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# CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE NONSELF MAPPINGS IN THE INTERMEDIATE SENSE IN UNIFORMLY CONVEX BANACH SPACES

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**Abstract.** Let $K$ be a nonempty closed convex nonexpansive retract of a uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. Let $T : K \to E$ be a nonself asymptotically nonexpansive in the intermediate sense mapping with $F(T) \neq \phi$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1,2,3$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{u_n^i\}$ for all $i = 1,2,3$ are bounded sequences in $K$ with $\sum_{n=1}^{\infty} u_n^i < \infty$. (i) If the dual $E^*$ of $E$ has the *Kadec-Klee* property, then $\{x_n\}$ converges weakly to a fixed point of $T$; (ii) if $T$ satisfies condition (A), then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of $T$. The results presented in this paper extend and improve the results in [1, 4, 6, 11, 12, 17, 18, 22, 32] and many others

## 1. Introduction and Preliminaries

Let $K$ be a nonempty closed convex subset of a Banach space $E$. A self mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$; $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the

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following inequality holds:

$$||T^n x - T^n y|| \leq k_n ||x - y||, \forall n \geq 1. \tag{1.1}$$

$T$ is called uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$||T^n x - T^n y|| \leq L||x - y||, \quad \forall n \geq 1. \tag{1.2}$$

$T$ is called asymptotically nonexpansive type [29] if the following inequality holds:

$$\limsup_{n\to\infty} \sup_{y\in K}(||T^n x - T^n y|| - ||x - y||) \leq 0. \tag{1.3}$$

for every $x \in K$, and that $T^N$ is continuous for some $N \geq 1$.

$T$ is called asymptotically nonexpansive in the intermediate sense [21] if $T$ is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} (||T^n x - T^n y|| - ||x - y||) \leq 0. \tag{1.4}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [13] as an important generalization of the class of nonexpansive maps (i.e., mappings $T : K \to K$ such that $||Tx - Ty|| \leq ||x - y||$, $\forall x, y \in K$) who proved that if $K$ is a nonempty closed convex subset of a real uniformly convex Banach space and $T$ is an asymptotically nonexpansive self-mapping of $K$, then $T$ has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (see e.g., [27], [2, 3], [5], [25], [18], [7], [26], [1], [11, 12], [8], [14, 15, 16, 17]) using the Mann iteration method (see e.g., [31]) or the Ishikawa iteration method (see e.g., [24]).

In 1978, Bose [23] proved that if $K$ is a bounded closed convex nonempty subset of a uniformly convex Banach space $E$ satisfying Opial's [33] condition and $T : K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of $T$ provided $T$ is asymptotically regular at $x \in K$, i.e., $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$. Passty [7] and also Xu [9] proved that the requirement that $E$ satisfies Opial's condition can be replaced by the condition that $E$ has a Fréchet differentiable norm. Furthermore, Tan and Xu [14, 15] later proved that the asymptotic regularity of $T$ can be weakened to the weakly asymptotic regularity of $T$ at $x$, i.e., $\omega - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$.

In [11, 12], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space $H$. In 1994, Rhoades [1] extended the Schu's result to uniformly convex Banach space using a modified Ishikawa iteration method. In all the above results, the operator $T$ remains a self-mapping of a nonempty closed convex subset $K$ of a uniformly convex Banach space. If, however, the domain of $T$, $D(T)$ is a proper subset of $E$ (and this is the case in several applications), and $T$ maps $D(T)$ into $E$, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

In 2003, Chidume et al [4] studied the iterative scheme defined by

$$\begin{aligned} & x_1 \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{1.5}$$

in the framework of uniformly convex Banach space, where $K$ is a closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retract. $T\colon K \to E$ is an asymptotically nonexpansive nonself map with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself maps.

In 2005, Shahzad [19] studied the sequence $\{x_n\}$ defined by

$$\begin{aligned} & x_1 \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n Tx_n]), \end{aligned} \tag{1.6}$$

where $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. He proved weak and strong convergence theorems for nonself nonexpansive mappings in Banach spaces.

Recently, Su and Qin [32] studied the sequence $\{x_n\}$ defined by

$$\begin{aligned} & x_1 \in K, \\ z_n &= P(\alpha_n'' T(PT)^{n-1}x_n + (1-\alpha_n'')x_n), \\ y_n &= P(\alpha_n' T(PT)^{n-1}z_n + (1-\alpha_n')x_n), \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}y_n + (1-\alpha_n)x_n), \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$ and $\{\alpha_n''\}$ are real sequences in $(0,1)$ and $K$ is a nonempty closed convex nonexpansive retract of a uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. They proved weak and strong convergence theorems for asymptotically nonexpansive nonself mappings in uniformly convex Banach space.

Motivated by Su and Qin [32] and some others, the purpose of this paper is to construct a three step iterative scheme with errors for approximating fixed point of asymptotically nonexpansive nonself mappings in the intermediate sense (when such a fixed point exists) and to prove weak and strong convergence theorems for such maps.

Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ and $T\colon K \to E$ is asymptotically nonexpansive nonself mappings in the intermediate sense. In this paper, the following iteration scheme is studied:

$$x_1 \in K,$$

$$\begin{aligned} z_n &= P(\alpha_n^3 T(PT)^{n-1} x_n + \beta_n^3 x_n + \gamma_n^3 u_n^3), \\ y_n &= P(\alpha_n^2 T(PT)^{n-1} z_n + \beta_n^2 x_n + \gamma_n^2 u_n^2), \\ x_{n+1} &= P(\alpha_n^1 T(PT)^{n-1} y_n + \beta_n^1 x_n + \gamma_n^1 u_n^1), \end{aligned} \tag{1.8}$$

where$\{\alpha_n^1\}, \{\alpha_n^2\}, \alpha_n^3\}$,$\{\beta_n^1\}, \{\beta_n^2\}, \{\beta_n^3\}$, $\{\gamma_n^1\}, \{\gamma_n^2\}, \{\gamma_n^3\}$ are sequences in $[0, 1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3$, and $\{u_n^1\}, \{u_n^2\}, \{u_n^3\}$ are bounded sequences in $K$.

Our theorems improve and generalize some previous results. Our weak convergence result applies not only to $L^p$-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. More precisely, we prove weak convergence of the above defined iteration scheme with errors (1.8) in a uniformly convex Banach space whose dual has the *Kadec-Klee* property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial's property; however their dual does have the *Kadec-Klee* property (see, e.g., [10, 28]).

Let $E$ be a real Banach space. A subset $K$ of $E$ is said to be a retract of $E$ if there exists a continuous map $P\colon E \to E$ such that $Px = x$ for all $x \in K$. A map $P\colon E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$. A set $K$ is optimal if each point outside $K$ can be moved to be closure to all points of $K$. It is well known (see, e.g., [30]) that

(i) if $E$ is a separable, strictly convex, smooth, reflexive Banach space, and if $K \subset E$ is an optimal set with interior, then $K$ is a nonexpansive retract of $E$;

(ii) a subset of $\ell^p$, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$.

A Banach space $E$ is said to have the *Kadec-Klee* property if for every sequence $\{x_n\}$ in $E$, $x_n \to x$ weakly and $||x_n|| \to ||x||$ strongly together imply $||x_n - x| \to 0$.

Recall that the following:

A mapping $T\colon K \to K$ with $F(T) \neq \phi$ is said to satisfy condition (A) [8] on $K$ if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$,

$$||x - Tx|| \geq f(d(x, F(T)))$$

where $d(x, F(T)) = \inf\{||x - p|| : p \in F(T)\}$.

In order to prove our main results, we will make use of the following lemmas:

**Lemma 1.1.**(see [16]): Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$s_{n+1} \leq s_n + t_n \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists. Moreover, if there exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} \to 0$ as $j \to \infty$, then $s_n \to 0$ as $n \to \infty$.

**Lemma 1.2.** ( Schu [12]): Let $E$ be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $E$ satisfying

$$\limsup_{n\to\infty} ||x_n|| \leq r, \qquad \limsup_{n\to\infty} ||y_n|| \leq r,$$

$$\lim_{n\to\infty} ||t_n x_n + (1 - t_n) y_n|| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} ||x_n - y_n|| = 0.$$

**Lemma 1.3.** (Demiclosed principle for nonselfmap [6]): Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$. Let $T\colon K \to E$ be a mapping which is asymptotically nonexpansive in the intermediate sense.

If the sequence $\{x_n\} \subset K$ converges weakly to $x^*$ and if

$$\lim_{j\to\infty}(\limsup_{n\to\infty}||x_n - T(PT)^{j-1}x_n||) = 0,$$

then $Tx^* = x^*$.

**Lemma 1.4.** (see [10]): Let $E$ be a real reflexive Banach space such that its dual $E^*$ has the *Kadec-Klee* property. Let $\{x_n\}$ be a bounded sequence in $E$ and $x^*, y^* \in w_w(x_n)$; here $w_w(x_n)$ denotes the weak $w$-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty}||tx_n + (1-t)x^* - y^*||$ exists for all $t \in [0,1]$. Then $x^* = y^*$.

## 2. Main Results

**Definition 2.1.** (see [4]): Let $E$ be a real normed linear space, $K$ a nonempty subset of $E$. Let $P : E \to K$ be the nonexpansive retraction of $E$ onto $K$. A map $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$; $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq k_n||x-y||, \quad \forall n \geq 1. \tag{2.1}$$

$T$ is called uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq L||x-y||, \quad \forall n \geq 1. \tag{2.2}$$

$T$ is called asymptotically nonexpansive type if the following inequality holds:

$$\limsup_{n\to\infty}\sup_{y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x-y||) \leq 0, \tag{2.3}$$

for every $x \in K$, and that $T^N$ be continuous for some $N \geq 1$.

$T$ is called asymptotically nonexpansive in the intermediate sense (Chidume et al [6]) if $T$ is uniformly continuous and

$$\limsup_{n\to\infty}\sup_{x,y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x-y||) \leq 0. \tag{2.4}$$

**Lemma 2.2.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$

be an asymptotically nonexpansive nonself mapping in the intermediate sense. Put

$$G_n = \max\{\sup_{x,y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x-y||), 0\}, \quad \forall n \geq 1$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1,2,3$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{u_n^i\}$ are bounded sequences in $K$ for all $i = 1,2,3$ with $\sum_{n=1}^{\infty} u_n^i < \infty$, $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$. Then for any $x^* \in F(T)$, $\lim_{n\to\infty} ||x_n - x^*||$ exists.

*Proof.* For any given $x^* \in F(T)$, and since $\{u_n^i\}$ for $i = 1,2,3$ is bounded sequence in $K$, so we put

$$M = \max\{\sup_{n\geq 1} ||u_n^i - x^*|| : i = 1,2,3\},$$

it follows from scheme (1.8) that

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||P(\alpha_n^1 T(PT)^{n-1}y_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - Px^*|| \\ &\leq ||(\alpha_n^1 T(PT)^{n-1}y_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - x^*|| \\ &\leq \alpha_n^1 ||T(PT)^{n-1}y_n - x^*|| + \beta_n^1 ||x_n - x^*|| + \gamma_n^1 ||u_n^1 - x^*|| \\ &\leq \alpha_n^1 ||y_n - x^*|| + G_n + \beta_n^1 ||x_n - x^*|| + \gamma_n^1 ||u_n^1 - x^*||, \end{aligned} \tag{2.5}$$

$$\begin{aligned} ||y_n - x^*|| &= ||P(\alpha_n^2 T(PT)^{n-1}z_n + \beta_n^2 x_n + \gamma_n^2 u_n^2) - Px^*|| \\ &\leq ||(\alpha_n^2 T(PT)^{n-1}z_n + \beta_n^2 x_n + \gamma_n^2 u_n^2) - x^*|| \\ &\leq \alpha_n^2 ||T(PT)^{n-1}z_n - x^*|| + \beta_n^2 ||x_n - x^*|| + \gamma_n^2 ||u_n^2 - x^*|| \\ &\leq \alpha_n^2 ||z_n - x^*|| + G_n + \beta_n^2 ||x_n - x^*|| + \gamma_n^2 ||u_n^2 - x^*|| \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \|z_n - x^*\| &= \|P(\alpha_n^3 T(PT)^{n-1}x_n + \beta_n^3 x_n + \gamma_n^3 u_n^3) - Px^*\| \\ &\leq ||(\alpha_n^3 T(PT)^{n-1}x_n + \beta_n^3 x_n + \gamma_n^3 u_n^3) - x^*|| \\ &\leq \alpha_n^3 ||T(PT)^{n-1}x_n - x^*|| + \beta_n^3 ||x_n - x^*|| + \gamma_n^3 ||u_n^3 - x^*|| \\ &\leq \alpha_n^3 ||x_n - x^*|| + G_n + \beta_n^3 ||x_n - x^*|| + \gamma_n^3 ||u_n^3 - x^*||. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6),

$$\begin{aligned}
||y_n - x^*|| &\leq \alpha_n^2[\alpha_n^3||x_n - x^*|| + G_n + \beta_n^3||x_n - x^*|| + \gamma_n^3||u_n^3 - x^*|| \quad (2.8) \\
&\quad + G_n + \beta_n^2||x_n - x^*|| + \gamma_n^2||u_n^2 - x^*|| \\
&\leq \alpha_n^2\alpha_n^3||x_n - x^*|| + \alpha_n^2 G_n + \alpha_n^2\beta_n^3||x_n - x^*|| \\
&\quad + \alpha_n^2\gamma_n^3||u_n^3 - x^*|| + G_n + \beta_n^2||x_n - x^*|| + \gamma_n^2||u_n^2 - x^*|| \\
&\leq (1 - \beta_n^2 - \gamma_n^2)\alpha_n^3||x_n - x^*|| + (1 - \beta_n^2 - \gamma_n^2)\beta_n^3||x_n - x^*|| \\
&\quad + \beta_n^2||x_n - x^*|| + m_n \\
&\leq \beta_n^2||x_n - x^*|| + (1 - \beta_n^2)\alpha_n^3||x_n - x^*|| \\
&\quad + (1 - \beta_n^2)\beta_n^3||x_n - x^*|| + m_n \\
&\leq \beta_n^2||x_n - x^*|| + (1 - \beta_n^2)(\alpha_n^3 + \beta_n^3)||x_n - x^*|| + m_n \\
&\leq \beta_n^2||x_n - x^*|| + (1 - \beta_n^2)||x_n - x^*|| + m_n \\
&\leq ||x_n - x^*|| + m_n
\end{aligned}$$

where $m_n = 2G_n + \gamma_n^2||u_n^2 - x^*|| + \gamma_n^3||u_n^3 - x^*||$. Note that $\sum_{n=1}^{\infty} m_n < \infty$. Now substituting (2.8) into (2.5), we have

$$\begin{aligned}
||x_{n+1} - x^*|| &\leq \alpha_n^1[||x_n - x^*|| + m_n] + G_n + \beta_n^1||x_n - x^*|| \quad (2.9) \\
&\quad + \gamma_n^1||u_n^1 - x^*|| \\
&\leq (\alpha_n^1 + \beta_n^1)||x_n - x^*|| + \alpha_n^1 m_n + G_n + \gamma_n^1||u_n^1 - x^*|| \\
&\leq ||x_n - x^*|| + m_n + G_n + \gamma_n^1||u_n^1 - x^*|| \\
&\leq ||x_n - x^*|| + 3G_n + (\gamma_n^1 + \gamma_n^2 + \gamma_n^3)M \\
&\leq ||x_n - x^*|| + b_n
\end{aligned}$$

where $b_n = 3G_n + (\gamma_n^1 + \gamma_n^2 + \gamma_n^3)M$. Since $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$, it follows that $\sum_{n=1}^{\infty} b_n < \infty$. Therefore, by Lemma 1.1, we have $\lim_{n\to\infty} ||x_n - x^*||$ exists. This completes the proof. □

**Lemma 2.3.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T\colon K \to E$ be asymptotically nonexpansive nonself mapping in the intermediate sense with $F(T) \neq \phi$. Put

$$G_n = \max\{\sup_{x,y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x - y||), 0\}, \quad \forall n \geq 1$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\alpha_n^3\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\beta_n^3\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ and $\{\gamma_n^3\}$ are real sequences in $[0, 1]$ and $\{u_n^1\}$, $\{u_n^2\}$ and $\{u_n^3\}$ are bounded sequences in $K$ such that

$(i)$ $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2 = \alpha_n^3 + \beta_n^3 + \gamma_n^3 = 1.$

$(ii)$ $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$.

$(iii)$ $0 \leq \alpha < \alpha_n^1, \alpha_n^2 \leq \beta < 1$.

Then

$(a)$ $\lim_{n\to\infty} ||T(PT)^{n-1}y_n - x_n|| = 0$;

$(b)$ $\lim_{n\to\infty} ||T(PT)^{n-1}z_n - x_n|| = 0$.

*Proof.* For any $x^* \in F(T)$, it follows from Lemma 2.2, we have $\lim_{n\to\infty} ||x_n - x^*||$ exists. Let $\lim_{n\to\infty} ||x_n - x^*|| = c$ for some $c \geq 0$. From (2.8), we have

$$||y_n - x^*|| \leq ||x_n - x^*|| + m_n, \quad \forall n \geq 1.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\begin{aligned}\limsup_{n\to\infty} ||y_n - x^*|| &\leq \limsup_{n\to\infty} ||x_n - x^*|| \\ &= \lim_{n\to\infty} ||x_n - x^*|| \\ &= c.\end{aligned}$$

Note that

$$\begin{aligned}\limsup_{n\to\infty} ||T(PT)^{n-1}y_n - x^*|| &\leq \limsup_{n\to\infty}(||y_n - x^*|| + G_n) \\ &= \limsup_{n\to\infty} ||y_n - x^*|| \\ &\leq c.\end{aligned}$$

Next consider

$$||T(PT)^{n-1}y_n - x^* + \gamma_n^2(u_n^2 - x_n)|| \leq ||T(PT)^{n-1}y_n - x^*|| + \gamma_n^2||u_n^2 - x_n||.$$

Thus

$$\limsup_{n\to\infty} ||T(PT)^{n-1}y_n - x^* + \gamma_n^2(u_n^2 - x_n)|| \leq c.$$

Also,

$$||x_n - x^* + \gamma_n^1(u_n^1 - x_n)|| \leq ||x_n - x^*|| + \gamma_n^1||u_n^1 - x_n||$$

gives that

$$\limsup_{n\to\infty} ||x_n - x^* + \gamma_n^1(u_n^1 - x_n)|| \leq c$$

and

$$\begin{aligned} c &= \lim_{n\to\infty} ||x_{n+1} - x^*|| \\ &\le \liminf_{n\to\infty} ||\alpha_n^1 T(PT)^{n-1} y_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x^*|| \\ &= \liminf_{n\to\infty} ||\alpha_n^1 [(T(PT)^{n-1} y_n - x^*) + \frac{\gamma_n^1}{2\alpha_n^1}(u_n^1 - x^*)] \\ &\quad + \beta_n^1 [(x_n - x^*) + \frac{\gamma_n^1}{2\beta_n^1}(u_n^1 - x^*)]|| \\ &\le \liminf_{n\to\infty} \alpha_n^1 ||T(PT)^{n-1} y_n - x^*|| + \liminf_{n\to\infty} \beta_n^1 ||x_n - x^*|| \\ &\le \liminf_{n\to\infty} \alpha_n^1 [||y_n - x^*|| + G_n] + \liminf_{n\to\infty} \beta_n^1 ||x_n - x^*|| \\ &\le \liminf_{n\to\infty} [\alpha_n^1 (||x_n - x^*|| + G_n) + (1 - \alpha_n^1)||x_n - x^*||] \\ &\le \liminf_{n\to\infty} [||x_n - x^*|| + \alpha_n^1 G_n] = c. \end{aligned}$$

Hence

$$\begin{aligned} c &= \lim_{n\to\infty} ||\alpha_n^1 [(T(PT)^{n-1} y_n - x^*) + \frac{\gamma_n^1}{2\alpha_n^1}(u_n^1 - x^*)] \\ &\quad + \beta_n^1 [(x_n - x^*) + \frac{\gamma_n^1}{2\beta_n^1}(u_n^1 - x^*)]|| \\ &= \lim_{n\to\infty} ||\alpha_n^1 [(T(PT)^{n-1} y_n - x^*) + \frac{\gamma_n^1}{2\alpha_n^1}(u_n^1 - x^*)] \\ &\quad + (1 - \alpha_n^1)[(x_n - x^*) + \frac{\gamma_n^1}{2\beta_n^1}(u_n^1 - x^*)]||. \end{aligned}$$

By Lemma 1.2, we have

$$\lim_{n\to\infty} ||T(PT)^{n-1} y_n - x_n + (\frac{\gamma_n^1}{2\alpha_n^1} - \frac{\gamma_n^1}{2\beta_n^1})(u_n^1 - x^*)|| = 0.$$

Since

$$\lim_{n\to\infty} ||(\frac{\gamma_n^1}{2\alpha_n^1} - \frac{\gamma_n^1}{2\beta_n^1})(u_n^1 - x^*)|| = 0,$$

we obtain that

$$\lim_{n\to\infty} ||T(PT)^{n-1} y_n - x_n|| = 0.$$

This completes the proof of $(a)$.

$(b)$ For each $n \ge 1$,

$$\begin{aligned} ||x_n - x^*|| &\le ||x_n - T(PT)^{n-1} y_n|| + ||T(PT)^{n-1} y_n - x^*|| \\ &\le ||x_n - T(PT)^{n-1} y_n|| + ||y_n - x^*|| + G_n. \end{aligned}$$

Since $\lim_{n\to\infty}||x_n - T(PT)^{n-1}y_n|| = 0 = \lim_{n\to\infty} G_n$, we obtain that

$$c = \lim_{n\to\infty}||x_n - x^*|| \leq \liminf_{n\to\infty}||y_n - x^*||.$$

It follows that

$$c = \liminf_{n\to\infty}||y_n - x^*|| \leq \limsup_{n\to\infty}||y_n - x^*|| \leq c.$$

This implies that

$$\lim_{n\to\infty}||y_n - x^*|| = c.$$

On the other hand, we note that

$$\begin{aligned}
||z_n - x^*|| &= ||P(\alpha_n^3 T(PT)^{n-1}x_n + \beta_n^3 x_n + \gamma_n^3 u_n^3) - Px^*|| \\
&\leq ||(\alpha_n^3 T(PT)^{n-1}x_n + \beta_n^3 x_n + \gamma_n^3 u_n^3) - x^*|| \\
&\leq \alpha_n^3||T(PT)^{n-1}x_n - x^*|| + \beta_n^3||x_n - x^*|| + \gamma_n^3||u_n^3 - x^*|| \\
&\leq \alpha_n^3(||x_n - x^*|| + G_n) + \beta_n^3||x_n - x^*|| + \gamma_n^3||u_n^3 - x^*|| \\
&\leq \alpha_n^3||x_n - x^*|| + G_n + (1-\alpha_n^3)||x_n - x^*|| + \gamma_n^3||u_n^3 - x^*|| \\
&\leq ||x_n - x^*|| + G_n + \gamma_n^3||u_n^3 - x^*||
\end{aligned}$$

By boundedness of $\{u_n^3\}$ and $\lim_{n\to\infty} G_n = 0 = \lim_{n\to\infty}\gamma_n^3$, we have

$$\limsup_{n\to\infty}||z_n - x^*|| \leq \limsup_{n\to\infty}||x_n - x^*|| = c,$$

and

$$\limsup_{n\to\infty}||T(PT)^{n-1}z_n - x^*|| \leq \limsup_{n\to\infty}(||z_n - x^*|| + G_n) \leq c.$$

Next, consider

$$||T(PT)^{n-1}z_n - x^* + \gamma_n^3(u_n^3 - x_n)|| \leq ||T(PT)^{n-1}z_n - x^*|| + \gamma_n^3||u_n^3 - x_n||.$$

Thus

$$\limsup_{n\to\infty}||T(PT)^{n-1}z_n - x^* + \gamma_n^3(u_n^3 - x_n)|| \leq c.$$

Also,

$$||x_n - x^* + \gamma_n^3(u_n^3 - x_n)|| \leq ||x_n - x^*|| + \gamma_n^3||u_n^3 - x_n||,$$

gives that

$$\limsup_{n\to\infty}||x_n - x^* + \gamma_n^3(u_n^3 - x_n)|| \leq c$$

and

$$\begin{aligned} c &= \lim_{n\to\infty} ||y_n - x^*|| \\ &\leq \liminf_{n\to\infty} ||\alpha_n^2 T(PT)^{n-1} z_n + \beta_n^2 x_n + \gamma_n^2 u_n^2 - x^*|| \\ &= \liminf_{n\to\infty} ||\alpha_n^2[(T(PT)^{n-1} z_n - x^*) + \frac{\gamma_n^2}{2\alpha_n^2}(u_n^2 - x^*)] \\ &\quad + \beta_n^2[(x_n - x^*) + \frac{\gamma_n^2}{2\beta_n^2}(u_n^2 - x^*)]|| \\ &\leq \liminf_{n\to\infty} \alpha_n^2 ||T(PT)^{n-1} z_n - x^*|| + \liminf_{n\to\infty} \beta_n^2 ||x_n - x^*|| \\ &\leq \liminf_{n\to\infty} \alpha_n^2[||z_n - x^*|| + G_n] + \liminf_{n\to\infty} \beta_n^2 ||x_n - x^*|| \\ &\leq \liminf_{n\to\infty} [\alpha_n^2(||x_n - x^*|| + G_n) + (1 - \alpha_n^2)||x_n - x^*||] \\ &\leq \liminf_{n\to\infty} [||x_n - x^*|| + \alpha_n^2 G_n] = c. \end{aligned}$$

Hence

$$\begin{aligned} c &= \lim_{n\to\infty} ||\alpha_n^2[(T(PT)^{n-1} z_n - x^*) + \frac{\gamma_n^2}{2\alpha_n^2}(u_n^2 - x^*)] \\ &\quad + \beta_n^2[(x_n - x^*) + \frac{\gamma_n^2}{2\beta_n^2}(u_n^2 - x^*)]|| \\ &= \lim_{n\to\infty} ||\alpha_n^2[(T(PT)^{n-1} z_n - x^*) + \frac{\gamma_n^2}{2\alpha_n^2}(u_n^2 - x^*)] \\ &\quad + (1 - \alpha_n^2)[(x_n - x^*) + \frac{\gamma_n^2}{2\beta_n^2}(u_n^2 - x^*)]||. \end{aligned}$$

By Lemma 1.2, we have

$$\lim_{n\to\infty} ||T(PT)^{n-1} z_n - x_n + (\frac{\gamma_n^2}{2\alpha_n^2} - \frac{\gamma_n^2}{2\beta_n^2})(u_n^2 - x^*)|| = 0.$$

Since

$$\lim_{n\to\infty} ||(\frac{\gamma_n^2}{2\alpha_n^2} - \frac{\gamma_n^2}{2\beta_n^2})(u_n^2 - x^*)|| = 0,$$

we obtain that

$$\lim_{n\to\infty} ||T(PT)^{n-1} z_n - x_n|| = 0.$$

This completes the proof of $(b)$. □

**Lemma 2.4.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T\colon K \to E$ be asymptotically nonexpansive nonself mapping in the intermediate sense with $F(T) \neq \phi$. Put

$$G_n = \max\{\sup_{x,y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x-y||), 0\}, \qquad \forall n \geq 1$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\alpha_n^3\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\beta_n^3\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ and $\{\gamma_n^3\}$ are real sequences in $[0,1]$ and $\{u_n^1\}$, $\{u_n^2\}$ and $\{u_n^3\}$ are bounded sequences in $K$ such that

(*i*) $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2 = \alpha_n^3 + \beta_n^3 + \gamma_n^3 = 1.$

(*ii*) $\sum_{n=1}^{\infty}\gamma_n^1 < \infty$, $\sum_{n=1}^{\infty}\gamma_n^2 < \infty$, $\sum_{n=1}^{\infty}\gamma_n^3 < \infty$.

(*iii*) $0 \leq \alpha < \alpha_n^1, \alpha_n^2, \alpha_n^3 \leq \beta < 1.$

Then for all $u, v \in F(T)$, the limit

$$\lim_{n\to\infty} ||tx_n + (1-t)u - v||$$

exists for all $t \in [0,1]$.

*Proof.* By Lemma 2.2, we have $\lim_{n\to\infty}||x_n - x^*||$ exists for all $x^* \in F(T)$. This implies that $\{x_n\}$ is bounded. Observe that there exists $R > 0$ such that $\{x_n\} \subset C = B_R(0) \cap K$, where $B_R(0) = \{x \in E : ||x|| \leq R\}$. Then $C$ is a nonempty closed convex bounded subset of $E$. Let $a_n(t) = ||tx_n + (1-t)u - v||$. Then $\lim_{n\to\infty} a_n(0) = ||u-v||$ and from Lemma 2.2, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}||x_n - v||$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty}||x_n - u|| = r > 0$ and $t \in (0,1)$. Define $T_n\colon C \to C$ by

$$\begin{aligned} T_n x \quad &= \quad P(\alpha_n^1 T(PT)^{n-1}(P(\alpha_n^2 T(PT)^{n-1})(P(\alpha_n^3 T(PT)^{n-1}x) \\ &\quad\quad +\beta_n^3 x + \gamma_n^3 u_n^3) + \beta_n^2 x + \gamma_n^2 u_n^2) + \beta_n^1 x + \gamma_n^1 u_n^1), \end{aligned}$$

$x \in K$ and set $S_{n,m} = T_{n+m-1}T_{n+m-2}\dots T_n$, $m \geq 1$. Then

$$\begin{aligned} ||S_{n,m}x - S_{n,m}y|| \quad &\leq \quad ||x-y|| + G_{n+m-1} + G_{n+m-2} + \cdots + G_n \\ &\leq \quad ||x-y|| + \sum_{j=n}^{n+m-1} G_j \end{aligned}$$

Observe that $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}y = y$, for all $y \in F(T)$.

Set

$$b_{n,m} := ||x_n - u||[S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u];$$

$$D_{n,m} := [S_{n,m}u + S_{n,m}x_n - 2S_{n,m}(tx_n + (1-t)u)] \sum_{j=n}^{n+m-1} G_j;$$

$$M_{n,m} := \big[t||x_n - u|| + \sum_{j=n}^{n+m-1} G_j\big] \times \big[(1-t)||x_n - u|| + \sum_{j=n}^{n+m-1} G_j\big];$$

$$F_{n,m} := [tS_{n,m}u + (1-2t)S_{n,m}(tx_n + (1-t)u) - (1-t)S_{n,m}x_n] \times \sum_{j=n}^{n+m-1} G_j$$

and

$$L_n := [t||x_n - u|| + L] \times [(1-t)||x_n - u|| + L], \text{ where } L = \sum_{j=n}^{n+m-1} G_j.$$

It is well known (see, for example, Bruck [20], p.108) that if $E$ is uniformly convex,

$$\begin{aligned} ||tx + (1-t)y|| &\leq 1 - 2\min\{t, (1-t)\}\delta_E(||x-y||) \\ &\leq 1 - 2t(1-t)\delta_E(||x-y||) \end{aligned} \tag{2.10}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $||x|| \leq 1$, $||y|| \leq 1$. Set

$$W_{n,m} := \frac{S_{n,m}u - S_{n,m}(tx_n + (1-t)u)}{t||x_n - u|| + \sum_{j=n}^{n+m-1} G_j};$$

$$Z_{n,m} := \frac{S_{n,m}(tx_n + (1-t)u) - S_{n,m}x_n}{(1-t)||x_n - u|| + \sum_{j=n}^{n+m-1} G_j}.$$

Then $||W_{n,m}|| \leq 1$ and $||Z_{n,m}|| \leq 1$ so that it follows from (2.10) that

$$2t(1-t)\delta_E(||W_{n,m} - Z_{n,m}||) \leq 1 - ||tW_{n,m} + (1-t)Z_{n,m}||. \tag{2.11}$$

Observe that

$$||W_{n,m} - Z_{n,m}|| = \frac{||b_{n,m} - D_{n,m}||}{M_{n,m}}$$

and

$$||tW_{n,m} + (1-t)Z_{n,m}|| = \frac{||t(1-t)||x_n - u||[S_{n,m}u - S_{n,m}x_n] + F_{n,m}||}{M_{n,m}}.$$

From (2.11), we then obtain that

$$2M_{n,m}\delta_E\big(\frac{||b_{n,m}-D_{n,m}||}{M_{n,m}}\big) \leq ||x_n-u||^2 + \frac{1}{t(1-t)}\sum_{j=n}^{n+m-1} G_j\big[||x_n-u|| + \sum_{j=n}^{n+m-1} G_j\big] - ||x_n-u||\cdot||x_{n+m}-u|| + \frac{||F_{n,m}||}{t(1-t)}. \tag{2.12}$$

Observe that $L_n \geq M_{n,m}$ for all $n, m$. Since $E$ is uniformly convex, then $\frac{\delta_E(s)}{s}$ is nondecreasing and hence from (2.12) that

$$2L_n\delta_E\big(\frac{||b_{n,m}-D_{n,m}||}{L_n}\big) \leq ||x_n-u||^2 + \frac{1}{t(1-t)}\sum_{j=n}^{n+m-1} G_j\big[||x_n-u|| + \sum_{j=n}^{n+m-1} G_j\big] - ||x_n-u||\cdot||x_{n+m}-u|| + \frac{||F_{n,m}||}{t(1-t)}.$$

Since $\lim_{n\to\infty}||x_n-u||$ exists, we have that

$$\lim_{n\to\infty}||x_n-u|| = \lim_{n\to\infty}||x_{n+m}-u||.$$

Moreover, $\delta_E(0)=0$, the continuity of $\delta_E$ gives from inequality (2.13) that $\liminf_n(\limsup_m ||b_{n,m}-D_{n,m}||)=0$. Observe that

$$\begin{aligned}\limsup_m ||b_{n,m}|| &\leq \limsup_m ||b_{n,m}-D_{n,m}|| + \limsup_m ||D_{n,m}|| \\ &= \limsup_m ||b_{n,m}-D_{n,m}|| + K_n\sum_{j=n}^{\infty} G_j\end{aligned}$$

for some bounded sequence $\{K_n\}$. Since

$$\sum_{j=n}^{\infty} G_j \to 0 \text{ as } n\to\infty \quad and \quad \liminf_n(\limsup_m ||b_{n,m}-D_{n,m}||)=0,$$

it follows that $\liminf_n(\limsup_m ||b_{n,m}||)=0$. Hence,

$$\liminf_n(\limsup_m ||S_{n,m}(tx_n+(1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u||)=0.$$

Clearly,

$$\begin{aligned} a_{n+m}(t) \leq & ||tx_{n+m} + (1-t)u - v \\ & + (S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u)|| \\ & + || - [S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u]|| \\ = & ||S_{n,m}(tx_n + (1-t)u) - v|| + ||S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n \\ & - (1-t)S_{n,m}u|| \\ \leq & ||tx_n + (1-t)u) - v|| + \sum_{j=n}^{n+m-1} G_j \\ & + ||S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u|| \\ \leq & a_n(t) + \sum_{j=n}^{n+m-1} G_j \\ & + ||S_{n,m}(tx_n + (1-t)u) - tS_{n,m}x_n - (1-t)S_{n,m}u||. \end{aligned}$$

Hence $\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t)$. This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty} ||tx_n + (1-t)u - v||$$

exists for all $t \in [0,1]$. This completes the proof.

□

**Theorem 2.5.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be asymptotically nonexpansive nonself mapping in the intermediate sense with $F(T) \neq \phi$. Put

$$G_n = \max\{\sup_{x,y\in K}(||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x-y||), 0\}, \quad \forall n \geq 1$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\alpha_n^3\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\beta_n^3\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ and $\{\gamma_n^3\}$ are real sequences in $[0,1]$ and $\{u_n^1\}$, $\{u_n^2\}$ and $\{u_n^3\}$ are bounded sequences in $K$ such that

(*i*) $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2 = \alpha_n^3 + \beta_n^3 + \gamma_n^3 = 1$.

(*ii*) $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$.

(*iii*) $0 \leq \alpha < \alpha_n^1, \alpha_n^2 \leq \beta < 1$.

Suppose $T$ satisfies condition $(A)$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of $T$.

*Proof.* It follows from Lemma 2.3, that

$$\lim_{n\to\infty} ||T(PT)^{n-1}y_n - x_n|| = 0 = \lim_{n\to\infty} ||T(PT)^{n-1}z_n - x_n||$$

and this implies that

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq \alpha_n^1||T(PT)^{n-1}y_n - x_n|| + \gamma_n^1||u_n^1 - x_n|| \\ &\to 0 \ as\ n \to \infty. \end{aligned} \tag{2.13}$$

Thus

$$\begin{aligned} ||T(PT)^{n-1}x_n - x_n|| &\leq ||T(PT)^{n-1}x_n - T(PT)^{n-1}y_n|| \\ &\quad +||T(PT)^{n-1}y_n - x_n|| \\ &\leq ||x_n - y_n|| + G_n + ||T(PT)^{n-1}y_n - x_n|| \\ &\leq \alpha_n^2||x_n - T(PT)^{n-1}z_n|| + \gamma_n^2||u_n^2 - x_n|| + G_n \\ &\quad +||T(PT)^{n-1}y_n - x_n|| \\ &\quad \to 0 \ as\ n \to \infty. \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} ||x_n - Tx_n|| &\leq ||x_{n+1} - x_n|| + ||x_{n+1} - T(PT)^n x_{n+1}|| \\ &\quad +||T(PT)^n x_{n+1} - T(PT)^n x_n|| + ||T(PT)^n x_n - Tx_n|| \end{aligned}$$

by uniform continuity of $T$ and from (2.13) and (2.14), we have

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{2.15}$$

By Lemma 2.2, $\lim_{n\to\infty} ||x_n - x^*||$ exists for all $x^* \in F(T)$. Let $\lim_{n\to\infty} ||x_n - x^*|| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By (2.15), we know that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, and Lemma 2.2 gives that

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| + b_n,$$

where $b_n = 3G_n + (\gamma_n^1 + \gamma_n^2 + \gamma_n^3)M$. That is,

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + b_n.$$

Since $\sum_{n=1}^{\infty} b_n < \infty$, so by Lemma 1.1 gives that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Since $T$ satisfies condition $(A)$, we have $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f$ is a nondecreasing function and $f(0) = 0$, we conclude that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Now we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{y_j\} \subset F(T)$ such that $||x_{n_j} - y_j|| < 2^{-j}$. Then we have

$$\begin{aligned} ||x_{n_j+1} - y_j|| &\leq ||x_{n_j} - y_j|| + G_n \\ &\leq ||x_{n_j} - y_j|| + \frac{M}{2^j} \\ &< 2^{-j} + \frac{M}{2^j} \\ &< \frac{M+1}{2^j}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} G_n < \infty$, so there exists $M > 0$ such that $G_n < \frac{M}{2^j}$ and hence

$$\begin{aligned} ||y_{j+1} - y_j|| &\leq ||y_{j+1} - x_{n_j+1}|| + ||x_{n_j+1} - y_j|| \\ &\leq 2^{-(j+1)} + \frac{M+1}{2^j} \\ &< \frac{2M+3}{2^{j+1}}. \end{aligned}$$

This shows that $\{y_j\}$ is a Cauchy sequence and therefore converges strongly to an element of $E$. Assume that $y_j \to y$ as $j \to \infty$. Then $y \in F(T)$ since $F(T)$ is closed, which implies that $x_{n_j} \to y$ as $j \to \infty$. This shows that $\{x_n\}$ converges strongly to some fixed point of $T$. Again since

$$||y_n - x_n|| \leq \alpha_n^2 ||T(PT)^{n-1} z_n - x_n|| + \gamma_n^2 ||u_n^2 - x_n|| \to 0, \ as \ n \to \infty$$

and

$$||z_n - x_n|| \leq \alpha_n^3 ||T(PT)^{n-1} x_n - x_n|| + \gamma_n^3 ||u_n^3 - x_n|| \to 0 \ as \ n \to \infty.$$

Therefore $\lim_{n\to\infty} y_n = y = \lim_{n\to\infty} z_n$. This completes the proof. □

**Theorem 2.6.** Let $E$ be a uniformly convex Banach space such that its dual $E^*$ has the *Kadec-Klee* property and $K$ a nonempty closed convex subset which is also a nonexpansive retract of $E$. Let $T : K \to E$ be an asymptotically nonexpansive nonself mapping in the intermediate sense with $F(T) \neq \phi$. Put

$$G_n = \max\{ \sup_{x,y\in K} (||T(PT)^{n-1}x - T(PT)^{n-1}y|| - ||x - y||), 0\}, \quad \forall n \geq 1$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1.8), where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\alpha_n^3\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\beta_n^3\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ and $\{\gamma_n^3\}$ are real sequences in $[0, 1]$ and $\{u_n^1\}$, $\{u_n^2\}$ and $\{u_n^3\}$ are bounded sequences in $K$ such that

(*i*) $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2 = \alpha_n^3 + \beta_n^3 + \gamma_n^3 = 1.$

(*ii*) $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$.

(*iii*) $0 \leq \alpha < \alpha_n^1, \alpha_n^2, \alpha_n^3 \leq \beta < 1$.

Then $\{x_n\}$ converges weakly to some fixed point of $T$.

*Proof.* By Lemma 2.2, we have $\lim_{n\to\infty} ||x_n - x^*||$ exists for all $x^* \in F(T)$. This implies that $\{x_n\}$ is bounded. Since $E$ is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to some $x^* \in K$.

By Theorem 2.5 (equ. (2.15)), we have $\lim_{j\to\infty} ||x_{n_j} - Tx_{n_j}|| = 0$. Since $T$ is uniformly continuous, we can get that

$$\lim_{m\to\infty} (\limsup_{j\to\infty} ||T(PT)^{m-1}x_{n_j} - x_{n_j}||) = 0.$$

Now Lemma 1.3 guarantees that $Tx^* = x^*$, hence this means that $x^* \in F(T)$. It remains to show that $\{x_n\}$ converges weakly to $x^*$. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $y^*$. Then $y^* \in K$ and so $x^*, y^* \in \omega_w(x_n) \cap F(T)$. By Lemma 2.4, the limit

$$\lim_{n\to\infty} ||tx_n + (1-t)x^* - y^*||$$

exists for all $t \in [0, 1]$. By Lemma 1.4, we have $x^* = y^*$. As a result, $\omega_w(x_n) \cap F(T)$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of $T$. This completes the proof. □

**Remark 2.7.** Our results extend the corresponding results of Su and Qin [32] to the case of three step iterative sequences with errors for more general class of asymptotically nonexpansive nonself mappings. Also our iteration scheme generalizes the scheme of [32].

**Remark 2.8.** Our results also extend the corresponding results of Chidume et al [6] to the case of three step iterative sequences with errors.

**Remark 2.9.** Our results also extend the corresponding results of Plubtieng and Wangkeeree ( [22]) to the case of nonself mappings.

**Remark 2.10.** Our results also extend the corresponding results of Chidume et al [4] to the case of three step iterative sequences with errors and more general class of asymptotically nonexpansive nonself mappings.

**Remark 2.11.** [6]: It is well known that duals of reflexive Banach spaces with Fréchet differentiable norm have the *Kadec-Klee* property. However, it is worth mentioning that there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor satisfy Opial's condition but their duals do have the *Kadec-Klee* property. To see this, consider $X = \mathbb{R}^2$ with the norm given by $|x| = \sqrt{||x_1||^2 + ||x_2||^2}$ and $Y = L^p[0,1]$ with $1 < p < \infty$ and $p \neq 2$. Then the Cartesian product $X \times Y$ equipped with the $\ell^2$-norm is uniformly convex, it does not satisfy Opial's condition, and its norm is not Fréchet differentiable. However, its dual does have the *Kadec-Klee* property. For details, see Falset et al. [10] and Kaczor [28].

**Remark 2.12.** Theorem 2.4 extends Theorem 1.5 of Schu [11] and the corresponding result of Rhoades [1], and Osilike and Aniagbosor [18] to the case of more general class of nonself mappings and three step iteration scheme with errors. Furthermore, no boundedness condition is imposed on $K$. Under the additional hypothesis that the dual $E^*$ of $E$ has the *Kadec-Klee* property, Theorem 2.6 generalizes Theorem 2.1 of Schu [12] to the case of nonself maps and three step iteration scheme with errors in Banach spaces that includes $L_p$ spaces $(1 < p < \infty)$, with Opial's condition and boundedness of $K$ dispensed with. Since duals of reflexive Banach spaces with Fréchet differentiable norms have the *Kadec-Klee* property, Theorem 2.6 extends Theorem 3.1 of Tan and Xu [17] to the case of nonself maps which are asymptotically nonexpansive in the intermediate sense and three step iteration scheme with errors, with boundedness of $K$ dispensed with.

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