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SIMULTANEOUS FARTHEST POINTS IN VECTOR VALUED FUNCTION SPACES

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Abstract. Let X be a Banach space, (I,μ) be a finite measure space and G be a closed bounded subset of X. Let φ be an increasing subadditive continuous function on $[0,\infty)$ with $\varphi(0)=0$, let us denote by $L^{\varphi}(I,X)$, the space of all X-valued strongly measurable functions on I with $\int_I \varphi \|f(t)\|\, dt < \infty$. In this paper, we show that for a separable simultaneously remotal set G in X, $L^{\varphi}(I,G)$ is simultaneously remotal in $L^{\varphi}(I,X)$. Further, we study Banach space X with subspace Y such that $L^{\varphi}(I,B[Y])$ is remotal in $L^{\varphi}(I,X)$, where $B[Y]$ is the unit ball of Y.

1. Introduction

A function $\varphi:[0,\infty)\to[0,\infty)$ is called a modulus function if the following hold:

(1) φ is continuous and increasing function
(2) $\varphi(x)=0$ if and only if $x=0$.
(3) $\varphi(x+y)\le\varphi(x)+\varphi(y)$.

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The functions $\varphi(x) = x^p$, $0 < p < 1$ and $\varphi(x) = \ln(x+1)$ are modulus functions. In fact if φ is a modulus function, then also $\psi(x) = \frac{\varphi(x)}{1+\varphi(x)}$.

For a modulus function φ and a Banach space X, let us denote by $L^\varphi(I,X)$ the space of all X-valued strongly measurable functions on the interval I with $\int_I \varphi \|f(t)\| \, dt < \infty$. For $f \in L^\varphi(I,X)$, set $\|f\|_\varphi = \int_I \varphi \|f(t)\| \, dt$. It is known that $L^\varphi(I,X)$ is a metric space. For more on $L^\varphi(I,X)$ we refer to [5], [6], [7].

Let X be a Banach space and G be a bounded subset of X. For $x \in X$, set $\rho(x,G) = \sup\{\|x-y\| : y \in G\}$. A point $g_0 \in G$ is called a farthest point of G from the point $x \in X$, if it satisfies: $\|x - g_0\| = \rho(x,G)$. For $x \in X$ the farthest point map $F_G(x) = \{g \in G : \|x-g\| = \rho(x,G)\}$ i.e. the set of points of G farthest from x. Note that, this set may be empty. Let $R(G,X) = \{x \in X : F_G(x) \neq \phi\}$. Call a closed bounded set G remotal if $R(G,X) = X$ and densely remotal if $R(G,X)$ is norm dense in X. The concept of remotal sets in Banach spaces goes back to the sixties. However, almost all the results on remotal sets are concerned with the topological properties of such sets, see [1], [3], [8], [9], [11]. The study of remotal sets is little more difficult and less developed than that of proximal sets. While best approximation has applications in many areas of mathematics, remotal sets and farthest points have applications in the study of geometry of Banach spaces. Remotal sets in vector valued continuous functions were considered in [4]. Related results on Bochner integrable function spaces, $L^p(I,X)$, $1 < p \leq \infty$, are given in [2], [10] and [13].

The problem of approximating a set of points $x_1, x_2, ..., x_m$ simultaneously by a point g (farthest point) in a subset G of X can be done in several ways, see [12]. Here we will follow the following definition:

Definition 1.1. *Let φ be a modulus function and G a closed bounded subset of X. A point $g \in G$ is called a simultaneous farthest point of $(x_1, x_2, ..., x_m) \in X^m$, if*

$$\sum_{i=1}^{m} \varphi \|x_i - g\| = \sup_{h \in G} \sum_{i=1}^{m} \varphi \|x_i - h\| .$$

We call a closed bounded set G of a Banach space X simultaneously remotal if each m-tuple $(x_1, x_2, ..., x_m) \in X^m$ admits a farthest point in G and simultaneously densely remotal if the set of points of G such that $\{x \in X : F_G(x) \neq \phi\}$ where $F_G(x) = \{g \in G : \rho(x_1, x_2, ..., x_m, G) = \sum_{i=1}^{m} \varphi \|x_i - g\|\}$ is norm dense in X. If $m = 1$ and $\varphi(x) = x$, simultaneously remotal is precisely remotal.

In section 2 of this paper, we calculate a formula for the farthest distance in L^φ-spaces. Section 3, studies the problem of simultaneous farthest points for bounded sets of the form $L^\varphi(I,G)$ in the metric space $L^\varphi(I,X)$, when G

is separable in X. In section 4, we give some results about remotality when the subspaces possess some M-structure. Throughout this paper, consider $(I, \sum, \mu,)$ to be a finite measure space, X a Banach space, G a closed subset of X and $L^{\varphi}(I, X)$ the space of all X-valued strongly measurable functions on I with $\int_I \varphi \|f(t)\| \, dt < \infty$, where φ is a modulus function.

2. Distance Formula

Progress in the discussion of the farthest points when X does not possess pleasant properties is greatly facilitated by the fact that the φ-farthest distance from an element $f \in L^{\varphi}(I, X)$ to a bounded subset $L^{\varphi}(I, G)$ can be computed by the following theorem:

Theorem 2.1. *Let G be a closed bounded subset of X. Then for each m-tuple $(f_1, f_2, ..., f_m) \in (L^{\varphi}(I, G))^m$*

$$\rho_{\varphi}(f_1, f_2, ..., f_m, L^{\phi}(I, G)) = \int_I \rho_{\varphi}(f_1(t), f_2\,(t), ..., f_m(t), G) dt.$$

Proof. Let $(f_1, f_2, ..., f_m) \in (L^{\varphi}(I, X))^m$. Then for each $i, 1 \le i \le m$, f_i is strongly measurable and so is the limit of almost everywhere of a sequence of simple functions (f_{n_i}) in $L^{\varphi}(I, X)$ such that $\|f_{n_i}(s) - f_i(s)\| \to 0$. The inequality

$$\sum_{i=1}^{n} \varphi \|f_{n_i}(s) - g\| \le \sum_{i=1}^{n} \varphi \|f_{n_i}(s) - f_i(s)\| + \sum_{i=1}^{n} \varphi \|f_i(s) - g\|,$$

implies that

$$\begin{aligned}\rho_{\varphi}(f_{n_1}(s), f_{n_2}(s), ..., f_{n_m}(s), G) \quad \le \quad & \sum_{i=1}^{m} \varphi \|f_{n_i}(s) - f_i(s)\| \\ & + \rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G)\end{aligned}$$

and hence we get

$$|\rho_{\varphi}(f_{n_1}(s), f_{n_2}(s), ..., f_{n_m}(s), G) - \rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G)| \to 0$$

as $n \to \infty$ for almost all s in I.

Now, set $H_n(s) = \rho_{\varphi}(f_{n_1}(s), f_{n_2}(s), ..., f_{n_m}(s))$. Then for each n, H_n is a measurable function. Thus $\rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G)$ is measurable and

$$\rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) \ge \sum_{i=1}^{m} \varphi \|f_i(s) - z\| \text{ ,for all } z \text{ in } G.$$

Therefore, for all $g \in L^{\varphi}(I,G)$ we have

$$\rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) \geq \sum_{i=1}^{m} \varphi \left\| f_i(s) - g(s) \right\|$$

Further,

$$\begin{aligned} \left\| \rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) \right\|_1 &= \int_I \rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) ds \\ &\geq \int_I \sum_{i=1}^{m} \varphi \left\| f_i(s) - g(s) \right\| ds \\ &= \sum_{i=1}^{m} \left\| f_i - g \right\|_{\varphi} \ , \text{for every } g \in L^{\varphi}(I,G). \end{aligned}$$

Since G is bounded it follows that $\rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) \in L^1(I)$ and

$$\int_I \rho_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) ds \geq \rho_{\varphi}(f_1, f_2, ..., f_m, L^{\varphi}(I,G)). \tag{2.1}$$

For the other direction, we will use the fact that simple functions are dense in $L^{\varphi}(I,X)$. So, for $f_i \in L^{\varphi}(I,X)$ and $\epsilon > 0$, there exists a simple function γ_i such that $\left\| f_i - \gamma_i \right\|_{\varphi} < \frac{\epsilon}{2m}$. With no loss of generality, we can write

$$\gamma_i(t) = \sum_{j=1}^{n} \chi_{A_j}(t) y_{ij},$$

where χ_{A_j} is the characteristic function of the set $A_j \subseteq I$ and $y_{ij} \in X$. We may assume that $\sum_{j=1}^{n} \chi_{A_j} = 1$ and $\mu(A_j) > 0$ for all j. Now, for each m-tuple $(y_{1j}, y_{2j}, ..., y_{mj})$, $1 \leq j \leq n$, there exists $g_j \in G$ such that

$$\rho_{\varphi}(y_{1j}, y_{2j}, ..., y_{mj}, G) \leq \sum_{i=1}^{m} \varphi \left\| y_{ij} - g_j \right\| + \frac{\epsilon}{2n\mu(A_j)}.$$

Set $g(t) = \sum\limits_{i=1}^{n} \chi_{A_i}(t) g_i$. It is clear that $g \in L^{\varphi}(I, X)$ and

$$\begin{aligned}
\int\limits_I \rho_\varphi(\gamma_1(t), \gamma_2(t), .., \gamma_m(t), G)dt &= \sum_{j=1}^{n} \int\limits_{A_j} \rho_\varphi(\gamma_1(t), \gamma_2(t), .., \gamma_m(t), G)dt \\
&= \sum_{j=1}^{n} \int\limits_{A_j} \rho_\varphi(y_{1j}, y_{2j}, ..., y_{mj}, G)dt \\
&= \sum_{j=1}^{n} \rho_\varphi(y_{1j}, y_{2j}, ..., y_{mj}, G)\mu(A_j) \\
&\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \varphi \|y_{ij} - g_j\| + \frac{\epsilon}{2n\mu(A_j)} \right) \mu(A_j) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} \varphi(\|y_{ij} - g_j\|)\mu(A_j) + \frac{\epsilon}{2}. \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \int\limits_{A_j} \varphi(\|\gamma_i(t) - g(t)\|)dt + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{m} \int\limits_I \varphi \|\gamma_i(t) - g(t)\| \, dt + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{m} \|\gamma_i - g\|_\varphi + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{m} (\|\gamma_i - f_i\|_\varphi + \|f_i - g\|_\varphi) + \frac{\epsilon}{2} \\
&< \epsilon + \sum_{i=1}^{m} \|f_i - g\|_\varphi .
\end{aligned}$$

The inequality

$$\sum_{j=1}^{m} \varphi \|f_j(t) - a\| \leq \sum_{j=1}^{m} \varphi \|\gamma_j(t) - a\| + \sum_{j=1}^{m} \varphi \|\gamma_j(t) - f_j(t)\|$$

implies that

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) \leq \rho_\varphi(\gamma_1(t), \gamma_2(t), ..., \gamma_m(t), G) + \sum_{j=1}^{m} \varphi \|\gamma_j(t) - f_j(t)\|$$

and this implies

$$\begin{aligned}\int_I \rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G)dt &\leq \int_I \rho_\varphi(\gamma_1(t), \gamma_2(t), ..., \gamma_m(t), G)dt + \frac{\epsilon}{2}. \\ &< \frac{3\epsilon}{2} + \sum_{i=1}^{m} \|f_i - g\|_\varphi \\ &\leq \frac{3\epsilon}{2} + \rho_\varphi(f_1, f_2, ..., f_m, L^\varphi(I, G)).\end{aligned}$$

Since ϵ was arbitrary we get:

$$\int_I \rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G)dt \leq \rho_\varphi(f_1, f_2, ..., f_m, L^\varphi(I, G)) \tag{2.2}$$

by (2.1) and (2.2) above, we get the result. □

Corollary 2.2. *Let G be a simultaneously remotal subset of X. Then $g \in L^\varphi(I, G)$ is a simultaneous farthest point in $L^\varphi(I, G)$ from $(f_1, f_2, ..., f_m) \in (L^\varphi(I, X))^m$ if and only if for almost all $t \in I$, $g(t)$ is a simultaneous farthest point in G from $(f_1(t), f_2(t), ..., f_m(t))$ in X^m.*

Proof. Let $g(t)$ be a simultaneous farthest point from $(f_1(t), f_2(t), ..., f_m(t))$. Then

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) = \sum_{i=1}^{m} \varphi \|f_i(t) - g(t)\|.$$

By Theorem 2.1, we have:

$$\begin{aligned}\rho_\varphi(f_1, f_2, ..., f_m, L^\varphi(I, G)) &= \int_I \rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G))dt \\ &= \sum_{i=1}^{m} \|f_i - g\|_\varphi.\end{aligned}$$

Therefore g is a simultaneous farthest point from $(f_1, f_2, ..., f_m) \in (L^\varphi(I, X))^m$. For the other direction, let g be a simultaneous farthest point to $(f_1, f_2, ..., f_m)$. Then

$$\begin{aligned}\rho_\varphi(f_1, f_2, ..., f_m, L^\varphi(I, G)) &= \sum_{i=1}^{m} \|f_i - g\|_\varphi \\ &= \sum_{i=1}^{m} \int_I \varphi\left(\|f_i(t) - g(t)\|\right) dt.\end{aligned}$$

By using Theorem 2.1, we get:

$$\int_I \sum_{i=1}^{m} \varphi\left(\|f_i\left(t\right) - g(t)\|\right) dt = \int_I \rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G)dt.$$

But

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) - \sum_{i=1}^{m} \varphi(\|f_i(t) - g(t)\|) \geq 0.$$

Consequently , for almost all $t \in I$

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) - \sum_{i=1}^{m} \varphi\left(\|f_i\left(t\right) - g(t)\|\right) = 0$$

and then

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) = \sum_{i=1}^{m} \varphi\left(\|f_i\left(t\right) - g(t)\|\right)$$

Hence $g(t)$ is a simultaneous farthest point from $(f_1(t), f_2(t), ..., f_m(t))$ for almost all $t \in I$. □

3. Simultaneous farthest points in $L^\varphi(I, X)$

Let G be a closed bounded subset of a Banach space X. In [10], it has been proved that if G is a remotal subset of X and $\overline{\text{span}G}$ is finite dimensional, then $L^1(I, G)$ is remotal in $L^1(I, X)$. In [2] this result was extended to the case of separable sets. It has been proved if G is a separable remotal subset of X, then $L^1(I, G)$ is remotal in $L^1(I, X)$. In this section, we prove that if G is a separable simultaneously remotal subset of X, then $L^\varphi(I, G)$ is simultaneously remotal in $L^\varphi(I, X)$. Some other results are presented.

Theorem 3.1. *Let G be a simultaneously remotal subset of X. Then every m-tuple of simple functions admits a simultaneous farthest point in $L^\varphi(I, G)$.*

Proof. Let $f_1, f_2, ..., f_m$ be a set of simple functions in $L^\varphi(I, X)$. So we can write $f_i(t) = \sum_{j=1}^{n} \chi_{A_j}(t)x_{ij}$ where χ_{A_j} is the characteristic function of the set $A_j \subseteq I$ and $x_{ij} \in X$. Let e_j be a simultaneous farthest point of the m-tuple $(x_{1j}, x_{2j}, ..., x_{mj})$ and consider the simple function $f_0 = \sum_{j=1}^{n} \chi_{A_j}(t)e_j$.

Then $f_0 \in L^{\varphi}(I,X)$ and

$$\begin{aligned}\sum_{i=1}^{m} \|f_i - f_0\|_{\varphi} &= \sum_{i=1}^{m} \int_I \varphi \|f_i(t) - f_0(t)\| \, dt \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{A_j} \varphi \|f_i(t) - f_0(t)\| \, dt \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \|x_{ij} - e_j\| \, \mu(A_j) \\ &\geq \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \|x_{ij} - a\| \, \mu(A_j), \text{ for all } a \in G.\end{aligned}$$

In particular

$$\begin{aligned}\sum_{i=1}^{m} \|f_i - f_0\|_{\varphi} &\geq \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{A_j} \varphi \|f_i(t) - g(t)\| \, dt \\ &= \sum_{i=1}^{m} \|f_i(t) - g(t)\|_{\varphi}\end{aligned}$$

for all $g \in L^{\varphi}(I,G)$. This shows that f_0 is a simultaneous farthest point to $f_1, f_2, ..., f_m$. □

Theorem 3.2. *If G is simultaneously remotal in X, then $L^{\varphi}(I,G)$ is densely simultaneously remotal in $L^{\varphi}(I,X)$.*

Proof. Let $(f_1, f_2, ..., f_m) \in (L^{\varphi}(I,X))^m$. Then there exist $\gamma_1, \gamma_2, ..., \gamma_m$ simple functions such that $\|f_i - \gamma_i\|_{\varphi} < \frac{\epsilon}{m}$. By Theorem 3.1, the m-tuple $(\gamma_1, \gamma_2, ..., \gamma_m)$ admits a simultaneous farthest point and $\sum_{i=1}^{m} \|f_i - \gamma_i\|_{\varphi} \leq \epsilon$. □

Theorem 3.3. *For a bounded set G in X, if $L^{\varphi}(I,G)$ is simultaneously remotal in $L^{\varphi}(I,X)$ then, G is simultaneously remotal in X.*

Proof. Let $(x_1, x_2, ..., x_m) \in X^m$. Set $f_1, f_2, ..., f_m$ such that $f_i(t) = x_i \otimes 1$, where 1 is the constant function. Clearly $f_i \in L^{\varphi}(I,X), 1 \leq i \leq m$. By assumption there exists $g \in L^{\varphi}(I,G)$ such that

$$\sum_{i=1}^{m} \|f_i - g\|_{\varphi} \geq \sum_{i=1}^{m} \|f_i - h\|_{\varphi}.$$

By Theorem 2.1, $g(t)$ is a simultaneous farthest point in G from $(f_1(t), ..., f_m(t))$. Equivalently, for every $h \in L^{\varphi}(I, G)$, we have

$$\sum_{i=1}^{m} \varphi \|f_i(t) - g(t)\| \geq \sum_{i=1}^{m} \varphi \|f_i(t) - h(t)\|$$

But since $f_i(t) = x_i \otimes 1$ for all $t \in I$, then

$$\sum_{i=1}^{m} \varphi \|x_i - g(t)\| \geq \sum_{i=1}^{m} \varphi \|x_i - h(t)\| \ \text{, for every } h \in L^{\varphi}(I, G).$$

Now, letting h runs over all constant functions, we get

$$\sum_{i=1}^{m} \varphi \|x_i - g(t_0)\| \geq \sum_{i=1}^{m} \varphi \|x_i - g\| \,,$$

for every $g \in G$. □

Now we will prove the main result in this paper. To do so we need the following lemma:

Lemma 3.4 ([14], Lemma 3)**.** *Assume $\mu(I) < +\infty$. Suppose (M, d) is a metric space and A is a subset of I such that $\mu^*(A) = \mu(I)$, where μ^* denotes the outer measure associated to μ. If g is a mapping from I to M with separable range, then for any $\epsilon > 0$ there exists a countable partition $\{E_n\}$ of I in measurable sets such that $A_n \subset A \cap E_n$, $\mu^*(A_n) = \mu(E_n)$ and diam$(g(A_n)) < \epsilon$ for all n.*

Theorem 3.5. *Let G be a closed bounded separable simultaneously remotal subset of X and $f_1, f_2, ..., f_m : I \to X$ be measurable functions. Then there is a measurable function $g : I \to G$ such that $g(t)$ is a simultaneous farthest point of $(f_1(t), f_2(t), ..., f_m(t))$ in G for almost all $t \in I$.*

Proof. Let $(f_1, f_2, ..., f_m) \in (L^{\varphi}(I, X))^m$. Then $f_1, f_2, ..., f_m$ are strongly measurable. We may assume that $f_1(I), f_2(I), ..., f_m(I)$ are separable sets in X. So, for each i, $1 \leq i \leq m$, there exists a countable partition of I, $\{I_{n_i}\}_{n_i=1}^{\infty}$, such that $\text{diam}(f_i(I_{n_i})) < \frac{1}{2}$, for all n_i; where

$$\text{diam}(S) = \sup\{\|x - y\| : x, y \in S\}.$$

Consider the partition $I_{k_1 k_2 ... k_m} = \bigcap_{i=1}^{m} I_{k_i}$, where $(I_{k_1}, I_{k_2}, ..., I_{k_m}) \in \prod_{i=1}^{m} \{I_{n_i}\}$. Then $\text{diam}(f_i(I_{k_1 k_2 ... k_m})) < \frac{1}{2}$. For simplicity write $I_{k_1 k_2 ... k_m}$ as $\{I_n\}_{n=1}^{\infty}$. For each $t \in I$, let $g_0(t)$ be a simultaneous farthest point from $f_1(t), f_2(t), ..., f_m(t)$ in G. Define the map g_0 from I into G by $g_0(t)$ with $t \in I$. Apply Lemma 3.4 with $\epsilon = \frac{1}{2}$ and $I = I_n = A$, we get countable partitions in each I_n and therefore a countable partition in the whole of I in measurable sets $\{E_n\}_{n=1}^{\infty}$ and a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned} A_n &\subseteq E_n,\ \mu^*(A_n) = \mu(E_n), \\ \operatorname{diam}(g_0(A_n)) &< \frac{1}{2} \text{ and } \operatorname{diam}(f_i(E_n)) < \frac{1}{2},\ 1 \leq i \leq m. \end{aligned}$$

Repeat the same argument in each E_n with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each n we get a countable partition $\{E_{(n,k)} : 1 \leq k < \infty\}$ of E_n in measurable sets and a sequence $\{A_{(n,k)} : 1 \leq k < \infty\}$ of subsets of I such that

$$\begin{aligned} A_{(n,k)} &\subseteq E_{(n,k)} \cap A_n,\ \mu^*(A_{(n,k)}) = \mu(E_{(n,k)}), \\ \operatorname{diam}(g_0(A_{(n,k)})) &< \frac{1}{2^2} \text{ and } \operatorname{diam}(f_i(E_{(n,k)})) < \frac{1}{2^2}, i = 1, 2, ..., m. \end{aligned}$$

Now, we will use mathematical induction for each n, let $\triangle_n$ be the set of n-tuples of natural numbers and $\triangle = \bigcup\{\triangle_n : 1 \leq n < \infty\}$. On this $\triangle$ consider the partial order defined by $(m_1, m_2, ..., m_i) < (n_1, n_2, ..., n_j)$ iff $i \leq j$ and $m_k = n_k$, $k = 1, 2, ..., i$. Then by induction for each n , we can find a countable partition $\{E_\alpha : \alpha \in \triangle_n\}$ of I of measurable sets and a collection $\{A_\alpha : \alpha \in \triangle_n\}$ of subsets of I such that:

(1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$
(2) $E_\beta \subseteq E_\alpha$ and $A_\beta \subseteq A_\alpha$ if $\alpha \leq \beta$
(3) $\operatorname{diam}(f_i(E_\alpha)) < \frac{1}{2^n}$ for $i = 1, 2, ..., m$ and $\operatorname{diam}(g_0(A_\alpha)) < \frac{1}{2^n}$ for $\alpha \in \triangle_n$.

We may assume $A_\alpha \neq \phi$, for all α. For each $\alpha \in \triangle$, let $t_\alpha \in A_\alpha$ and define g_n from I into G by $g_n(t) = \sum_{\alpha \in \triangle_n} \chi_{E_\alpha}(t) g_0(t_\alpha)$. Then for each $t \in I$ and $n \leq m$ we have

$$\begin{aligned} \|g_n(t) - g_m(t)\| &= \left\| \sum_{\alpha \in \triangle_n} \chi_{E_\alpha}(t) g_0(t_\alpha) - \sum_{\beta \in \triangle_m} \chi_{E_\beta}(t) g_0(t_\beta) \right\| \\ &\leq \left\| \sum_{\beta \in \triangle_m} \chi_{E_\beta}(t)(g_0(t_\alpha) - g_0(t_\beta)) \right\| \\ &\leq \sum_{\beta \in \triangle_m} \|(g_0(t_\alpha) - g_0(t_\beta))\| \, \mu\,(E_\beta) \\ &\leq \frac{1}{2^n} \end{aligned}$$

Hence $(g_n(t))$ is a Cauchy sequence in G for all $t \in I$Therefore $(g_n(t))$ is a convergent sequence. Let $g : I \to G$ be defined to be the pointwise limit of (g_n). Since g_n is strongly measurable for each n, we have g is strongly measurable. Further for $t \in I$, $1 \leq n < \infty$, and $t \in E_\alpha$ for some $\alpha \in \triangle_n$ we have:

$$\begin{aligned}
\sum_{i=1}^{m} \varphi \|f_i(t) - g_n(t)\| &= \sum_{i=1}^{m} \varphi \|f_i(t) - g_0(t_\alpha)\| \\
&\geq \sum_{i=1}^{m} \varphi \,|\|f_i(t_\alpha) - g_0(t_\alpha)\| - \|f_i(t) - f_i(t_\alpha)\||\\
&\geq \sum_{i=1}^{m} \varphi \left| \|f_i(t_\alpha) - g_0(t_\alpha)\| - \frac{1}{2^n} \right| \\
&\geq \sum_{i=1}^{m} \varphi \|f_i(t_\alpha) - g_0(t_\alpha)\| - \varphi(\frac{1}{2^n}) \\
&\geq \rho_\varphi(f_1(t_\alpha), f_2(t_\alpha), G) - m\varphi(\frac{1}{2^n}).
\end{aligned}$$

The inequality

$$\begin{aligned}
\sum_{i=1}^{m} \varphi \|f_i(t) - a\| &\leq \sum_{i=1}^{m} \varphi \|f_i(t) - f_i(t_\alpha)\| + \varphi \|f_i(t_\alpha) - a\| \\
&\leq \sum_{i=1}^{m} \varphi(\frac{1}{2^n}) + \varphi \|f_i(t_\alpha) - a\| \\
&= m\varphi(\frac{1}{2^n}) + \sum_{i=1}^{m} \varphi \|f_i(t_\alpha) - a\|
\end{aligned}$$

implies that

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) \leq m\varphi(\frac{1}{2^n}) + \rho_\varphi(f_1(t_\alpha), f_2(t_\alpha), ..., f_m(t_\alpha), G).$$

Therefore

$$\sum_{i=1}^{m} \varphi \|f_i(t) - g_n(t)\| \geq \rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) - 2m\varphi(\frac{1}{2^n}).$$

Taking limits as $n \to \infty$ we get

$$\rho_\varphi(f_1(t), f_2(t), ..., f_m(t), G) = \sum_{i=1}^{m} \varphi \|f_i(t) - g(t)\| .$$

□

Theorem 3.6. *Let G be a separable subset of X. Then $L^\varphi(I, G)$ is simultaneously remotal in $L^\varphi(I, X)$ iff G is simultaneously remotal in X.*

Proof. Necessity is in Theorem 3.3. Let us show sufficiency: Suppose that G is simultaneously remotal in X, and let $(f_1, f_2, ..., f_m) \in (L^\varphi(I,X))^m$. Theorem 3.5 guarantees that there exists a measurable function g defined on I with values in X such that $g(t)$ is a simultaneous farthest point of $(f_1(t), f_2(t), ..., f_m(t))$ in G for almost all t. It follows from Corollary 2.2 that g a simultaneous farthest point of $(f_1, f_2, ...f_m)$ in $L^\varphi(I,G)$. □

4. Further results

It is known that the M-structure theory plays a great role in Approximation theory. One direction of the M-structure theory in Banach spaces, studies the M-summand (L^1-summand or φ-summand) subspaces of the Banach space X and their projections.

A subspace Y of a Banach space X is called L^1-summand (M-summand or φ-summand) if there exists a bounded projection $P : X \to Y$ such that for any $x \in X$, $x = P(x) + (I - P)(x)$, where I is the identity function and $\|x\| = \|P(x)\| + \|(I-P)(x)\|$ (resp. $\|x\| = \max\{\|P(x)\|, \|(I-P)(x)\|\}$ or $\varphi\|x\| = \varphi\|P(x)\| + \varphi\|(I-P)(x)\|$).

In this section, we study the remotality of $L^\varphi(I, B[Y])$ in $L^\varphi(I,X)$, whenever the subspace Y possesses an M-structure in the Banach space X where $B[Y]$ is the unit ball of Y.

Theorem 4.1. *Let Y be a φ-summand subspace of a Banach space X. If $B[Y]$ is simultaneously remotal in Y, then $B[Y]$ is simultaneously remotal in X.*

Proof. Let $(x_1, x_2, ..., x_m)$ be an m-tuple in X^m. Then there exists a bounded projection P such that for all i, $1 \leq i \leq m, x_i = P(x_i) + (I-P)(x_i)$. Put $y_i = P(x_i)$ and $w_i = (I-P)(x_i)$. Then $\varphi\|x_i\| = \varphi\|y_i\| + \varphi\|w_i\|$. If $e \in B[Y]$ is a simultaneous farthest point to the m-tuple $(y_1, y_2, ..., y_m) \in Y^m$, then

$$\begin{aligned}
\sum_{i=1}^m \varphi\|x_i - e\| &= \sum_{i=1}^m \varphi\|y_i + w_i - e\| \\
&= \sum_{i=1}^m \varphi\|y_i - e + w_i\| \\
&= \sum_{i=1}^m \varphi\|y_i - e\| + \varphi\|w_i\| \\
&\geq \sum_{i=1}^m \varphi\|y_i - z\| + \varphi\|w_i\|
\end{aligned}$$

$$= \sum_{i=1}^{m} \varphi \left\| x_i - z \right\|,$$

for all $z \in B[Y]$. Consequently e is a simultaneous farthest point in $B[Y]$ of $(x_1, x_2, ..., x_m)$. □

Corollary 4.2. *Let Y be an L^1-summand subspace of X. Then $B[Y]$ is remotal in X.*

Proof. Follows from Theorem 4.1 and the fact that $B[Y]$ is remotal in Y, then by Lemma 3.1 in [10], taking $\varphi(x) = x$. □

Corollary 4.3. *If Y is a φ-summand subspace of X and $B[Y]$ is simultaneously remotal in Y, then $L^\varphi(I, B[Y])$ is densely simultaneously remotal in $L^\varphi(I, X)$.*

Proof. Follows from Theorem 4.1 and Theorem 3.2. □

Theorem 4.4. *If Y is a φ-summand subspace of X, then $B[L^\varphi(I, Y)]$ is remotal in $L^\varphi(I, X)$.*

Proof. For $f \in L^\varphi(I, X)$, then $f(t) \in X$ for every $t \in I$, since Y is φ-summand, there exists a bounded projection $P : X \to Y$ such that

$$\begin{aligned} f(t) &= P(f(t)) + (I - P)(f(t)), \\ \varphi \left\| f(t) \right\| &= \varphi \left\| P(f(t)) \right\| + \varphi \left\| (I - P)(f(t)) \right\|. \end{aligned}$$

Therefore

$$\begin{aligned} \int_I \varphi \left\| f(t) \right\| dt &= \int_I \varphi \left\| P(f(t)) \right\| dt + \int_I \varphi \left\| (I - P)(f(t)) \right\| dt, \\ \left\| f \right\|_\varphi &= \left\| Pf \right\|_\varphi + \left\| (I - P)f \right\|_\varphi. \end{aligned}$$

Since P is continuous and $\left\| Pf \right\|_\varphi \leq \left\| f \right\|_\varphi < \infty$, it follows that $Pf \in L^\varphi(I, Y)$ and $L^\varphi(I, Y)$ is L^1-summand in $L^\varphi(I, X)$. The result follows from Corollary 4.2. □

Theorem 4.5. *If Y is M-summand in X, then $L^\varphi(I, B[Y])$ is densely remotal in $L^\varphi(I, X)$.*

Proof. Let $f \in L^\varphi(I, X)$. Then there exists a simple function γ such that $\left\| f - \gamma \right\| < \epsilon$. We can write $\gamma = \sum_{i=1}^{n} \chi_{A_i} x_i$, where χ_{A_i} is the characteristic function of the set $A_i \subseteq I$ and $x_i \in X$.

Now, since Y is M-summand in X then for all i, we can write $x_i = y_i + h_i$, where $y_i \in Y$. But $B[Y]$ is remotal in Y from ([10], Lemma 3.1), therefore for each i there exists $g_i \in B[Y]$ such that $\left\| y_i - g_i \right\| \geq \left\| y_i - z \right\|$, for all $z \in B[Y]$.

Set $g=\sum\limits_{i=1}^{n}\chi_{A_i}g_i$. Then $g\in L^{\varphi}(I,B[Y])$ and

$$\begin{aligned}\|\gamma-g\|_{\varphi} &= \int_I \varphi\,\|\gamma(t)-g(t)\|\,dt\\ &= \sum_{i=1}^{n}\int_{A_i}\varphi\,\|\gamma(t)-g(t)\|\,dt\\ &= \sum_{i=1}^{n}\int_{A_i}\varphi\,\|x_i-g_i\|\,dt\\ &= \sum_{i=1}^{n}\int_{A_i}\varphi\,\|y_i-g_i+h_i\|\,dt\\ &= \sum_{i=1}^{n}\int_{A_i}\varphi(\max\{\|y_i-g_i\|,\|h_i\|\})dt.\end{aligned}$$

Since Y is M-summand in X, and by remotality of $B[Y]$ in Y we have:

$$\begin{aligned}\|\gamma-g\|_{\varphi} &\geq \sum_{i=1}^{n}\int_{A_i}\varphi(\max\{\|y_i-z\|,\|h_i\|\})dt\\ &= \sum_{i=1}^{n}\int_{A_i}\varphi\,\|x_i-z\|\,dt,\end{aligned}$$

for all $z\in B[Y]$. (Again, by Y being M-summand in X and writing $h_i=x_i-y_i$, from above.)

In particular,

$$\|\gamma-g\|_{\varphi}\geq\sum_{i=1}^{n}\int_{A_i}\varphi\,\|\gamma(t)-h(t)\|\,dt\ =\|\gamma-h\|_{\varphi}$$

for all $h\in L^{\varphi}(I,B[Y])$. Hence, the result follows. □

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