,\left(8 - 3\,q + \sqrt{(q-2)\,(5\,q-14)}\right)(1-q)\,, \\ h_2 &= \frac{1}{8}\,\left(12 - 4\,q + \sqrt{2}\sqrt{(5\,q-14)\,(q-3)} + \sqrt{2}\sqrt{(q-2)\,(q-3)}\right)(1-q) \end{aligned}$$

and

$$h_k = (q-2)\,(q-1)\,.$$

Furthermore we obtain convergence interval

$$\begin{aligned} h_1 \leq 1 &\Longrightarrow q \in [0.6041, 2] \cup [2.8, 4.745]\,, \\ h_2 \leq 1 &\Longrightarrow q \in [0.5906, 2] \cup [3, -\infty)\,, \\ h_k \leq \frac{1}{2} &\Longrightarrow q \in \left[\frac{3}{2} - \frac{\sqrt{3}}{2}, \frac{3}{2} + \frac{\sqrt{3}}{2}\right]. \end{aligned}$$

Let us choose $q = 0.8$.

n	$t_{n+1}-t_n$	$\bar t_{n+1}-\bar t_n$	$s_{n+1}-s_n$
0	$1,000,000\cdot 10^{-01}$	$1,000,000\cdot 10^{-01}$	$1,000,000\cdot 10^{-01}$
1	$1,538,462\cdot 10^{-02}$	$1,410,256\cdot 10^{-02}$	$1,578,947\cdot 10^{-02}$
2	$3,806,503\cdot 10^{-04}$	$3,186,475\cdot 10^{-04}$	$4,143,011\cdot 10^{-04}$
3	$2,332,882\cdot 10^{-07}$	$1,628,328\cdot 10^{-07}$	$2,856,349\cdot 10^{-07}$
4	$8,762,471\cdot 10^{-14}$	$4,252,115\cdot 10^{-14}$	$1,357,695\cdot 10^{-13}$
5	$1,236,215\cdot 10^{-26}$	$2,899,554\cdot 10^{-27}$	$3,067,494\cdot 10^{-26}$
6	$2,460,531\cdot 10^{-52}$	$1,348,291\cdot 10^{-53}$	$1,565,839\cdot 10^{-51}$
7	$9,747,621\cdot 10^{-104}$	$2,915,333\cdot 10^{-106}$	$4,080,126\cdot 10^{-102}$
8	$1,529,812\cdot 10^{-206}$	$1,363,005\cdot 10^{-211}$	$2,770,300\cdot 10^{-203}$
9	$3,768,055\cdot 10^{-412}$	$2,979,313\cdot 10^{-422}$	$1,277,124\cdot 10^{-405}$

TABLE 1. Sequences $\{t_n\}$ and $\{\bar t_n\}$ are finer than $\{s_n\}$.

4.2. **Application - II.** Let us consider twice-Fréchet differentiable operators. Let γ_0, γ be the functions w_0, w, respectively. Moreover assume for all $\theta \in [0,1]$

$$\left\|\mathcal{F}'(x_0)^{-1}F''(x_0+\theta(x-x_0))\right\| \le \gamma_0\left(\theta\|x-x_0\|\right)$$

and

$$\left\|\mathcal{F}'(x_0)^{-1}F''(y+\theta(x-y))\right\| \le \gamma\left(\theta\|y-x_0\|+\theta\|x-y\|\right)$$

for all $x, y \in U(x_0, r)$, $0<r<R$. Then we can set

$$w_0(t)=\int_0^1 \gamma_0(\theta\, t)t^2\, d\theta \quad \text{and} \quad w(r,t)=\int_0^1 \gamma(r+\theta\, t)t^2(1-\theta)\, d\theta\,.$$

Similar choices for m-Fréchet-differentiable ($m>2$) operators can be found (see [books]).

Example 4.2. Let $\mathbf{X}=\mathbf{Y}=\mathcal{C}[0,1]$, equipped with the norm $\| \ x \ \|= \max_{0\le s\le 1}|x(s)|$. Consider the following nonlinear boundary value problem [9]

$$\begin{cases} u''=-u^3-\gamma\ u^2, \\ u(0)=0, \quad u(1)=1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s)=s+\int_0^1 Q(s,t)\ (u^3(t)+\gamma\ u^2(t))\ dt \tag{4.13}$$

where Q is the Green function:

$$Q(s,t)=\begin{cases} t\ (1-s), & t\le s \\ s\ (1-t), & s<t. \end{cases}$$

We observe that

$$\max_{0\leq s\leq 1} \int_0^1 |Q(s,t)|\, dt = \frac{1}{8}.$$

Then problem (4.13) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathbf{Y}$ is defined as

$$[\mathcal{F}(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\ (x^3(t) + \gamma\ x^2(t))\ dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[\mathcal{F}'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \|= 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. Then, we have

$$\eta = \frac{1+\gamma}{5-2\,\gamma}, \quad L = \frac{\gamma + 6\,R + 3}{4} \quad \text{and} \quad L_0 = \frac{2\,\gamma + 3\,R + 6}{8}.$$

Note that $L = L_0 + 9/8R$ thus $L > L_0$.

Example 4.3. Define the scalar function $\mathcal{F}$ by $\mathcal{F}(x) = c_0\,x + c_1 + c_2\,\sin e^{c_3\,x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, L/L_0 can be arbitrarily large. That is (2.28) may be satisfied but not (2.35).

Example 4.4. Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$, equipped with the max–norm $\| \,.\, \|$. Let $\theta \in [0,1]$ be a given parameter. Consider the "Cubic" Chandrasekhar integral equation [4, 9, 15, 28]

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, \mathrm{d}t + y(s) - \theta. \tag{4.14}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$. The parameter λ in (4.14) is a real number called the "albedo" for scattering, and $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$\mathcal{F}(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, \mathrm{d}t + y(s) - \theta, \tag{4.15}$$

for all $s \in [0,1]$, then every zero of $\mathcal{F}$ satisfies equation (4.14). We have the estimates

$$\max_{0\leq s\leq 1} |\int_0^1 \frac{s}{s+t}\, dt| = \ln 2.$$

Therefore, if we set $\xi = \| \mathcal{F}'(u_0)^{-1} \|$, then

$$\eta = \xi \, (|\lambda| \, \ln 2 + 1 - \theta),$$

$$L = 2 \, \xi \, (|\lambda| \, \ln 2 + 3 \, (2 - \theta)) \quad \text{and} \quad L_0 = \xi \, (2 \, |\lambda| \, \ln 2 + 3 \, (3 - \theta)).$$

It follows that if conditions (4.7) and (4.8) holds, then problem (4.14) has a unique solution near u_0. These assumptions are weaker than the one given before using the Newton–Kantorovich hypothesis (4.9). Since $L = L_0 + 3\xi(1 - \theta)$ thus $L_0 \leq L$ for all $\theta \in [0, 1]$.

References

[1] J. Appel, E. De Pascale, J. V. Lysenko, and P. P. Zabrejko, *New results on Newton-Kantorovich approximations with applications to nonlinear integral equations*, Numer. Funct. Anal. and Optim., 18(1 & 2) (1997), 1–17.

[2] S. Amat, S. Busquier, and M. Negra, *Adaptive approximation of nonlinear operators*, Numer. Funct. Anal. Optim. 25. 397–405.

[3] I. K. Argyros, *On the convergence of the Newton-Kantorovich method: The generalized Hölder case*, Nonlinear Studies, 14 (2007), 355–364.

[4] I. K. Argyros, Advances on Iterative Procedures (Mathematics Research Developments), Nova Science Publ., New York, 2011.

[5] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. 298 (2004), 374–397.

[6] I. K. Argyros, *On the Newton-Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. 169 (2004), 315–332.

[7] I. K. Argyros, *Concerning the "terra incognita" between convergence regions of two Newton methods*, Nonlinear Analysis 62 (2005), 179–194.

[8] I. K. Argyros, *Approximating solutions of equations using Newton's method with a modified Newton's method iterate as a starting point*, Rev. Anal. Numér. Théor. Approx. 36 (2007), 123–138.

[9] I. K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[10] I. K. Argyros, *On a class of Newton-like methods for solving nonlinear equations*, J. Comput. Appl. Math. 228 (2009), 115–122.

[11] I. K. Argyros, *A semilocal convergence analysis for directional Newton methods*, Math. Comput. 80 (2011), 327–343.

[12] I. K. Argyros and S. Hilout, Efficient methods for solving equations and variational inequalities, Polimetrica Publisher, Milano, Italy, 2009.

[13] I. K. Argyros and S. Hilout, *Enclosing roots of polynomial equations and their applications to iterative processes*, Surv. Math. Appl. v4. 119-132.

[14] I. K. Argyros and S. Hilout, *Extending the Newton-Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. 234 (2010), 2993–3006.

[15] I. K. Argyros, Y. J. Cho and S. Hilout, Numerical Methods for Equations and its Applications, CRC Press/Taylor & Francis, New York, 2012.

[16] W. Bi, Q. Wu and H. Ren, *Convergence ball and error analysis of Ostrowski-Traub's method*, Appl. Math. J. Chinese Univ. Ser. B 25 (2010), 374–378.

[17] E. Cătinaş, *The inexact, inexact perturbed, and quasi-Newton methods are equivalent models*, Math. Comp. 74 (2005), 291–301.

[18] X. Chen and T. Yamamoto, *Convergence domains of certain iterative methods for solving nonlinear equations*, Numer. Funct. Anal. Optim. 10 (1989), 37–48.

[19] P. Deuflhard, Newton methods for nonlinear problems. Affine invariance and adaptive algorithms, Springer Series in Computational Mathematics, 35, Springer-Verlag, Berlin, 2004.

[20] J. A. Ezquerro, J. M. Gutiérrez, M. A. Hernández, N. Romero and M. J. Rubio, *The Newton method: from Newton to Kantorovich*,(Spanish), Gac. R. Soc. Mat. Esp., 13 (2010), 53–76.

[21] J. A. Ezquerro and M. A. Hernández, *On the R-order of convergence of Newton's method under mild differentiability conditions*, J. Comput. Appl. Math. 197 (2006), 53–61.

[22] J. A. Ezquerro and M. A. Hernández, *An improvement of the region of accessibility of Chebyshev's method from Newton's method*, Math. Comp. 78 (2009), 1613–1627.

[23] J. A. Ezquerro, M. A. Hernández and N. Romero, *Newton-type methods of high order and domains of semilocal and global convergence*, Appl. Math. Comput. 214 (2009), 142–154.

[24] W. B. Gragg and R. A. Tapia, *Optimal error bounds for the Newton-Kantorovich theorem*, SIAM J. Numer. Anal. 11 (1974), 10–13.

[25] J. M. Gutiérrez, *A new semilocal convergence theorem for Newton's method*, J. Comp. Appl. Math. 79 (1997), 131–145.

[26] M. A. Hernández, *A modification of the classical Kantorovich conditions for Newton's method*, J. Comp. Appl. Math. 137 (2001), 201–205.

[27] Z. Huang, *A note on the Kantorovich theorem for Newton iteration*, J. Comp. Appl. Math. 47 (1993), 211 – 217.

[28] L. V. Kantorovich and G. P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.

[29] S. Krishnan and D. Manocha, *An efficient surface intersection algorithm based on lower-dimensional formulation*, ACM Trans. on Graphics 16 (1997), 74–106.

[30] G. Lukács, The generalized inverse matrix and the surface-surface intersection problem. Theory and practice of geometric modeling (Blaubeuren, 1988), 167–185, Springer, Berlin, 1989.

[31] L. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables. 1970. Academic press, New York.

[32] A. M. Ostrowski, Sur la convergence et l'estimation des erreurs dans quelques procédés de résolution des équations numériques. (French), Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], 213–234. publisher unknown, Moscow, 1940.

[33] A. M. Ostrowski, La méthode de Newton dans les espaces de Banach, C. R. Acad. Sci. Paris Sér. A-B 272 (1971) 1251–1253.

[34] A. M. Ostrowski, Solution of Equations in Euclidean and Banach Spaces, Academic press, New York, 1973.

[35] I. Păvăloiu, Introduction in the Theory of Approximation of Equations Solutions. Dacia Ed., Cluj-Napoca, 1976.

[36] F. A. Potra, *The rate of convergence of a modified Newton's process*, With a loose Russian summary., Apl. Mat. 26 (1981) 13–17.

[37] F. A. Potra, *An error analysis for the secant method*, Numer. Math. v38. 427-445.

[38] F. A. Potra, On the convergence of a class of Newton-like methods. Iterative solution of nonlinear systems of equations (Oberwolfach, 1982), 125–137, Lecture Notes in Math., 953, Springer, Berlin-New York, 1982.

[39] F. A. Potra, *Sharp error bounds for a class of Newton-like methods*, Libertas Mathematica 5 (1985), 71–84.

[40] F. A. Potra and V. Pták, *Sharp error bounds for Newton's process*, Numer. Math. 34 (1980), 63–72.

[41] F. A. Potra and V. Pták, Nondiscrete induction and iterative processes. Research Notes in Mathematics, 103. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[42] P. D. Proinov, *General local convergence theory for a class of iterative processes and its applications to Newton's method*, J. Complexity 25 (2009), 38–62.

[43] P. D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity 26 (2010), 3–42.

[44] L. B. Rall, Computational solution of nonlinear operator equations, Wiley, New–York, 1969.

[45] W. C. Rheinboldt, *A unified convergence theory for a class of iterative processes*, SIAM J. Numer. Anal. 5 (1968), 42–63.

[46] W. C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, Polish Academy of Science, Banach Ctr. Publ. 3 (1977) 129–142.

[47] S. Smale, Newtons method estimate from data at one point, in The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics (eds., Ewing, R. et al.), Springer-Verlag, New York, (1986).

[48] R. A. Tapia, *Classroom Notes: The Kantorovich Theorem for Newton's Method*, Amer. Math. Monthly 78 (1971), 389–392.

[49] J. F. Traub and H. Woźniakowski, *Convergence and complexity of Newton iteration for operator equations*, J. Assoc. Comput. Mach. 26 (1979), 250–258.

[50] D. Wang and F. Zhao, *The theory of Smales point estimation and its applications*, J. Comput. Appl. Math. 60 (1955), 253269.

[51] X. H. Wang and D. F. Han, *On dominating sequence method in the point estimate and Smale theorem*, Sci. Sinica Ser. A, 33 (1990), 135144.

[52] Q. Wu and H. Ren, *A note on some new iterative methods with third-order convergence*, Appl. Math. Comput. 188 (2007), 1790–1793.

[53] T. Yammoto, *A convergence theorem for Newton-like methods in Banach spaces*, Numer. Math. 51 (1987), 545–557.

[54] R. P. Zabrejko and D. F. Nguen, *The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates*, Numer. Funct. Anal. Optim. 9 (1987), 671–684.

[55] A. I. Zinčenko, *Some approximate methods of solving equations with non-differentiable operators*, (Ukrainian), Dopovidi Akad. Nauk Ukraïn. RSR (1963), 156–161.