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HYBRID METHOD FOR A FAMILY OF MAPPINGS WITH APPLICATIONS IN ZERO OF MAXIMAL MONOTONE OPERATORS

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Abstract. In this paper, we construct some algorithms for finding common fixed point of a family of mappings with some sufficient condition. In fact, let C be a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and let $\{T_n\}$ be a family of self-mappings on C such that the set of all common fixed points of $\{T_n\}$ is nonempty. We construct a sequence $\{x_n\}$ generated by the hybrid method in mathematical programming and also we give the conditions of $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$. Finally, we apply our results to zero of maximal monotone operators.

1. Introduction and preliminaries

Throughout this paper, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let X be a real Banach space with dual space X^*. For a set-valued mapping $T : X \multimap Y$, the *domain* of

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T is *Dom*$(T) = \{x \in X : T(x) \neq \emptyset\}$, *range* of T is $R(T) = \{y \in Y : \exists x \in X, (x, y) \in T\}$ and the *inverse* T^{-1} of T is $\{(y, x) : (x, y) \in T\}$. For a real number c, let $cT = \{(x, cy) : (x, y) \in T\}$. If S and T are any set-valued mappings, we define $S + T = \{(x, y + z) : (x, y) \in S, (x, z) \in T\}$. Set

$$G = \{g : [0, +\infty) \to [0, +\infty) \quad | \quad g(0) = 0, \ g \text{ is continuous, strictly increasing and convex}\}. \tag{1.1}$$

Lemma 1.1. [3] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $x \in X$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y \in C} \|y - x\|$. Putting $x_0 = P_C(x)$, we call P_C the metric projection onto C.*

Lemma 1.2. [10] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and let $x \in X$. Then $y = P_C(x)$ if and only if $\langle y - z, J(x - y)\rangle \geq 0$ for all $z \in C$.*

Lemma 1.3. [10] *Suppose X has a Gateaux differentiable norm. Then the duality mapping J is single-valued and $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, Jy\rangle$ for all $x, y \in X$.*

Lemma 1.4. [11] *The Banach space X is uniformly convex if and only if for every bounded subset B of X, there exists $g_B \in G$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g_B(\|x - y\|) \tag{1.2}$$

for all $x, y \in B$ and all $\lambda \in [0, 1]$.

Let $\{T_n\}_{n=0}^{+\infty}$ be a family of mappings of a real Hilbert space $\mathcal{H}$ into itself and let $F(T_n)$ be the set of all fixed points of T_n. By the assumption that $\bigcap_{n=0}^{+\infty} F(T_n) \neq \emptyset$, Haugazeau [4] introduced a sequence $\{x_n\}$ generated by the hybrid method, as following

$$\begin{cases} x_0 \in \mathcal{H} \\ y_n = T_n(x_n) \\ C_n = \{z \in \mathcal{H} : \langle x_n - y_n, y_n - z\rangle \geq 0\} \\ Q_n = \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

In case that C_i is a closed convex subset of $\mathcal{H}$ for $i = 1, \ldots, m$, $\bigcap_{i=1}^{m} C_i \neq \emptyset$ and $T_n = P_{C_{n(\text{mod } m+1)}}$, he proved a strong convergence theorem. Recently, Solodov and Svaiter [9], Bauschke and Combettes [2], Atsushiba and Takahashi [1], Nakajo and Takahashi [8], Iiduka, Takahashi and Toyoda [5], Nakajo, Shimoji and Takahashi [7], studied the hybrid method in a Hilbert spaces and also Nakajo, Shimoji and Takahashi [6] considered this method for families of mappings in Banach spaces.

Motivated and inspired by the mentioned results, in this paper we construct some algorithms for finding common fixed point of a family of mappings with generalized Takahashi's condition. Then we apply our results to zero of maximal monotone operators.

2. Convergence theorems

Let $\{x_n\}$ be a sequence, $w_w(x_n)$ will denote the set of all weak cluster points of $\{x_n\}$. Let $\{T_n\}$ be a family of self-mappings of C with $F := \bigcap_{n=0}^{+\infty} F(T_n) \neq \emptyset$ which satisfies the following condition, in the sequel we call it *generalized Takahashi's condition*, denoted by (GTC).
$\exists x_0 \in C\ \exists \{a_n\} \subseteq (0, +\infty)$ with $\liminf_n a_n > 0\ \exists \{\alpha_n\} \subseteq [0,1],\ \exists \{\beta_n\} \subseteq [0,1]$ such that

$$\langle v_n - z, J(v_n - w_n)\rangle \geq a_n \|v_n - w_n\|^2 \tag{2.1}$$

for all $x \in C$, $z \in F(T_n)$, where $v_n = \alpha_n x_0 + (1-\alpha_n)x$ and $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(v_n)$.

Algorithm 2.1. *Let $\{T_n\}$ be a family of self-mappings of C with $F \neq \emptyset$ which satisfies (GTC). Let $\{x_n\}_{n=1}^{+\infty}$ be a sequence generated by the following algorithm.*

$$\begin{cases} y_n = \alpha_n x_0 + (1-\alpha_n)x_n \\ z_0 = T_0(x_0) \\ z_n = \beta_n z_0 + (1-\beta_n)T_n(y_n)\ (n \geq 1) \\ C_n = \{z \in C : \langle y_n - z, J(y_n - z_n)\rangle \geq a_n\|y_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.2}$$

Theorem 2.2. *Suppose C is a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies (GTC). Assume that*

(∗) *for every bounded sequence $\{u_n\}$ in C, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ and $\sum_{n=0}^{+\infty} g(a\|v_n - w_n\| - \alpha_n\|x_0 - u_n\|) < +\infty$ for some $g \in G$ and $a > 0$, where $v_n = \alpha_n x_0 + (1-\alpha_n)u_n$ and $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(v_n)$, imply that $w_w(u_n) \subseteq F$.*

Then the sequence $\{x_n\}$ generated by Algorithm 2.1 *converges strongly to $P_F(x_0)$.*

Proof. We split the proof into six steps.

Step 1. $\{x_n\}$ *is well defined.*

Notice that C_n and Q_n are closed and convex sets for all $n \in \mathbb{N} \cup \{0\}$. On the other hand, condition (2.1) and the definition of C_n in Algorithm 2.1 imply that $F(T_n) \subseteq C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $F \subseteq C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $J(0) = 0$, it follows from the definition of Q_n in Algorithm 2.1 that $Q_0 = C$ which implies that $F \subseteq C_0 \cap Q_0$. Lemma 1.1 guarantees that there exists a unique element $x_1 = P_{C_0 \cap Q_0}(x_0)$. By Lemma 1.2,

$$\langle x_1 - z, J(x_0 - x_1) \rangle \geq 0$$

for all $z \in C_0 \cap Q_0$ and hence by $F \subseteq C_0 \cap Q_0$ we get

$$\langle x_1 - z, J(x_0 - x_1) \rangle \geq 0$$

for all $z \in F$. Therefore, $F \subseteq Q_1$ and so apply the fact that $F \subseteq C_n$ for all $n \in \mathbb{N} \cup \{0\}$ we have $F \subseteq C_1 \cap Q_1$. Again, Lemma 1.1 guarantees that there exists a unique element $x_2 = P_{C_1 \cap Q_1}(x_0)$. Inductively, we find that $\{x_n\}$ is well defined.

Step 2. $\{x_n\}$ *is a bounded sequence.*

From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $F \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}_0$ we have

$$\|x_{n+1} - x_0\| \leq \|x_0 - P_F(x_0)\| \tag{2.3}$$

for all $n \in \mathbb{N}_0$, which implies that $\{x_n\}$ is a bounded sequence.

Step 3. $\lim_n \|x_n - x_0\|$ *exists.*

Replace terms $x_{n+1} - x_0$ and $x_n - x_o$ respectively with x and y in Lemma 1.3, then we get

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_n, J(x_n - x_0) \rangle$$

and hence $x_{n+1} \in Q_n$ implies that $\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2$ for all $n \in \mathbb{N}_0$; i.e., $\|x_n - x_0\|$ is an increasing sequence and so by Step 2 we find that $\lim_n \|x_n - x_o\|$ exists.

Step 4. $\sum_{n=0}^{+\infty} g(\|x_{n+1} - x_n\|) < +\infty$ *for some* $g \in G$.

It follows from Lemma 1.4 that there exists $g \in G$ such that

$$\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \leq \frac{1}{2}\|x_n - x_0\|^2 + \frac{1}{2}\|x_{n+1} - x_0\|^2 - \frac{1}{4}g(\|x_{n+1} - x_n\|) \tag{2.4}$$

and hence

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_n - x_0\|^2 + 2\|x_{n+1} - x_0\|^2 - 4\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \tag{2.5}$$

for all $n \in \mathbb{N}_0$. On the other hand, it follows from Lemma 1.2 and the definition of Q_n that $x_n = P_{Q_n} x_0$ and so by $x_{n+1} \in Q_n$ and convexity of Q_n we get

$\frac{x_n+x_{n+1}}{2} \in Q_n$. Again, by $x_n = P_{Q_n}x_0$,

$$\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \geq \|x_n - x_0\|^2. \quad (2.6)$$

It follows from inequalities (2.5) and (2.6) that

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_{n+1} - x_0\|^2 - 2\|x_n - x_0\|^2 \text{ for all } n \in \mathbb{N}_0. \quad (2.7)$$

That $\sum_{n=0}^{+\infty} g(\|x_{n+1} - x_n\|) < +\infty$ follows from (2.7), Step 1 and Step 3.

Step 5. $\sum_{n=0}^{+\infty} g(a\|y_n - z_n\| - \alpha_n\|x_n - x_0\|) < +\infty$ *for some* $g \in G$ *and* $a > 0$.

Since $a_n > 0$ for all $n \in \mathbb{N}_0$ and $\liminf_n a_n > 0$, there exists $a > 0$ for which $a_n \geq a$ for all $n \geq k \in \mathbb{N}_0$. Now, $x_{n+1} \in C_n$ guarantees that

$$\|y_n - x_{n+1}\|\|y_n - z_n\| \geq \langle y_n - x_{n+1}, J(y_n - z_n)\rangle \geq a_n\|y_n - z_n\|^2$$

and thus

$$a\|y_n - z_n\| \leq \|y_n - x_{n+1}\| \leq \alpha_n\|x_n - x_0\| + \|x_{n+1} - x_n\|$$

and hence

$$a\|y_n - z_n\| - \alpha_n\|x_n - x_0\| \leq \|x_{n+1} - x_n\| \quad (2.8)$$

for all $n \in \mathbb{N}_0$. That $\sum_{n=0}^{+\infty} g(a\|y_n - x_n\| - \alpha_n\|x_n - x_0\|) < +\infty$ follows from (2.8), (1.1) and Step 4.

Step 6. $\{x_n\} \to P_F(x_0)$

It follows from (∗), Step 4 and Step 5 that $w_w(x_n) \subseteq F$. Let the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $w \in F$. Therefore, weak lower semicontinuity of the norm and (2.3) imply that

$$\|P_F(x_0) - x_0\| \leq \|w - x_0\| \leq \lim_{i\to+\infty} \|x_{n_i} - x_0\| \leq \|P_F(x_0) - x_0\|$$

and hence $x_{n_i} \to w = P_F(x_0)$. □

Corollary 2.3. *Suppose C is a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following conditions.*

(a) $\exists x_0 \in C$ $\exists\{a_n\} \subseteq (0, +\infty)$ with $\liminf_n a_n > 0$ $\exists\{\alpha_n\} \subseteq [0, 1]$ *such that*

$$\langle v_n - z, J(v_n - T_n(v_n))\rangle \geq a_n\|v_n - T_n(v_n))\|^2 \quad (2.9)$$

for all $x \in C$, $z \in F(T_n)$, where, $v_n = \alpha_n x_0 + (1 - \alpha_n)x$;

(b) *for every bounded sequence* $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ *and* $\sum_{n=0}^{+\infty} g(a\|v_n - T_n(v_n))\| - \alpha_n\|x_0 - u_n\|) < +\infty$ *for some* $g \in G$, $v_n = \alpha_n x_0 + (1-\alpha_n)u_n$ *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} y_n = \alpha_n x_0 + (1-\alpha_n)x_n \\ z_n = T_n(y_n) \\ C_n = \{z \in C : \langle y_n - z, J(y_n - z_n)\rangle \geq a_n\|y_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.10}$$

Corollary 2.4. *Suppose* C *is a nonempty closed convex subset of a uniformly convex Banach space* X *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following conditions.*

(a) $\exists x_0 \in C\ \exists\{a_n\} \subseteq (0, +\infty)$ with $\liminf_n a_n > 0\ \exists\{\beta_n\} \subseteq [0,1]$ *such that*

$$\langle x - z, J(x - w_n)\rangle \geq a_n\|x - w_n\|^2 \tag{2.11}$$

for all $x \in C$, $z \in F(T_n)$, *and* $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(x)$;

(b) *for every bounded sequence* $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ *and* $\sum_{n=0}^{+\infty} g(a\|u_n - w_n\|) < +\infty$ *for some* $g \in G$, $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(u_n)$, *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} z_0 = T_0(x_0) \\ z_n = \beta_n z_0 + (1-\beta_n)T_n(x_n)\ (n \geq 1) \\ C_n = \{z \in C : \langle x_n - z, J(x_n - z_n)\rangle \geq a_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.12}$$

Corollary 2.5. [6] *Suppose* C *is a nonempty closed convex subset of a uniformly convex Banach space* X *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following conditions.*

(a) $\exists\{a_n\} \subseteq (0, +\infty)$ with $\liminf_n a_n > 0$ *such that*

$$\langle x - z, J(x - T_n(x))\rangle \geq a_n\|x - T_n(x)\|^2 \tag{2.13}$$

for all $x \in C$ *and* $z \in F(T_n)$;

(b) *for every bounded sequence* $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ *and* $\sum_{n=0}^{+\infty} g(a\|u_n - T_n(u_n)\|) < +\infty$ *for some* $g \in G$ *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$.

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} x_0 \in C \\ y_n = T_n(x_n) \\ C_n = \{z \in C : \langle x_n - z, J(x_n - y_n)\rangle \geq a_n\|x_n - y_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.14}$$

Corollary 2.6. *Suppose* C *is a nonempty closed convex subset of a real Hilbert space* H *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following conditions.*

(a) $\exists x_0 \in C$ $\exists\{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *and* $\exists\{\alpha_n\} \subseteq [0, 1]$, $\exists\{\beta_n\} \subseteq [0, 1]$ *such that*

$$\|w_n - z\|^2 \leq \|v_n - z\|^2 - b_n\|v_n - w_n\|^2 \tag{2.15}$$

for all $x \in C$, $z \in F(T_n)$, *where,* $v_n = \alpha_n x_0 + (1 - \alpha_n)x$ *and* $w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(v_n)$;

(b) *for every bounded sequence* $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ *and* $\sum_{n=0}^{+\infty} (a\|w_n - u_n\| - \alpha_n\|x_0 - u_n\|)^2 < +\infty$, *where*

$$w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(\alpha_n x_0 + (1 - \alpha_n)u_n)$$

and $a > 0$ *imply that* $w_w(u_n) \subseteq F$.

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} y_n = \alpha_n x_0 + (1 - \alpha_n)x_n \\ z_0 = T_0(x_0) \\ z_n = \beta_n z_0 + (1 - \beta_n)T_n(y_n) \ (n \geq 1) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|y_n - z\|^2 - b_n\|y_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.16}$$

Proof. First we note that, for $x \in C$, $z \in F(T_n)$, $v_n = \alpha_n x_0 + (1 - \alpha_n)x$ and $w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(v_n)$, by (2.15) we have $\|w_n - z\|^2 \leq \|v_n - z\|^2 - b_n\|v_n - w_n\|^2$, if and only if

$$\|w_n - v_n\|^2 + 2\langle w_n - v_n, v_n - z\rangle + \|v_n - z\|^2 \leq \|v_n - z\|^2 - b_n\|v_n - w_n\|^2$$

if and only if $\langle v_n - z, v_n - w_n\rangle \geq \frac{1+b_n}{2}\|v_n - w_n\|^2$. Then condition (2.1) satisfies for $a_n = \frac{1+b_n}{2}$. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$, so, we can consider $g_B(t) = t^2$ for each bounded subset B of H in Lemma 1.4 and hence $(*)$ holds. Then all conditions of Theorem 2.2 hold which it implies that $\{x_n\}$ converges strongly to $P_F(x_0)$. $\square$

Corollary 2.7. *Suppose C is a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following conditions.*

(a) $\exists x_0 \in C$ $\exists \{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *and* $\exists \{\alpha_n\} \subseteq [0,1]$ *such that*

$$\|T_n(v_n) - z\|^2 \leq \|v_n - z\|^2 - b_n\|v_n - T_n(v_n)\|^2 \tag{2.17}$$

for all $x \in C$, $z \in F(T_n)$, where, $v_n = \alpha_n x_0 + (1-\alpha_n)x$;

(b) *for every bounded sequence $\{u_n\}$ in C, $\sum_{n=0}^{+\infty}\|u_{n+1} - u_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty}(a\|T_n(v_n) - u_n\| - \alpha_n\|x_0 - u_n\|)^2 < +\infty$, where $v_n = \alpha_n x_0 + (1-\alpha_n)u_n$ and $a > 0$ imply that $w_w(u_n) \subseteq F$.*
Then $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

$$\begin{cases} y_n = \alpha_n x_0 + (1-\alpha_n)x_n \\ z_n = T_n(y_n) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|y_n - z\|^2 - b_n\|y_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.18}$$

Corollary 2.8. *Suppose C is a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following conditions.*

(a) $\exists x_0 \in C$ $\exists \{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *and* $\exists \{\beta_n\} \subseteq [0,1]$ *such that*

$$\|w_n - z\|^2 \leq \|x - z\|^2 - b_n\|x - w_n\|^2 \tag{2.19}$$

for all $x \in C$, $z \in F(T_n)$, where $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(x)$;

(b) *for every bounded sequence $\{u_n\}$ in C, $\sum_{n=0}^{+\infty}\|u_{n+1} - u_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty}(a\|w_n - u_n\|)^2 < +\infty$, where $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(u_n)$ and $a > 0$*

imply that $w_w(u_n) \subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} z_0 = T_0(x_0) \\ z_n = \beta_n z_0 + (1-\beta_n)T_n(x_n) \ (n \geq 1) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{2.20}$$

By putting $\alpha_n = \beta_n = 0$, we get the following result of K. Nakajo, K. Shimoji and W. Takahashi.

Corollary 2.9. [6] *Suppose* C *is a nonempty closed convex subset of a real Hilbert space* H *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following conditions.*

(a) $\exists\{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *such that*

$$\|T_n(x) - z\|^2 \leq \|x - z\|^2 - b_n\|x - T_n(x)\|^2 \tag{2.21}$$

for all $x \in C$, $z \in F(T_n)$;

(b) *for every bounded sequence* $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ *and* $\sum_{n=0}^{+\infty} \|u_n - T_n u_n\|^2 < +\infty$ *imply that* $w_w(u_n) \subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} x_0 \in C \\ z_n = T_n(x_n) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{2.22}$$

3. Applications to zero of maximal monotone operators

An operator $A : X \multimap X^*$ is said to be *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ for every $(x, x^*), (y, y^*) \in Gph(A)$. A monotone operator $A : X \multimap X^*$ is called *maximal* if $Gph(A)$ is not properly contained in the graph of any other monotone operator. It is known that a monotone operator $A : X \multimap X^*$ is maximal if and only if for $(y, y^*) \in X \times X^*$, we have $\langle x^* - y^*, x - y\rangle \geq 0$ for every $(x, x^*) \in X \times X^*$ implies $(y, y^*) \in Gph(A)$. We remark the following facts.

Fact 1: Let X be a uniformly convex Banach space whose norm is Gteaux differentiable and let $A : X \multimap X^*$ be a monotone operator. Then, A is maximal if and only if $R(J + rA) = X^*$ for all $r > 0$.

Fact 2: Let X be a uniformly convex Banach space whose norm is Gteaux differentiable and A be a maximal monotone operator of X into X^*, for any $x \in X$ and $r > 0$, there exists a unique element $x_r \in Dom(A)$ such that $0 \in J(x_r - x) + rA(x_r)$.

Using Fact 2, we can define $J_r : X \to X$ by $J_r(x) = x_r$ for every $x \in X$ and $r > 0$ and such J_r is called *the resolvent of* A, for more details see [10].

Theorem 3.1. *Let X be a uniformly convex Banach space with a Gateaux differentiable norm and let T be a maximal monotone operator from X to X^* such that $T^{-1}(0) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm and* $\liminf_n r_n > 0$. *If* $\lim_n \alpha_n = 0$*, then $\{x_n\}$ converges strongly to $P_{T^{-1}(0)}(x_0)$ as $n \to \infty$.*

$$\begin{cases} x_0 \in X \\ y_n = \alpha_n x_0 + (1 - \alpha_n)x_n \\ z_n = J_{r_n}(y_n) \\ C_n = \{z \in X : \langle y_n - z, J(x_n - z_n)\rangle \geq 0\} \\ Q_n = \{z \in X : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{3.1}$$

Proof. Let $T_n = J_{r_n}$ for all $n \in \mathbb{N}_0$. It follows from Fact 2 that T_n is a mapping from X into $Dom(T)$ and $F(T_n) = T^{-1}(0) \neq \emptyset$ for all $n \in \mathbb{N}_0$. Fix $n \in \mathbb{N}_0$, $x \in C$ and $z \in F(T_n)$. Set $v_n = \alpha_n x_0 + (1 - \alpha_n)x$. By Fact 2 we have $\frac{1}{r_n} J(v_n - T_n(v_n)) \in T(T_n(v_n))$ and since T is monotone, one can deduce that $\langle T_n(v_n) - z, J(v_n - T_n(v_n))\rangle \geq 0$. Therefore,

$$\begin{aligned} \langle v_n - z, J(v_n - T_n(v_n))\rangle &= \langle v_n - T_n(v_n) + T_n(v_n) - z, J(v_n - T_n(v_n))\rangle \\ &\geq \langle v_n - T_n(v_n), J(v_n - T_n(v_n))\rangle \\ &= \|v_n - T_n(v_n)\|^2. \end{aligned}$$

Consequently, condition (2.9) satisfies with $a_n = 1$. Now, let for a bounded sequence $\{u_n\}$ in C, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ and $\sum_{n=0}^{+\infty} g(a\|v_n - T_n(v_n))\| - \alpha_n\|x_0 - u_n\|) < +\infty$ for some $g \in G$ where $v_n = \alpha_n x_0 + (1 - \alpha_n)u_n$ and $a > 0$. Then $\lim_n \|u_{n+1} - u_n\| = \lim_n \|v_n - T_n(v_n))\| = 0$. Let $(y, y^*) \in Gph(T)$ and

$u_n \xrightarrow{w} u$, so

$$\begin{aligned}
0 &\leq \langle T_n(v_n) - y, \frac{1}{r_n} J(v_n - T_n(v_n)) - y^* \rangle \\
&= \langle T_n(v_n) - v_n, \frac{1}{r_n} J(v_n - T_n(v_n)) - y^* \rangle + \langle v_n - y, \frac{1}{r_n} J(v_n - T_n(v_n)) - y^* \rangle \\
&= -\frac{1}{r_n} \|T_n(v_n) - v_n\|^2 + \langle T_n(v_n) - v_n, -y^* \rangle + \langle v_n - y, \frac{1}{r_n} J(v_n - T_n(v_n)) \rangle \\
&\quad + \langle v_n - y, -y^* \rangle \\
&\leq \|T_n(v_n) - v_n\| \|y^*\| + \frac{1}{r_n} \|v_n - y\| \|v_n - T_n(v_n)\| + \langle v_n - y, -y^* \rangle.
\end{aligned}$$

Then $\langle u - y, -y^* \rangle \geq 0$ and from maximality of T we get $0 \in T(u)$; i.e., $u \in T^{-1}(0) = F$ and hence $w_w(u_n) \subseteq F$. That $\{x_n\}$ converges strongly to $P_{T^{-1}}(0)$ follows from Corollary 2.3. □

Corollary 3.2. [6] *Let X be a uniformly convex Banach space with a Gateaux differentiable norm and let T be a maximal monotone operator from X to X^* such that $T^{-1}(0) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm and $\liminf_n r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{T^{-1}(0)}(x_0)$.*

$$\begin{cases}
x_0 \in X \\
y_n = J_{r_n}(x_n) \\
C_n = \{z \in X : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\} \\
Q_n = \{z \in X : \langle x_n - z, J(x_0 - x_n) \rangle \geq 0\} \\
x_{n+1} = P_{C_n \cap Q_n}(x_0).
\end{cases} \tag{3.2}$$

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