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A CONSTANT RELATED TO FIXED POINTS AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. Let X be a Banach space and $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. The parameter

$$V_X^*(\varepsilon) = \sup\{1 - \frac{\|x+y\|}{2} : x, y \in S_X \text{ and } \langle x-y, f\rangle \leq \varepsilon \text{ for some } f \in \nabla_x\},$$

where $0 \leq \varepsilon \leq 2$ and $\nabla_x \subseteq S_{X^*}$ is the set of norm 1 supporting functionals f at x, is introduced and investigated. The main result is that if $V_X^*(\varepsilon) < \frac{\varepsilon}{2}$ for some $0 < \varepsilon < 2$, then X is uniformly nonsquare and has a uniform normal structure.

1. Introduction

Let X be a Banach space with the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. For $x \in S_X$, let $\nabla_x \subseteq S_{X^*}$ be the set of norm 1 supporting functionals of S_X at x.

In [4], Gao introduced the parameter, the modulus of U-convexity defined by:

$$U_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \langle x-y, f\rangle \geq \varepsilon \text{ for some } f \in \nabla_x\},$$

where $0 \leq \varepsilon \leq 2$, to measure a certain type of geometric property of the unit sphere S_X of X. It was also proved that if there exists $\delta > 0$ such that

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$U_X(\frac{1}{2}-\delta) > 0$, then X has a uniform normal structure (the definition is given in the next section).

In [8], García-Falset introduced the parameter

$$R(X) = \sup\{\liminf_{n\to\infty} \|x_n + x\|\}$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all x in S_X. It was proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property, that is, for every bounded closed convex subset C of X and for every nonexpansive mapping $T : C \to C$, it follows that T has a fixed point.

In [10], Mazcunan-Navarro proved a relationship between two of above mentioned parameters. Namely, if there exists $\delta > 0$ such that $U_X(1-\delta) > 0$, then $R(X) < 2$. So, every Banach space X with $U_X(1-\delta) > 0$ enjoys the fixed point property.

In [11], Saejung proved that if a Banach space X is superreflexive, then the moduli of U-convexity of the ultrapower $X_{\mathcal{U}}$ of X and X itself coincide. By using ultrapower method he showed that a Banach space X and its dual X^* has uniform normal structure whenever $U_X(1) > 0$. He also gave an example showing that such a condition is sharp.

In this paper, the new parameter $V_X^*(\varepsilon)$ is introduced and the properties of this parameter are investigated. More precisely, sufficient condition for uniform nonsquareness and uniform normal structure is given in terms of this parameter.

2. Preliminary

Definition 2.1 ([2]). *A nonempty bounded and convex subset K of a Banach space X is said to have a normal structure if for every convex subset H of K that contains more than one point there is a point $x_0 \in H$ such that*

$$\sup\{\|x_0 - y\| : y \in H\} < \operatorname{diam} H,$$

where $\operatorname{diam} H = \sup\{\|x-y\| : x, y \in H\}$ *denotes the diameter of H. A Banach space X is said to have normal structure if every bounded convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K of X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that*

$$\sup\{\|x_0 - y\| : y \in K\} < c \cdot \operatorname{diam} K.$$

For a reflexive Banach space, normal structure and weak normal structure coincide.

Lemma 2.2 ([7]). *Suppose that $x, y \in S_X$ and $0 < \varepsilon < 1$ such that $\frac{\|x+y\|}{2} > 1 - \varepsilon$. If $z = cx + (1-c)y \in \operatorname{co}(\{x, y\}) = [x, y]$ for some $0 \leq c \leq 1$, then $\|z\| > 1 - 2\varepsilon$. That is, every point in the line segment connecting x and y has norm bigger than $1 - 2\varepsilon$.*

The following lemma is a geometric property of a Banach space without weak normal structure.

Lemma 2.3 ([7]). *Let X be a Banach space without weak normal structure. Then for any $0 < \varepsilon < 1$, there exist x_1, y_2, y_3 in S(X) satisfying*

(i) *$y_2 - y_3 = ax_1$ with $|a - 1| < \varepsilon$;*
(ii) *$|\|x_1 - y_2\| - 1|$, $|\|y_3 - (-x_1)\| - 1| < \varepsilon$; and*
(iii) *$\|\frac{x_1+y_2}{2}\|$, $\|\frac{y_3+(-x_1)}{2}\| > 1 - \varepsilon$.*

Definition 2.4 ([5]). *Let X be a Banach space. A hexagon H in X is called a normal hexagon if the length of each side is 1 and each pair of two opposite sides are parallel.*

Remark 2.5. *The concept of normal hexagon is different from the concept of regular hexagon in Euclidean spaces. We may consider the normal hexagon as an image of a regular hexagon under a bounded linear mapping from an Euclidean space to a Banach space.*

Lemma 2.3 can be refined for having an inscribed normal hexagon in S_X,

Lemma 2.6 ([5]). *Let X be a Banach space without weak normal structure. Then for any $0 < \delta < 1$, there are x_1, x_2, x_3 in S_X satisfying*

(i) *$x_2 - x_3 = x_1$;*
(ii) *$\|\frac{x_1+x_2}{2}\| > 1 - \delta$; and*
(iii) *$\|\frac{x_3+(-x_1)}{2}\| > 1 - \delta$.*

The geometric meaning of the lemma is that if a Banach space X fails to have weak normal structure then there are infinitely many inscribed normal hexagons which four sides are arbitrarily closed to the unit sphere S_X.

Proof. For $\delta > 0$, let $x_1, y_2, y_3, -x_1, -y_2$, and $-y_3 \in S_X$ satisfying the three conditions in Lemma 2.3 for $\varepsilon = \frac{\delta}{3}$. And let, $x_2, x_3 \in S_X$ and $x_2 - x_3 = x_1$. Then, the hexagon H with vertices $x_1, y_2, y_3, -x_1, -y_2$, and $-y_3$ is a normal hexagon. Let

$$z = \frac{x_1 + x_2}{2} + \alpha x_1 \in [x_1, y_2].$$

From Lemma 2.2, $\|z\| \geq 1 - \frac{2\delta}{3}$. And, it is easy to see that

$$|\alpha| \leq |a - 1|.$$

Therefore,

$$\|\frac{x_1+x_2}{2}\| = \|z-\alpha x_1\| \geq \|z\| - |\alpha| \geq 1 - \frac{2\delta}{3} - \frac{\delta}{3} = 1-\delta.$$

Similarly, we have

$$\|\frac{x_3+(-x_1)}{2}\| \geq 1-\delta.$$

This normal hexagon H satisfies the three conditions of this lemma. □

3. Main results

Definition 3.1. *For a Banach space X, the function $V_X^* : [0,2] \to [0,1]$ defined by*

$$V_X^*(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \langle x-y, f\rangle \leq \varepsilon \text{ for some } f \in \nabla_x\right\},$$

is called the modulus of V^*-convexity of X.

Definition 3.2. [6]. *For a normed linear space X, $C_X(\varepsilon) = \sup\{1 - \|\frac{x+y}{2}\|$: for any $x, y \in S_X$ with $\|x-y\| \leq \varepsilon$, $0 \leq \varepsilon \leq 2\}$.*

Definition 3.3. [3]. *For a normed linear space X, $\delta_X(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\|$: for any $x, y \in S_X$ with $\|x-y\| \geq \varepsilon$, $0 \leq \varepsilon \leq 2\}$, is called the modulus of convexity of X.*

The proofs of the following three remarks are part of, what is commonly referred as "standard argument":

Remark 3.4. *In the above definitions of $\delta_X(\varepsilon)$, $C_X(\varepsilon)$, $V_X^*(\varepsilon)$ and $U_X(\varepsilon)$, the condition "$\geq \varepsilon$" in "$\|x-y\| \geq \varepsilon$" and "$\langle x-y, f\rangle \geq \varepsilon$" may be replaced by "$> \varepsilon$" or replaced by "$= \varepsilon$". Similarly, the condition "$\leq \varepsilon$" in "$\|x-y\| \leq \varepsilon$" and "$\langle x-y, f\rangle \leq \varepsilon$" may be replaced by "$< \varepsilon$" or replaced by "$= \varepsilon$" too.*

Remark 3.5. *Suppose that X is a Banach space and $0 < \varepsilon \leq 2$. Then*

$$\begin{aligned} V_X^*(\varepsilon) &= \sup\left\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X \text{ and } \langle x-y, f\rangle \leq \varepsilon \text{ for some } f \in \nabla_x\right\} \\ &= \sup\left\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X \text{ and } \langle x-y, f\rangle = \varepsilon \text{ for some } f \in \nabla_x\right\} \\ &= \sup\left\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X \text{ and } \langle x-y, f\rangle < \varepsilon \text{ for some } f \in \nabla_x\right\}. \end{aligned}$$

Remark 3.6. *Suppose that X is a Banach space.*

(i) *$V_X^*(\varepsilon)$ is an increasing function of ε for $0 \leq \varepsilon \leq 2$.*
(ii) *$\delta_X(\varepsilon) \leq C_X(\varepsilon) \leq V_X^*(\varepsilon) \leq \frac{\varepsilon}{2}$.*
(iii) *$\delta_X(\varepsilon) \leq U_X(\varepsilon) \leq V_X^*(\varepsilon) \leq \frac{\varepsilon}{2}$.*

It is easy to see that the moduli of V^*-convexity and U-convexity of a Hilbert space H satisfy the following:

$$V_H^*(\varepsilon) = U_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon}{2}} \quad \text{for all } 0 \leq \varepsilon \leq 2.$$

Definition 3.7 ([9]). *A Banach space X is said to be uniformly nonsquare if there exists a $\delta > 0$ such that either $\frac{1}{2}\|x+y\| \leq 1-\delta$ or $\frac{1}{2}\|x-y\| \leq 1-\delta$ for any $x, y \in S_X$.*

Theorem 3.8. *For a Banach space X if $V_X^*(\varepsilon) < \frac{\varepsilon}{2}$ for some $0 < \varepsilon < 2$, then X is uniformly nonsquare.*

Proof. If X is not uniformly nonsquare, choose $\delta > 0$ and $0 \leq c \leq 1$ such that

$$2\delta < \sqrt{\delta} \quad \text{and} \quad 1 - c - \sqrt{\delta} > 0.$$

Then, there exist $x, y \in S_X$ such that both

$$\|\frac{x+y}{2}\| \quad \text{and} \quad \|\frac{x-y}{2}\| > 1 - \delta.$$

Let

$$z_1 = cy + (1-c)x, \qquad z = \frac{z_1}{\|z_1\|} \in S_X,$$
$$t_1 = \sqrt{\delta}x + (1-\sqrt{\delta})(-y), \qquad t = \frac{t_1}{\|t_1\|} \in S_X$$

and f_t be a norm 1 supporting functional of S_X at t. Then

$$\|z_1\| \geq 1 - 2\delta \quad \text{and} \quad \|t_1\| \geq 1 - 2\delta.$$

We have

$$\begin{aligned} 1 - 2\delta \leq \|t_1\| &= \langle \sqrt{\delta}x + (1-\sqrt{\delta})(-y), f_t \rangle \\ &= \langle \sqrt{\delta}x, f_t \rangle + \langle (1-\sqrt{\delta})(-y), f_t \rangle \leq 1. \end{aligned}$$

So,

$$\begin{aligned} \sqrt{\delta}\langle x, f_t \rangle &\geq 1 - 2\delta - (1-\sqrt{\delta})\langle -y, f_t \rangle \\ &\geq 1 - 2\delta - (1-\sqrt{\delta}) = \sqrt{\delta} - 2\delta, \end{aligned}$$

therefore,

$$\langle x, f_t \rangle \geq 1 - 2\sqrt{\delta}.$$

For any norm 1 supporting functional f_t of S_X at t,

$$\begin{aligned}\langle t-z, f_t\rangle &= 1-\langle z, f_t\rangle \\ &= 1-\langle z_1, f_t\rangle - \langle z-z_1, f_t\rangle \\ &\le 1-\langle cy+(1-c)x, f_t\rangle + 2\delta \\ &= 1+c\langle -y, f_t\rangle - (1-c)\langle x, f_t\rangle + 2\delta \\ &\le 1+c-(1-c)(1-2\sqrt{\delta})+2\delta \\ &= 1+c-1+c+2\sqrt{\delta}-2c\sqrt{\delta}+2\delta \\ &= 2c+2\sqrt{\delta}-2c\sqrt{\delta}+2\delta.\end{aligned}$$

But

$$\begin{aligned}\|t+z\| &\le \|t_1+z_1\| + \|z-z_1\| + \|t_1-t\| \\ &\le \|(1-c+\sqrt{\delta})x+(1-c-\sqrt{\delta})(-y)\| + 4\delta \\ &= (2-2c)\|\frac{(1-c+\sqrt{\delta})x+(1-c-\sqrt{\delta})(-y)}{2-2c}\| + 4\delta \\ &\le 2-2c+4\delta.\end{aligned}$$

So,

$$1-\frac{\|t+z\|}{2} \ge 1-(1-c+2\delta) = c-2\delta.$$

We have

$$\begin{aligned}&V_X^*(2c+2\sqrt{\delta}-2c\sqrt{\delta}+2\delta) \\ &= \sup\{1-\frac{\|x+y\|}{2} : x, y \in S_X, \langle x-y, f_x\rangle \quad \le 2c+2\sqrt{\delta}-2c\sqrt{\delta}+2\delta, f_x \in \nabla_x\} \\ &\ge c-2\delta.\end{aligned}$$

Since δ can be arbitrarily closed to 0, we have

$$V_X^*(2c^+) = \lim_{\varepsilon\to 2c^+} V_X^*(\varepsilon) \ge c \quad \text{for} \quad 0 \le c < 1.$$

Since $V_X^*(\varepsilon) \le \frac{\varepsilon}{2}$ is a non-decreasing function, we have

$$\lim_{\varepsilon\to 2c^+} V_X^*(\varepsilon) = c.$$

If there is $\alpha \in [0,2]$ such that $V_X^*(\alpha) = a < \frac{\alpha}{2}$, then there exists $b \in (2a, \alpha)$ with $V_X^*(b) \le a$. So, $\lim_{\varepsilon\to b^+} V_X^*(\varepsilon) \le V_X^*(\alpha) = a < \frac{b}{2}$. Therefore

$$V_X^*(\varepsilon) = \frac{\varepsilon}{2} \quad \text{for any} \quad \varepsilon \in (0,2).$$

□

Theorem 3.9. *For a Banach space X if $V_X^*(\varepsilon) < \frac{\varepsilon}{2}$ for some $0 < \varepsilon < 2$, then X has a normal structure.*

Proof. X satisfies this condition implies X is uniform nonsquare and hence reflexive. In particular, weak normal structure and normal structure coincide. Suppose that X does not have a normal structure, for a fixed c, where $0 \leq c \leq 1$ and any $0 < \delta < 1$, let x_1, x_2, x_3 be in S_X satisfying the conditions in Lemma 2.6.

First of all, let

$$\begin{aligned} u_1 &= cx_3 + (1-c)(-x_1), u = \frac{u_1}{\|u_1\|} \in S_X, \\ w_1 &= \sqrt{\delta}x_2 + (1-\sqrt{\delta})x_1, w = \frac{w_1}{\|w_1\|} \in S_X, \end{aligned}$$

and f_w be a norm 1 supporting functional of S_X at w. Then

$$\|w_1\| \geq 1 - 2\delta \quad \text{and} \quad \|u_1\| \geq 1 - 2\delta.$$

We have

$$\begin{aligned} 1 - 2\delta \leq \|w_1\| &= \langle w_1, f_w \rangle = \langle \sqrt{\delta}x_2 + (1-\sqrt{\delta})x_1, f_w \rangle \\ &= \langle \sqrt{\delta}x_2, f_w \rangle + \langle (1-\sqrt{\delta})x_1, f_w \rangle \leq 1. \end{aligned}$$

So,

$$\begin{aligned} \sqrt{\delta}\langle x_2, f_w \rangle &\geq 1 - 2\delta - (1-\sqrt{\delta})\langle x_1, f_w \rangle \\ &\geq 1 - 2\delta - (1-\sqrt{\delta}) \\ &= \sqrt{\delta} - 2\delta, \end{aligned}$$

then

$$\langle x_2, f_w \rangle \geq 1 - 2\sqrt{\delta}.$$

For any norm 1 supporting functional f_w of S_X at w,

$$\begin{aligned} \langle w - u, f_w \rangle &= \langle w - w_1, f_w \rangle + \langle w_1 - u_1, f_w \rangle + \langle u_1 - u, f_w \rangle \\ &\leq 2\delta + \langle (1 - \sqrt{\delta} + c + 1 - c)x_1 + (\sqrt{\delta} - c)x_2, f_w \rangle + 2\delta \\ &= \langle (2-\sqrt{\delta})x_1, f_w \rangle - \langle (c - \sqrt{\delta})x_2, f_w \rangle + 4\delta \\ &\leq 2 - \sqrt{\delta} - (c - \sqrt{\delta})(1 - 2\sqrt{\delta}) + 2\delta \\ &= 2 - c + 2c\sqrt{\delta}. \end{aligned}$$

But

$$\begin{aligned}\|w+u\| &\leq \|w_1+u_1\| + \|w-w_1\| + \|u-u_1\| \\ &\leq \|c(x_2-x_1)+(1-c)(-x_1)+\sqrt{\delta}x_2+(1-\sqrt{\delta})x_1\| + 4\delta \\ &= \|(-c+c-1+1-\sqrt{\delta})x_1+(c+\sqrt{\delta})x_2\| + 4\delta \\ &= \|-\sqrt{\delta}x_1+(c+\sqrt{\delta})x_2\| + 4\delta \\ &\leq (c+\sqrt{\delta})\|x_2\| + \|\sqrt{\delta}x_1\| + 4\delta \\ &= c+6\sqrt{\delta}.\end{aligned}$$

So,

$$1-\frac{\|w+u\|}{2} \geq 1-\frac{c+6\sqrt{\delta}}{2} = \frac{2-c}{2} - 3\sqrt{\delta}.$$

We have

$$\begin{aligned}&V_X^*(2-c+2c\sqrt{\delta}) \\ &= \sup\{1-\frac{\|x+y\|}{2} : x,y\in S_X, \langle x-y, f_x\rangle \leq 2-c+2c\sqrt{\delta}, f_x\in\nabla_x\} \\ &\geq \frac{2-c}{2} - 3\sqrt{\delta}.\end{aligned}$$

Since δ can be arbitrarily close to 0, we have

$$V_X^*((2-c)^+) \geq \frac{2-c}{2}, \quad \text{for} \quad 0 < c \leq 1. \tag{3.1}$$

Secondly, let

$$\begin{aligned}&v_1 = t(-x_3)+(1-t)x_1, v = \frac{v_1}{\|v_1\|} \in S_X, \quad \text{where} \quad 0\leq t\leq 1, \\ &w_1 = \sqrt{\delta}x_1 + (1-\sqrt{\delta})x_2, w = \frac{w_1}{\|w_1\|} \in S_X,\end{aligned}$$

and f_w be a norm 1 supporting functional of S_X at w. Then

$$\|w_1\| \geq 1-2\delta, \quad \text{and} \quad \|v_1\| \geq 1-2\delta.$$

Similarly we have

$$\begin{aligned}\langle x_1, f_w\rangle &\geq 1-2\sqrt{\delta} \quad \text{this time, and} \\ \langle x_3, f_w\rangle = \langle x_2, f_w\rangle - \langle x_1, f_w\rangle &\leq 1-(1-2\sqrt{\delta}) = 2\sqrt{\delta}.\end{aligned}$$

Then,

$$\begin{aligned}\langle w-v, f_w\rangle &= 1-\langle v-v_1, f_w\rangle - \langle v_1, f_w\rangle \\ &\leq 1+2\delta - t\langle -x_3, f_w\rangle - (1-t)\langle x_1, f_w\rangle \\ &\leq 1+2t\sqrt{\delta} - (1-t)(1-2\sqrt{\delta}) + 2\delta \\ &= 1-1+t+2t\sqrt{\delta}+2(1-t)\sqrt{\delta}+2\delta \\ &= t+2\sqrt{\delta}+2\delta.\end{aligned}$$

But,

$$\begin{aligned}\|w+v\| &\leq \|w-w_1\| + \|w_1-x_2\| + \|x_2+v_1\| + \|v-v_1\| \\ &\leq 2\delta + \sqrt{\delta}\|x_1-x_2\| + \|x_2+t(-x_3)+(1-t)x_1\| + 2\delta \\ &\leq \|x_1+(1-t)x_2\| + 6\sqrt{\delta} \\ &= (2-t)\|\frac{x_1+(1-t)x_2}{2-t}\| + 6\sqrt{\delta} \\ &\leq 2-t+6\sqrt{\delta}.\end{aligned}$$

So,

$$1-\frac{\|w+v\|}{2} \geq 1-(1-\frac{t}{2}) - 3\sqrt{\delta} = \frac{t}{2} - 3\sqrt{\delta}.$$

We have

$$\begin{aligned}&V_X^*(t+2\sqrt{\delta}+2\delta) \\ &= \sup\{1-\frac{\|x+y\|}{2} : x,y\in S_X, \langle x-y, f_x\rangle \leq t+2\sqrt{\delta}+2\delta, f_x\in\nabla_x\} \\ &\geq \frac{t}{2} - 3\sqrt{\delta}.\end{aligned}$$

Since δ can be arbitrarily close to 0, we have

$$V^*(t^+) \geq \frac{t}{2}, \quad \text{for} \quad 0\leq t<1. \tag{3.2}$$

Combining (3.1) and (3.2), if X does not have normal structure, $V^*(\varepsilon^+) \geq \frac{\varepsilon}{2}$ for all $0<\varepsilon<2$. Similar to the last part of proof of Theorem 3.8, we have $V^*(\varepsilon) = \frac{\varepsilon}{2}$, for all $0<\varepsilon<2$. □

4. The Parameter $V^*(\varepsilon)$ and Ultraproduct

Let $\mathcal{U}$ be an ultrafilter on index set $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i\in\mathbb{N}$ for some Banach space X. For the ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultraproduct.

Lemma 4.1 ([13])**.** *Suppose $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space, then $(X^*)_{\mathcal{U}} = (X_{\mathcal{U}})^*$ if and only if X is superreflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle \quad \textit{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}},$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Lemma 4.2 (Bishop-Phelps-Bollobás [1])**.** *Let X be a Banach space, and let $0 < \varepsilon < 1$. Then, for $z \in B_X$ and $h \in S_{X^*}$ with $h(z) > 1 - \varepsilon^2/4$, there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Theorem 4.3. *Let $X_{\mathcal{U}}$ be the Banach space ultrapower of a Banach space X with respect to a nontrivial ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Then, for all $0 < \varepsilon < 2$,*

$$V^*_{X_{\mathcal{U}}}(\varepsilon) = V^*_X(\varepsilon).$$

Proof. Since X can be isometrically embedded into $X_{\mathcal{U}}$,

$$V^*_X(\varepsilon) \le V^*_{X_{\mathcal{U}}}(\varepsilon).$$

Moreover, if X is not super-reflexive, then

$$\frac{\varepsilon}{2} = V^*_X(\varepsilon) \le V^*_{X_{\mathcal{U}}}(\varepsilon) \le \frac{\varepsilon}{2}.$$

From now on we may assume that X is super-reflexive. Let $\eta > 0$ be given. There are elements $\widetilde{x} = (x_n)_{\mathcal{U}}$, $\widetilde{y} = (y_n)_{\mathcal{U}}$ in $S_{X_{\mathcal{U}}}$ and $\widetilde{f} = (f_n)_{\mathcal{U}}$ in $S_{X^*_{\mathcal{U}}}$ such that

$$1 - \frac{1}{2}\|\widetilde{x} + \widetilde{y}\| > V^*_{X_{\mathcal{U}}}(\varepsilon) - \eta, \quad \langle \widetilde{x}, \widetilde{f}\rangle = 1, \quad \text{and} \quad \langle \widetilde{y}, \widetilde{f}\rangle > 1 - \varepsilon.$$

Consequently,

$$\lim_{\mathcal{U}} \|x_n\| = \lim_{\mathcal{U}} \|y_n\| = \lim_{\mathcal{U}} \|f_n\| = \lim_{\mathcal{U}} \langle x_n, f_n\rangle = 1,$$
$$1 - \frac{1}{2}\lim_{\mathcal{U}} \|x_n + y_n\| > \widetilde{u_1}(\varepsilon) - \eta, \quad \text{and} \quad \lim_{\mathcal{U}} \langle x_n, f_n\rangle > 1 - \varepsilon.$$

We now put

$$x'_n := \frac{x_n}{\|x_n\|}, \quad y'_n := \frac{y_n}{\|y_n\|}, \quad \text{and} \quad f'_n := \frac{f_n}{\|f_n\|}.$$

By Bishop–Phelps–Bollobás' theorem, there are sequences $\{x''_n\}$ in S_X and $\{f''_n\}$ in S_{X^*} such that

$$\langle x''_n, f''_n\rangle = 1 \quad \text{for all natural number } n,$$

$$\|x''_n - x'_n\| \to 0 \quad \text{and} \quad \|f''_n - f'_n\| \to 0.$$

Clearly, $\lim_{\mathcal{U}}\langle x''_n, f''_n\rangle = \lim_{\mathcal{U}}\langle x_n, f_n\rangle > 1-\varepsilon$. Consequently,

$$\begin{aligned} V^*_{X_{\mathcal{U}}}(\varepsilon) - \eta &< 1 - \frac{1}{2}\lim_{\mathcal{U}}\|x_n + y_n\| \\ &= 1 - \frac{1}{2}\lim_{\mathcal{U}}\|x'_n + y'_n\| \\ &= 1 - \frac{1}{2}\lim_{\mathcal{U}}\|x''_n + y'_n\| \\ &\le V^*_X(\varepsilon). \end{aligned}$$

□

Remark 4.4. *We can prove that the moduli of W^*-convexity (and of U-convexity, resp.) of the space and its corresponding ultrapower are the same without assuming super-reflexivity as was the case in* [12, Theorem 4] *(*[11, Theorem 3.1]*, resp.).*

Theorem 4.5. *For a Banach space X if $V^*_X(\varepsilon) < \frac{\varepsilon}{2}$ for some $0 < \varepsilon < 2$, then X has a uniform normal structure.*

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