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A DISCRETE FORM OF THE BECKMAN-QUARLES THEOREM FOR TWO-DIMENSIONAL STRICTLY CONVEX NORMED SPACES

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Abstract. Let $\rho > 0$ be a fixed real number. Let X be a real normed vector space, dim $X \geq 2$. We prove that if $x, y \in X$ and $\|x - y\|/\rho$ is a rational number then there exists a finite set $\{x, y\} \subseteq S_{xy} \subseteq X$ with the following property: for each strictly convex Y of dimension 2 each map $f : S_{xy} \to Y$ preserving the distance ρ satisfies $\|f(x) - f(y)\| = \|x - y\|$. It implies that each map from X to Y that preserves the distance ρ is an affine isometry.

Let $\mathbb{Q}$ denote the field of rational numbers. All vector spaces mentioned in this article are assumed to be real. A normed vector space E is called *strictly convex* ([6]), if for each pair a, b of nonzero elements in E such that $\|a+b\| = \|a\| + \|b\|$, it follows that $a = \gamma b$ for some $\gamma > 0$. It is known ([17]) that two-dimensional strictly convex normed spaces satisfy the following condition $(*)$:

$(*)$ for any $a \neq b$ on line L and any c, d on the same side of L, if $\|a - c\| = \|a - d\|$ and $\|b - c\| = \|b - d\|$, then $c = d$.

Conversely ([17]), for any two-dimensional normed space the condition $(*)$ implies that the space is strictly convex.

The classical Beckman-Quarles theorem states that any map from $\mathbb{R}^n$ to $\mathbb{R}^n$ $(2 \leq n < \infty)$ preserving unit distance is an isometry, see [2], [3] and [7]. Various unanswered questions and counterexamples concerning the Beckman-Quarles theorem and isometries are discussed by Ciesielski and Rassias [5]. For more open problems and new results on isometric mappings the reader is referred to [8]-[14]. The Theorem below may be viewed as a discrete form of the Beckman-Quarles theorem for two-dimensional strictly convex normed spaces.

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Theorem. *Let $\rho > 0$ be a fixed real number. Let X be a normed vector space, $\dim X \geq 2$.*
1. *If $x, y \in X$ and $\|x-y\|/\rho$ is a rational number then there exists a finite set $\{x,y\} \subseteq S_{xy} \subseteq X$ with the following property: for each strictly convex Y of dimension 2 each map $f : S_{xy} \to Y$ preserving the distance ρ satisfies $\|f(x)-f(y)\| = \|x-y\|$.*
2. *If $x, y \in X$ and $\varepsilon > 0$ then there exists a finite set $\{x,y\} \subseteq T_{xy}(\varepsilon) \subseteq X$ with the following property: for each strictly convex Y of dimension 2 each map $f : T_{xy}(\varepsilon) \to Y$ preserving the distance ρ satisfies*

$$\big|\|f(x)-f(y)\| - \|x-y\|\big| \leq \varepsilon.$$

Proof. The proof is divided into three parts.
Part 1. We prove items 1 and 2 for injective maps. Let D denote the set of all non-negative numbers d with the following property ($\diamond$):

($\diamond$) if $x, y \in X$ and $\|x-y\| = d$ then there exists a finite set $\{x,y\} \subseteq S_{xy} \subseteq X$ such that each injective $f : S_{xy} \to Y$ preserving the distance ρ satisfies $\|f(x)-f(y)\| = \|x-y\|$.

Obviously $0, \rho \in D$. We first prove that if $d \in D$, then $2 \cdot d \in D$. Assume that $d \in D$, $d > 0$, $x, y \in X$, $\|x-y\| = 2 \cdot d$. Using the notation of Figure 1

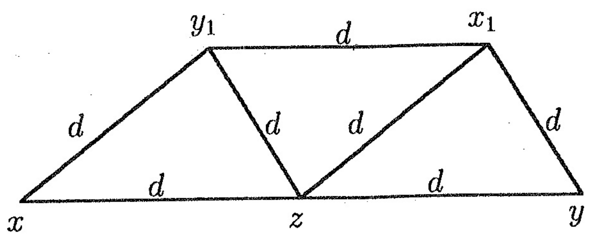

Figure 1

$$\|x-y\| = 2 \cdot d$$
$$z := \frac{x+y}{2}$$
$$\|x-z\| = \|x-y_1\| = \|z-y_1\| = d$$
$$x_1 := y_1 + (z-x)$$

we show that

$$S_{xy} := S_{xz} \cup S_{zy} \cup S_{y_1x_1} \cup S_{xy_1} \cup S_{zx_1} \cup S_{zy_1} \cup S_{yx_1}$$

satisfies the condition ($\diamond$). Let an injective $f : S_{xy} \to Y$ preserves the distance ρ. By the injectivity of f: $f(x) \neq f(x_1)$ and $f(y) \neq f(y_1)$. According to ($*$): $f(y_1) - f(x_1) = f(x) - f(z)$ and $f(y_1) - f(x_1) = f(z) - f(y)$. Hence $f(x) - f(z) = f(z) - f(y)$. Therefore $\|f(x) - f(y)\| = \|2(f(x) - f(z))\| = 2 \cdot \|f(x) - f(z)\| = 2 \cdot \|x - z\| = 2 \cdot d = \|x - y\|$.

From Figure 2, the previous step and the property that defines strictly convex normed spaces it is clear that if $d \in D$, then all distances $k \cdot d$ (k a positive integer) belong to D.

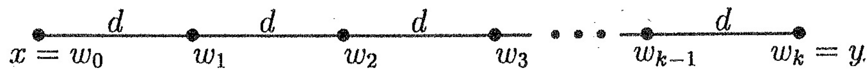

Figure 2

$$\|x - y\| = k \cdot d$$

$$S_{xy} = \bigcup \{S_{ab} : a, b \in \{w_0, w_1, ..., w_k\}, \|a - b\| = d \vee \|a - b\| = 2 \cdot d\}$$

From Figure 3, the previous step and the property that defines strictly convex normed spaces it is clear that if $d \in D$, then all distances d/k (k a positive integer) belong to D. Hence $D/\rho := \{d/\rho : d \in D\} \supseteq \mathbb{Q} \cap [0, \infty)$. This completes the proof of item 1 for injective maps.

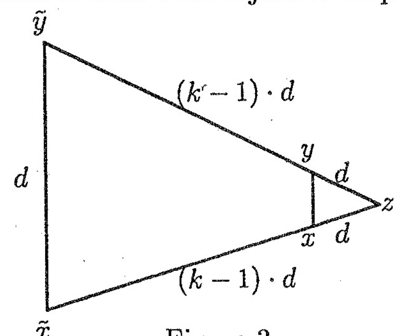

Figure 3

$$\|x - y\| = d/k$$

$$\tilde{x} := x + (k - 1)(x - z)$$

$$\tilde{y} := y + (k - 1)(y - z)$$

$$\tilde{x} - \tilde{y} = x - y + (k - 1)((x - z) - (y - z)) = k(x - y)$$

$$S_{xy} = S_{\tilde{x}\tilde{y}} \cup S_{\tilde{x}x} \cup S_{xz} \cup S_{\tilde{x}z} \cup S_{\tilde{y}y} \cup S_{yz} \cup S_{\tilde{y}z}$$

From Figure 4 follows item 2 for injective maps.

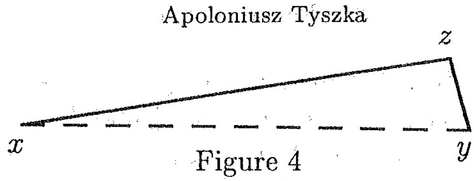

Figure 4

$$|x-z|/\rho, |z-y|/\rho \in \mathbb{Q} \cap [0,\infty), \quad |z-y| \le \varepsilon/2$$

$$T_{xy}(\varepsilon) = S_{xz} \cup S_{zy}$$

Note. From Figure 1 follows that instead of injectivity in Part 1 we may assume that

$$\forall u, v \in \mathrm{dom}(f)(\|u-v\|/\rho \in \mathbb{Q} \cap (0,\infty) \Rightarrow \|f(u)-f(v)\| \ne \|u-v\|/2).$$

Part 2. Let $X = \mathbb{R}^n$ $(2 \le n < \infty)$ be equipped with euclidean norm. We prove that the assumption of injectivity is unnecessary to prove items 1 and 2. In proofs of items 1 and 2 we used injectivity only in the first step for distances $2 \cdot d$, $d \in D$. Let D is defined without the assumption of injectivity. Let $d \in D$, $d > 0$. We need to prove that $2 \cdot d \in D$. Let us see at configuration from Figure 5, all segments have the length d.

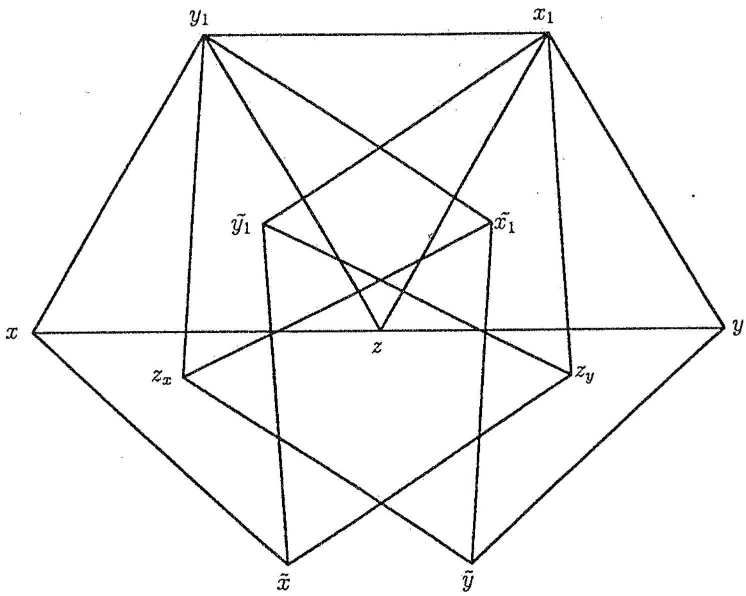

Figure 5

$$\|x-y\| = 2 \cdot d$$

$$z := \frac{x+y}{2}$$

$$S_{xy} = \bigcup \{S_{ab} : a, b \in \{x, \tilde{x}, x_1, \tilde{x_1}, y, \tilde{y}, y_1, \tilde{y_1}, z, z_x, z_y\}, \|a-b\| = d\}$$

Assume that $f : S_{xy} \to Y$ preserves the distance ρ. It is sufficient to prove that $f(x) \neq f(x_1)$ and similarly $f(y) \neq f(y_1)$. Suppose, on the contrary, that $f(x) = f(x_1)$, the proof of $f(y) \neq f(y_1)$ is similar. Hence four points: $f(\tilde{x})$, $f(z_y)$, $f(\tilde{y_1})$, $f(x_1)$ have the distance d from each other. We prove that it is impossible in two-dimensional strictly convex normed spaces. Suppose, on the contrary, that $a_1, a_2, a_3, a_4 \in Y$ and $\|a_1 - a_2\| = \|a_1 - a_3\| = \|a_1 - a_4\| = \|a_2 - a_3\| = \|a_2 - a_4\| = \|a_3 - a_4\| = d > 0$. Let us consider the segment a_2a_3. According to $(*)$ a_1 and a_4 lie on the opposite sides of the line $L(a_2, a_3)$ and $a_2 - a_1 = a_4 - a_3$. Let us consider the segment a_1a_3. According to $(*)$ a_2 and a_4 lie on the opposite sides of the line $L(a_1, a_3)$ and $a_1 - a_2 = a_4 - a_3$. Hence $a_4 - a_3 = 0$, a contradiction. This completes the proof.

Part 3. We prove that for each normed space X the assumption of injectivity is unnecessary to prove items 1 and 2. Analogously as in Part 2 it suffices to prove that for each $x, y \in X$, $x \neq y$ there exist points forming the configuration from Figure 5 where all segments have the length $\|x - y\|/2$. Let us consider $x, y \in X$, $x \neq y$. We choose two-dimensional subspace $\tilde{X} \subseteq X$ containing x and y.

First case: the norm induced on $\tilde{X}$ is strictly convex. Obviously $\tilde{X}$ is isomorphic to $\mathbb{R}^2$ as a linear space. Let us consider $\mathbb{R}^2$ with a strictly convex norm $\| \ \|$. It suffices to prove that for each $a, b \in \mathbb{R}^2$ satisfying $\|a\| = \|b\| = \|a - b\| = d > 0$ there exist $\tilde{a}, \tilde{b} \in \mathbb{R}^2$ satisfying $\|\tilde{a}\| = \|\tilde{b}\| = \|\tilde{a} - \tilde{b}\| = \|(\tilde{a} + \tilde{b}) - (a + b)\| = d$. We fix $a = (a_x, a_y)$ and $b = (b_x, b_y)$. Let $S := \{x \in \mathbb{R}^2 : \|x\| = d\}$. According to $(*)$ for each $u = (u_x, u_y) \in S$ there exists a unique $h(u) = (h(u)_x, h(u)_y) \in S$ such that $\|u - h(u)\| = d$ and

$$\det \begin{bmatrix} u_x & u_y \\ h(u)_x & h(u)_y \end{bmatrix} \cdot \det \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} > 0.$$

Obviously $h(a) = b$. The mapping $h : S \to S$ is continuous. For each $u \in S$ $h(-u) = -h(u)$ and $\|u + h(u)\| = \|2u - (u - h(u))\| \geq |\|2u\| - \|u - h(u)\|| = d$. The following function

$$S \ni x \xrightarrow{g} \|x + h(x) - a - h(a)\| \in [0, \infty)$$

is continuous. We have:

$$g(a) = 0,$$

$$g(-a) = \| - a + h(-a) - a - h(a)\| = 2 \cdot \|a + h(a)\| \geq 2 \cdot d.$$

Since g is continuous there exists $\tilde{a} \in S$ such that $g(\tilde{a}) = d$. From this $\tilde{a}$ and $\tilde{b} := h(\tilde{a})$ satisfy $\|\tilde{a}\| = \|\tilde{b}\| = \|\tilde{a} - \tilde{b}\| = \|(\tilde{a} + \tilde{b}) - (a + b)\| = d$. This completes the proof of the Theorem in the case where the norm induced on $\tilde{X}$ is strictly convex.

Second case: we assume only that $\| \ \|$ is a norm on $\tilde{X}$. The graph Γ from Figure 5 (11 vertices, 19 edges) has the following matrix representation:

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
$v_0 := x$	0	0	1	1	0	0	0	1	0	0	0
$v_1 := y$	0	0	1	0	1	0	1	0	0	0	0
$v_2 := z = \frac{x+y}{2}$	1	1	0	0	1	0	0	1	0	0	0
$v_3 := \tilde{x}$	1	0	0	0	0	0	0	0	1	0	1
$v_4 := x_1$	0	1	1	0	0	0	0	1	1	0	1
$v_5 := \tilde{x}_1$	0	0	0	0	0	0	1	1	0	1	0
$v_6 := \tilde{y}$	0	1	0	0	0	1	0	0	0	1	0
$v_7 := y_1$	1	0	1	0	1	1	0	0	0	1	0
$v_8 := \tilde{y}_1$	0	0	0	1	1	0	0	0	0	0	1
$v_9 := z_x$	0	0	0	0	0	1	1	1	0	0	0
$v_{10} := z_y$	0	0	0	1	1	0	0	0	1	0	0

Let $u_0 := v_0 = x$, $u_1 := v_1 = y$, $u_2 := v_2 = z = \frac{x+y}{2}$. We define the following function ψ:

$$\tilde{X}^8 \ni (u_3, ..., u_{10}) \xrightarrow{\psi} (\|u_i - u_j\| : 0 \le i < j \le 10, (v_i, v_j) \in \Gamma) \in \mathbb{R}^{19}.$$

The image of ψ is a closed subset of $\mathbb{R}^{19}$. For each $\varepsilon > 0$ and each bounded $B \subseteq \tilde{X}$ the norm $\| \ \|$ may be approximate on B with ε-accuracy by a strictly convex norm on $\tilde{X}$, for example by a norm $\| \ \| + t\| \ \|_{\text{euclidean}}$ for sufficiently small positive t. Therefore according to the first case for each $x, y \in X$, $x \neq y$ and each $\varepsilon > 0$ there exist points forming the configuration from Figure 5 where all segments have $\| \ \|$-lengths belonging to the interval $(\frac{\|x-y\|}{2} - \varepsilon, \frac{\|x-y\|}{2} + \varepsilon)$. Therefore:

$$(\|x-y\|/2, ..., \|x-y\|/2) \in \overline{\psi(\tilde{X}^8)} \text{ (the closure of } \psi(\tilde{X}^8)).$$

Since $\psi(\tilde{X}^8)$ is closed we conclude that

$$(\|x-y\|/2, ..., \|x-y\|/2) \in \psi(\tilde{X}^8).$$

This completes the proof of the Theorem. □

From item 2 of the Theorem we obtain the following Corollary.

Corollary. *Let $\rho > 0$ be a fixed real number. Let X and Y be normed vector spaces satisfying:* $\dim X \geq 2$, $\dim Y = 2$, *Y is strictly convex. Let $f : X \to Y$ preserves the distance ρ. Then f is an isometry, by Remark* 1 *below f is an affine isometry.*

Remark 1. *J. A. Baker* ([1]) *proved that an isometry from one normed vector space into a strictly convex normed vector space is affine.*

Remark 2. *W. Benz and H. Berens* ([4], *see also* [3] *and* [11]) *proved the following theorem: Let X and Y be normed vector spaces such that Y is strictly convex and such that the dimension of X is at least* 2. *Let $\rho > 0$ be a fixed real number and let $N > 1$ be a fixed integer. Suppose that $f : X \to Y$ is a mapping satisfying:*

$$\|a - b\| = \rho \Rightarrow \|f(a) - f(b)\| \leq \rho$$
$$\|a - b\| = N\rho \Rightarrow \|f(a) - f(b)\| \geq N\rho$$

for all $a, b \in X$. Then f is an affine isometry.

Remark 3. *A. Tyszka* ([15],[16]) *proved the following theorem: if $x, y \in \mathbb{R}^n$ $(2 \leq n < \infty)$ and $|x - y|$ is an algebraic number then there exists a finite set $\{x, y\} \subseteq S_{xy} \subseteq \mathbb{R}^n$ such that each unit distance preserving mapping $f : S_{xy} \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| = |x - y|$.*

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