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$q-$ INEQUALITIES

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis
Memphis, TN 38152, U.S.A.
e-mail: ganastss@gmail.com

Abstract. We give here forward and reverse $q-$Hölder inequality, $q-$Poincaré inequality, $q-$Sobolev inequality, $q-$reverse Poincaré inequality, $q-$reverse Sobolev inequality, $q-$Ostrowski inequality, $q-$Opial inequality and $q-$Hilbert-Pachpatte inequality. Some interesting background is mentioned and built in the introduction.

1. Introduction

Here we follow [3], [6].

Let $q \in (0,1)$, $n \in \mathbb{N}$. A $q-$natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + ... + q^{n-1}. \tag{1}$$

In general, a $q-$real number $[\alpha]_q$ is

$$[\alpha]_q := \frac{1-q^{\alpha}}{1-q}, \quad \alpha \in \mathbb{R}. \tag{2}$$

We define

$$[0]_q! := 1, \quad [n]_q! = [n]_q [n-1]_q ... [1]_q ,$$

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \tag{3}$$

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Also, the $q-$Pochhammer symbol is defined by

$$(z-a)^{(0)}=1,\quad (z-a)^{(k)}=\prod_{i=0}^{k-1}\left(z-aq^{i}\right),\ \ k\in\mathbb{N},\ z,a\in\mathbb{R}. \tag{4}$$

The $q-$derivative of a function $f(x)$ is

$$(D_qf)(x):=\frac{f(x)-f(qx)}{x-qx}\quad (x\neq 0), \tag{5}$$

$$(D_qf)(0):=\lim_{x\to 0}(D_qf)(x),$$

and the high $q-$derivatives

$$D_q^0f:=f,\ \ D_q^kf:=D_q\left(D_q^{k-1}f\right),\quad k=1,2,3,... \tag{6}$$

From the above definition it is clear that a continuous function on an interval, which does not include 0 is continuously $q-$differentiable.

Here we assume that the $q-$derivatives we use always exist up to n^{th} order.

Notice that if f is differentiable then $\lim_{q\to 1}D_qf(x)=f'(x)$.

The $q-$integral is defined by

$$(I_{q,0}f)(x)=\int_0^x f(t)\,d_qt=x(1-q)\sum_{k=0}^{\infty}f\left(xq^k\right)q^k,\quad (0<q<1). \tag{7}$$

We call f $q-$integrable on $[0,a]$, iff $\int_0^x |f(t)|\,d_qt$ exists for all $x\in[0,a]$, $a>0$.

If f is such that, for some $C>0$, $\alpha>-1$, $|f(x)|<Cx^{\alpha}$ in a right neighborhood of $x=0$, then f is $q-$integrable, see [3].

All functions considered in this article are assumed to be $q-$integrable.

By [2] it holds

$$(If)(x)=\int_0^x f(t)\,dt=\lim_{q\uparrow 1}(I_{q,0}f)(x), \tag{8}$$

given that f is Riemann integrable on $[0,x]$.

Also it holds

$$(D_qI_{q,0}f)(x)=f(x) \tag{9}$$

and

$$(I_{q,0}(D_qf))(x)=f(x)-f(0).$$

One can define

$$I_{q,0}^nf=I_{q,0}\left(I_{q,0}^{n-1}f\right),\quad n=1,2,... \tag{10}$$

Let $x>0$. Then one has ([2], [4], [5]) the $q-$Taylor formula

$$f(x)=\sum_{k=0}^{n-1}\frac{\left(D_q^kf\right)(0)}{[k]_q!}x^k+\frac{1}{[n-1]_q!}\int_0^x (x-qt)^{(n-1)}D_q^nf(t)\,d_qt. \tag{11}$$

Assuming $\left(D_q^k f\right)(0) = 0,\ k = 0, 1, ..., n-1$ we get

$$f(x) = \frac{1}{[n-1]_q!} \int_0^x (x - qt)^{(n-1)} D_q^n f(t)\, d_q t. \tag{12}$$

Let $u(x) = \alpha x^{\beta}$. Then we get the change of variable formula ([3]),

$$\int_{u(0)}^{u(a)} f(u)\, d_q u = \int_0^a f(u(x))\, D_{q^{\frac{1}{\beta}}} u(x)\, d_{q^{\frac{1}{\beta}}} x. \tag{13}$$

In this article double $q-$integrals are meant in an iterative way.

Lemma 1.1. *([3]) Let $n \in \mathbb{Z}_+$; $x, t, s, a, b, A, B \in \mathbb{R}$. Then*

$$D_q x^t = [t]_q\, x^{t-1}, \tag{14}$$

$$D_q (Ax + b)^{(n)} = [n]_q\, A\, (Ax + b)^{(n-1)} \tag{15}$$

and

$$D_q (a + Bx)^{(n)} = [n]_q\, B\, (a + Bqx)^{(n-1)}. \tag{16}$$

We get the $q-$power rule

$$\int_0^x (At + b)^{(n)}\, d_q t = \frac{(Ax + b)^{(n+1)} - b^{(n+1)}}{[n+1]_q\, A}, \tag{17}$$

where $b^{(n+1)} = b^{n+1} q^{\frac{n(n+1)}{2}}$.

Furthermore, it holds another $q-$power rule,

$$\int_0^x (a + Bqt)^{(n-1)}\, d_q t = \frac{(a + Bx)^{(n)} - a^n}{[n]_q\, B}. \tag{18}$$

Let $f(x) \geq 0$ and f increasing. Then

$$\int_0^x f(t)\, d_q t \leq f(x) \cdot x. \tag{19}$$

We easily observe that $(a > 0,\ 0 < q < 1)$

$$\left| \int_0^a f(x)\, d_q x \right| \leq \int_0^a |f(x)|\, d_q x \tag{20}$$

and

$$\int_0^a (c_1 f_1(x) + c_2 f_2(x))\, d_q x = c_1 \int_0^a f_1(x)\, d_q x + c_2 \int_0^a f_2(x)\, d_q x, \quad c_1, c_2 \in \mathbb{R}. \tag{21}$$

Let $0 < x \leq y$ and f increasing. Then

$$x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k \leq y(1-q)\sum_{k=0}^{\infty} f\left(yq^k\right)q^k,$$

so that

$$\int_0^x f(t)\,d_qt \leq \int_0^y f(t)\,d_qt. \tag{22}$$

Let $f \leq g$. Then

$$f\left(xq^k\right)q^k \leq g\left(xq^k\right)q^k$$

and

$$x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k \leq x(1-q)\sum_{k=0}^{\infty} g\left(xq^k\right)q^k,$$

that is

$$\int_0^x f(t)\,d_qt \leq \int_0^x g(t)\,d_qt \tag{23}$$

$(x > 0,\ 0 < q < 1)$.

Next comes the $q-$Hölder's inequality.

Proposition 1.2. *Let $x > 0$, $0 < q < 1$, $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then*

$$\int_0^x |f(t)|\,|g(t)|\,d_qt \leq \left(\int_0^x |f(t)|^{p_1}\,d_qt\right)^{\frac{1}{p_1}} \left(\int_0^x |g(t)|^{q_1}\,d_qt\right)^{\frac{1}{q_1}}. \tag{24}$$

Proof. By the discrete Hölder's inequality we have

$$\begin{aligned}
&\int_0^x |f(t)|\,|g(t)|\,d_qt \\
&= x(1-q)\sum_{k=0}^{\infty}\left|f\left(xq^k\right)\right|\left|g\left(xq^k\right)\right|q^k \\
&= x(1-q)\sum_{k=0}^{\infty}\left(\left|f\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{p_1}}\right)\left(\left|g\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{q_1}}\right) \\
&\leq \left(x(1-q)\sum_{k=0}^{\infty}\left|f\left(xq^k\right)\right|^{p_1}q^k\right)^{\frac{1}{p_1}}\left(x(1-q)\sum_{k=0}^{\infty}\left|g\left(xq^k\right)\right|^{q_1}q^k\right)^{\frac{1}{q_1}} \\
&= \left(\int_0^x |f(t)|^{p_1}\,d_qt\right)^{\frac{1}{p_1}}\left(\int_0^x |g(t)|^{q_1}\,d_qt\right)^{\frac{1}{q_1}}.
\end{aligned}$$

□

Clearly it holds that

$$\int_0^x 1 d_q t = x. \tag{25}$$

It follows the reverse $q-$Hölder's inequality.

Proposition 1.3. *Let $x>0$, $0<q<1$; $0<p_1<1$, $q_1<0 : \frac{1}{p_1}+\frac{1}{q_1}=1$. Let $f,g\geq 0$ with $\int_0^x (g(t))^{q_1} d_q t>0$. Then*

$$\int_0^x f(t)\, g(t)\, d_q t \geq \left(\int_0^x (f(t))^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^x (g(t))^{q_1} d_q t\right)^{\frac{1}{q_1}}. \tag{26}$$

Proof. Notice that $\int_0^x (g(t))^{q_1} d_q t>0$, iff $x(1-q)\sum_{k=0}^{\infty}\left(g\left(xq^k\right)\right)^{q_1} q^k>0$, iff $\sum_{k=0}^{\infty}\left(g\left(xq^k\right)\right)^{q_1} q^k>0$.

By the discrete reverse Hölder's inequality we have

$$\begin{aligned}
& x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right) g\left(xq^k\right) q^k \\
& = x(1-q)\sum_{k=0}^{\infty}\left(f\left(xq^k\right)\left(q^k\right)^{\frac{1}{p_1}}\right)\left(g\left(xq^k\right)\left(q^k\right)^{\frac{1}{q_1}}\right) \\
& \geq \left(x(1-q)\sum_{k=0}^{\infty}\left(f\left(xq^k\right)\right)^{p_1} q^k\right)^{\frac{1}{p_1}}\left(x(1-q)\sum_{k=0}^{\infty}\left(g\left(xq^k\right)\right)^{q_1} q^k\right)^{\frac{1}{q_1}},
\end{aligned}$$

proving the claim. □

2. Main Results

We present the $q-$Poincaré inequality.

Theorem 2.1. *Let $\alpha,\beta>1 : \frac{1}{\alpha}+\frac{1}{\beta}=1$, $x>0$. Assume $\left(D_q^k f\right)(0)=0$, $k=0,1,...,n-1$ and $\left|D_q^n f\right|$ be increasing. Then*

$$\begin{aligned}
& \int_0^x |f(w)|^{\beta} d_q w \\
& \leq \frac{1}{\left([n-1]_q!\right)^{\beta}}\left(\int_0^x\left(\int_0^w\left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{\beta}{\alpha}} d_q w\right) \qquad (27) \\
& \times\left(\int_0^x\left|D_q^n f(t)\right|^{\beta} d_q t\right).
\end{aligned}$$

Proof. For $0 \leq w \leq x$, we have

$$f(w) = \frac{1}{[n-1]_q!} \int_0^w (w - qt)^{(n-1)} D_q^n f(t) \, d_q t.$$

Hence, from the $q-$Hölder's inequality,

$$\begin{aligned} |f(w)| &\leq \frac{1}{[n-1]_q!} \int_0^w (w - qt)^{(n-1)} \left| D_q^n f(t) \right| d_q t \\ &\leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w - qt)^{(n-1)} \right)^\alpha d_q t \right)^{\frac{1}{\alpha}} \left(\int_0^w \left| D_q^n f(t) \right|^\beta d_q t \right)^{\frac{1}{\beta}} \\ &\leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w - qt)^{(n-1)} \right)^\alpha d_q t \right)^{\frac{1}{\alpha}} \left(\int_0^x \left| D_q^n f(t) \right|^\beta d_q t \right)^{\frac{1}{\beta}}. \end{aligned}$$

Hence

$$|f(w)|^\beta \leq \frac{1}{\left([n-1]_q! \right)^\beta} \left(\int_0^w \left((w - qt)^{(n-1)} \right)^\alpha d_q t \right)^{\frac{\beta}{\alpha}} \left(\int_0^x \left| D_q^n f(t) \right|^\beta d_q t \right). \tag{28}$$

Then applying $q-$integration on (28) over $[0, x]$, we prove (27). ☐

We present the $q-$Sobolev inequality.

Theorem 2.2. *Let* $\alpha, \beta > 1 : \frac{1}{\alpha} + \frac{1}{\beta} = 1$, $x > 0$, $r \geq 1$. *Assume* $\left(D_q^k f \right)(0) = 0$, $k = 0, 1, ..., n-1$ *and* $\left| D_q^n f \right|$ *be increasing. Denote*

$$\|f\|_{q,r,[0,x]} = \left(\int_0^x |f(w)|^r \, d_q w \right)^{\frac{1}{r}}.$$

Then

$$\begin{aligned} \|f\|_{q,r,[0,x]} &\leq \frac{1}{[n-1]_q!} \left(\int_0^x \left(\int_0^w \left((w - qt)^{(n-1)} \right)^\alpha d_q t \right)^{\frac{r}{\alpha}} d_q w \right)^{\frac{1}{r}} \\ &\quad \times \left\| D_q^n f \right\|_{q,\beta,[0,x]}. \end{aligned} \tag{29}$$

Proof. As in the proof of Theorem 2.1 we get

$$|f(w)|^r \leq \frac{1}{\left([n-1]_q! \right)^r} \left(\int_0^w \left((w - qt)^{(n-1)} \right)^\alpha d_q t \right)^{\frac{r}{\alpha}} \left(\int_0^x \left| D_q^n f(t) \right|^\beta d_q t \right)^{\frac{r}{\beta}}.$$

Hence, from (23)

$$\int_0^x |f(w)|^r d_q w \le \frac{1}{\left([n-1]_q!\right)^r}\left(\int_0^x\left(\int_0^w\left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{r}{\alpha}} d_q w\right) \times\left(\int_0^x \left|D_q^n f(t)\right|^{\beta} d_q t\right)^{\frac{r}{\beta}}. \tag{30}$$

Next raise both sides of (30) to power $\frac{1}{r}$. Thus proving the claim. $\square$

Next we give the reverse $q-$Poincaré inequality.

Theorem 2.3. *Let* $0 < p_1 < 1$, $q_1 < 0 : \frac{1}{p_1} + \frac{1}{q_1} = 1$, $x > 0$. *Assume* $\left(D_q^k f\right)(0) = 0$, $k = 0, 1, ..., n-1$; $\left|D_q^n f\right|$ *be decreasing, and* $D_q^n f(t)$ *of fixed strict sign on* $[0, x]$. *Then*

$$\int_0^x |f(w)|^{-q_1} d_q w \ge \frac{1}{\left([n-1]_q!\right)^{-q_1}}\left(\int_0^x\left(\int_0^w\left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{-q_1}{p_1}} d_q w\right) \times\left(\int_0^x \left|D_q^n f(t)^{q_1} d_q t\right|\right)^{-1}. \tag{31}$$

Proof. Clearly here we have

$$\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t > 0,$$

for all $0 < w \le x$. Also we have

$$f(w) = \frac{1}{[n-1]_q!}\int_0^w (w-qt)^{(n-1)} D_q^n f(t)\, d_q t,$$

all $0 \le w \le x$. Hence

$$|f(w)| = \frac{1}{[n-1]_q!}\int_0^w (w-qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t,$$

all $0 \le w \le x$. By $q-$reverse Hölder inequality, we obtain

$$|f(w)| \ge \frac{1}{[n-1]_q!}\left(\int_0^w\left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}}\left(\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}}.$$

Since $\left|D_q^n f\right|$ is decreasing, we have that $\left|D_q^n f\right|^{q_1}$ is increasing on $[0, x]$. Thus

$$\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t \le \int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t,$$

and

$$\left(\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}} \geq \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}},$$

for all $0 \leq w \leq x$. Therefore we derive

$$|f(w)| \geq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}},$$

all $0 \leq w \leq x$. Hence

$$|f(w)|^{-q_1} \geq \frac{1}{\left([n-1]_q!\right)^{-q_1}} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{-q_1}{p_1}} \times \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{-1}, \tag{32}$$

all $0 \leq w \leq x$. At last $q-$integrating (32) on $[0,x]$ we obtain (31). □

It follows the reverse $q-$Sobolev inequality.

Theorem 2.4. *All assumptions were as in Theorem 2.3 and $r \geq 1$. Then*

$$\|f\|_{q,r,[0,x]} \geq \frac{1}{[n-1]_q!} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} d_q w\right)^{\frac{1}{r}} \times \left\|D_q^n f\right\|_{q,q_1,[0,x]}. \tag{33}$$

Proof. As in the proof of Theorem 2.3 we obtain:

$$|f(w)|^r \geq \frac{1}{\left([n-1]_q!\right)^r} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} \times \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{r}{q_1}},$$

all $0 \leq w \leq x$. Hence

$$\int_0^x |f(w)|^r d_q w \geq \frac{1}{\left([n-1]_q!\right)^r} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} d_q w\right) \times \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{r}{q_1}},$$

proving the claim. □

We continue with a $q-$Ostrowski inequality.

Theorem 2.5. *Assume* $\left(D_q^k f\right)(0) = 0,\ k = 1, ..., n-1,\ x > 0,\ 0 < q < 1.$ *Then*

$$\left|\frac{1}{x}\int_0^x f(w)\, d_q w - f(0)\right| \leq \left\|D_q^n f\right\|_{\infty,[0,x]} \cdot \frac{x^n}{[n+1]_q!}. \tag{34}$$

Proof. By assumptions we obtain

$$f(w) - f(0) = \frac{1}{[n-1]_q!}\int_0^w (w-qt)^{(n-1)} D_q^n f(t)\, d_q t, \quad \text{all } 0 \leq w \leq x.$$

Hence

$$\begin{aligned}\Delta(x) &:= \frac{1}{x}\int_0^x f(w)\, d_q w - f(0) \\ &= \frac{1}{x}\int_0^x f(w)\, d_q w - \frac{1}{x}\int_0^x f(0)\, d_q w \\ &= \frac{1}{x}\left(\int_0^x (f(w) - f(0))\, d_q w\right).\end{aligned}$$

Thus

$$|\Delta(x)| \leq \frac{1}{x}\int_0^x |f(w) - f(0)|\, d_q w. \tag{35}$$

However we observe that

$$\begin{aligned}|f(w) - f(0)| &\leq \frac{1}{[n-1]_q!}\int_0^w (w-qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t \\ &\leq \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n-1]_q!}\int_0^w (w-qt)^{(n-1)}\, d_q t.\end{aligned} \tag{36}$$

Next we apply (13) for $u(t) := -t$.

We notice that $D_q u(t) = -1$. Therefore it holds from (18),

$$\begin{aligned}\int_0^w (w-qt)^{(n-1)}\, d_q t &= -\int_0^w (w+qu(t))^{(n-1)} D_q u(t)\, d_q t \\ &= -\int_0^{-w} (w+qu(t))^{(n-1)}\, d_q u(t) \\ &= -\int_0^{-w} (w+qy)^{(n-1)}\, d_q y \\ &= -\left[\frac{(w+(-w))^{(n)} - w^n}{[n]_q}\right] \\ &= \frac{w^n}{[n]_q}.\end{aligned}$$

By (36), then we have

$$|f(w)-f(0)| \leq \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!} w^n, \quad \text{all } 0 \leq w \leq x. \tag{37}$$

Consequently by (35) we derive from (14),

$$\begin{aligned} |\Delta(x)| &\leq \frac{1}{x}\left(\int_0^x w^n d_q w\right) \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!} \\ &= \frac{1}{x} \frac{x^{n+1}}{[n+1]_q} \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!} \\ &= \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n+1]_q!} x^n, \end{aligned}$$

proving the claim. □

Next we give a $q-$Opial type inequality.

Theorem 2.6. *Assume* $\left(D_q^k f\right)(0)=0$, $n \in \mathbb{N}$, $k=0,1,\ldots,n-1$, $x>0$, $0<q<1$; $\alpha, \beta>1: \frac{1}{\alpha}+\frac{1}{\beta}=1$. *Also suppose* $\left|D_q^n f\right|$ *is increasing on* $[0,x]$. *Then*

$$\begin{aligned} &\int_0^x |f(w)|\left|D_q^n f(w)\right| d_q w \\ &\leq \frac{x^{\frac{1}{\beta}}}{[n-1]_q!}\left(\int_0^x\left(\int_0^w\left((w-qt)^{(n-1)}\right)^\alpha d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \\ &\times\left(\int_0^x\left(D_q^n f(w)\right)^{2\beta} d_q w\right)^{\frac{1}{\beta}}. \end{aligned} \tag{38}$$

Proof. It holds

$$f(w)=\frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)}\left(D_q^n f\right)(t) d_q t, \quad \text{all } 0 \leq w \leq x.$$

Hence, from the $q-$Hölder's inequality,

$$\begin{aligned} |f(w)| &\leq \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)}\left|D_q^n f(t)\right| d_q t \\ &\leq \frac{1}{[n-1]_q!}\left(\int_0^w\left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}}\left(\int_0^w\left|D_q^n f(t)\right|^\beta d_q t\right)^{\frac{1}{\beta}}. \end{aligned}$$

Set

$$z(w):=\int_0^w\left|D_q^n f(t)\right|^\beta d_q t, \quad (z(0)=0), \quad \text{all } 0 \leq w \leq x.$$

That is,

$$|f(w)| \leq \frac{1}{[n-1]_q!}\left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} (z(w))^{\frac{1}{\beta}},$$

with

$$z(w) \leq \left|D_q^n f(w)\right|^{\beta} w,$$

and

$$(z(w))^{\frac{1}{\beta}} \leq \left|D_q^n f(w)\right| w^{\frac{1}{\beta}}, \quad \text{for all } 0 \leq w \leq x.$$

Consequently we have

$$|f(w)| \leq \frac{1}{[n-1]_q!}\left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} \left|D_q^n f(w)\right| w^{\frac{1}{\beta}},$$

and

$$|f(w)|\left|D_q^n f(w)\right| \leq \frac{1}{[n-1]_q!}\left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} \left(D_q^n f(w)\right)^2 w^{\frac{1}{\beta}},$$

all $0 \leq w \leq x$. Finally we derive, from the $q-$Hölder's inequality,

$$\begin{aligned}
&\int_0^x |f(w)|\left|D_q^n f(w)\right| d_q w \\
&\leq \frac{1}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} \left(D_q^n f(w)\right)^2 w^{\frac{1}{\beta}}\right) d_q w \\
&\leq \frac{1}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_0^x \left(D_q^n f(w)\right)^{2\beta} w d_q w\right)^{\frac{1}{\beta}} \\
&\leq \frac{x^{\frac{1}{\beta}}}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_0^x \left(D_q^n f(w)\right)^{2\beta} d_q w\right)^{\frac{1}{\beta}},
\end{aligned}$$

proving the claim. □

We finish with a $q-$Hilbert-Pachpatte type inequality.

Theorem 2.7. *Assume* $\left(D_q^k f\right)(0) = \left(D_q^k g\right)(0) = 0$, $k = 0, 1, ..., n-1$, $n \in \mathbb{N}$; $x, y > 0$, $0 < q < 1$; $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. *Also suppose* $\left|D_q^n f\right|$, $\left|D_q^n g\right|$ *are* *increasing on* $[0,x]$, $[0,y]$, *respectively. Define*

$$F(s) = \int_0^s \left((s-q\sigma)^{(n-1)}\right)^{p_1} d_q\sigma, \quad 0 \leq s \leq x,$$
$$G(t) = \int_0^t \left((t-q\tau)^{(n-1)}\right)^{q_1} d_q\tau, \quad 0 \leq t \leq y.$$

Then

$$\int_0^x \int_0^y \frac{|f(s)|\,|g(t)|}{\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)} d_qs d_qt \leq \frac{xy}{\left([n-1]_q!\right)^2} \left(\int_0^x \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}} \left(\int_0^y \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}}. \tag{39}$$

Proof. We have

$$f(s) = \frac{1}{[n-1]_q!} \int_0^s (s-q\sigma)^{(n-1)} D_q^n f(\sigma)\, d_q\sigma, \quad \text{all } 0 \leq s \leq x;$$
$$g(t) = \frac{1}{[n-1]_q!} \int_0^t (t-q\tau)^{(n-1)} D_q^n g(\tau)\, d_q\tau, \quad \text{all } 0 \leq t \leq y.$$

Hence

$$|f(s)| \leq \frac{1}{[n-1]_q!} \int_0^s (s-q\sigma)^{(n-1)} \left|D_q^n f(\sigma)\right| d_q\sigma$$
$$\leq \frac{1}{[n-1]_q!} \left(\int_0^s \left((s-q\sigma)^{(n-1)}\right)^{p_1} d_q\sigma\right)^{\frac{1}{p_1}} \times \left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}}.$$

Also it holds

$$|g(t)| \leq \frac{1}{[n-1]_q!} \int_0^t (t-q\tau)^{(n-1)} \left|D_q^n g(\tau)\right| d_q\tau$$
$$\leq \frac{1}{[n-1]_q!} \left(\int_0^t \left((t-q\tau)^{(n-1)}\right)^{q_1} d_q\tau\right)^{\frac{1}{q_1}} \times \left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}}.$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p_1}} b^{\frac{1}{q_1}} \leq \frac{a}{p_1} + \frac{b}{q_1}.$$

Therefore we have

$$\begin{aligned}
|f(s)||g(t)| &\leq \frac{1}{\left([n-1]_q!\right)^2}(F(s))^{\frac{1}{p_1}}(G(t))^{\frac{1}{q_1}} \\
&\times\left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}} \\
&\leq \frac{1}{\left([n-1]_q!\right)^2}\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right) \\
&\times\left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}}.
\end{aligned}$$

Hence it holds $(0<s\leq x,\ 0<t\leq y)$

$$\frac{|f(s)||g(t)|}{\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)} \leq \frac{1}{\left([n-1]_q!\right)^2}\left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}}.$$

Therefore

$$\begin{aligned}
&\int_0^x\int_0^y \frac{|f(s)||g(t)|}{\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)} d_q s d_q t \\
&\leq \frac{1}{\left([n-1]_q!\right)^2}\left(\int_0^x\left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}} d_q s\right) \\
&\times\left(\int_0^y\left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}} d_q t\right) \\
&\leq \frac{1}{\left([n-1]_q!\right)^2} x^{\frac{1}{p_1}}\left(\int_0^x\left(\int_0^s \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right) d_q s\right)^{\frac{1}{q_1}} \\
&\times y^{\frac{1}{q_1}}\left(\int_0^y\left(\int_0^t \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right) d_q t\right)^{\frac{1}{p_1}} \\
&\leq \frac{1}{\left([n-1]_q!\right)^2}\left(x^{\frac{1}{p_1}} y^{\frac{1}{q_1}}\right)\left(\int_0^x\left(\int_0^x \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right) d_q s\right)^{\frac{1}{q_1}} \\
&\times\left(\int_0^y\left(\int_0^y \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right) d_q t\right)^{\frac{1}{p_1}}
\end{aligned}$$

$$= \frac{xy}{\left([n-1]_q!\right)^2} \left(\int_0^x \left| D_q^n f(\sigma) \right|^{q_1} d_q\sigma \right)^{\frac{1}{q_1}} \left(\int_0^y \left| D_q^n g(\tau) \right|^{p_1} d_q\tau \right)^{\frac{1}{p_1}},$$

proving the claim. □

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