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FIXED POINT OF INVOLUTION MAPPINGS IN CONVEX METRIC SPACES

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Abstract. We establish some fixed point theorems in convex metric spaces for $(k, L)-$ Lipschitzian mappings. Our results generalize and extend corresponding results in the existing literature.

1. Introduction

The notion of convexity in metric spaces was introduced by Takahashi [16] and he established that all normed spaces and their convex subsets are convex metric spaces. In addition, he also gave several examples of convex metric spaces which are not imbedded in any normed space or Banach space. Subsequently several papers have been devoted to the study of fixed points in convex metric spaces in the literature (see Agarwal et al. [1], Beg [2, 3], Ciric [5], Gajic and Stojakovic [9], Guay et al. [10] and Shimizu and Takahashi [15].)

Definition 1.1. ([3, 16]) *Let (X, d) be a metric space. A mapping $W\colon X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x,\ y,\ \lambda) \in X \times X \times [0,1]$ and $u \in X$,*

$$d(u, W(x,\ y,\ \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y). \tag{1.1}$$

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A metric space X having the convex structure W is called a convex metric space.

Let (X, d, W) be a convex metric space. A nonempty subset E of (X, d, W) is said to be convex if $W(x,\ y,\ \lambda) \in E$ whenever $(x,\ y,\ \lambda) \in E \times E \times [0,1]$.Clearly, we have from (1.1) that $W(x, x, \lambda) = x$. Takahashi [16] has also shown that the open ball $B(x, r) = \{x \in X \mid d(x, y) < r\}$ and the closed ball $\overline{B(x, r)} = \{x \in X \mid d(x, y) \leq r\}$, are convex sets.

Definition 1.2. ([1, 3, 6]) *Let E be a nonempty subset of a metric space X. A mapping $T\colon E \to E$ is said to be $k-$Lipschitzian if there exists a $k \in [0, \infty)$ such that*

$$d(Tx, Ty) \leq kd(x, y),\ \forall\ x,\ y \in E. \tag{1.2}$$

Definition 1.3. *Let (X, d, W) be a complete convex metric space and E a nonempty closed convex subset of X. A mapping $T\colon E \to E$ is said to be $(k, L)-$Lipschitzian if there exists a $k \in [1, \infty),\ L \in [0, 1)$ such that*

$$d(Tx, Ty) \leq Ld(x, Tx) + kd(x, y),\ \forall\ x,\ y \in E. \tag{1.3}$$

Definition 1.4. ([3, 7]) *Let (X, d) be a complete metric space and $E \subset X$. A mapping $T\colon E \to E$ is said to be an involution if $T^2(x) = x$.*

Definition 1.5. ([4]) *A function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if:*
(i) ϕ is monotone increasing; and
(ii) $\lim_{n\to\infty} \phi^n(t) = 0,\ \forall\ t \in \mathbb{R}^+$.

Several iterative processes for approximating fixed points of various mappings in normed spaces are available in the literature. Three of the well-known iterative processes are those of Krasnoselskii [12], Schaefer [14] and Mann [13]. We now state these iterative processes in the context of convex metric space: For $x_0 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = W(x_n, Tx_n; \lambda),\ \lambda \in [0, 1],\ (n = 0, 1, 2, \cdots) \tag{1.4}$$

is called the *Schaefer's iterative process.*
If $\lambda = \frac{1}{2}$, then the process (1.4) reduces to the *Krasnoselskii' iterative* process:

$$x_{n+1} = W\left(x_n, Tx_n; \frac{1}{2}\right),\ (n = 0, 1, 2, \cdots). \tag{1.5}$$

For $x_0 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = W(x_n, Tx_n; \alpha_n),\ \alpha_n \in [0, 1],\ (n = 0, 1, 2, \cdots) \tag{1.6}$$

is called the *Mann iterative process.*
Beg [3] employed Krasnoselskii [12] iteration defined in (1.5) to obtain some fixed point theorems in convex metric spaces. In this paper, we shall use the

Mann iteration to establish results for $(k, L)-$ Lipschitzian mappings.

2. Main results

Suppose that E is a nonempty closed convex subset of a complete convex metric space X and $T\colon E \to E$ is a mapping. For $x_0 \in E$, let $\{x_n\}$ be defined by

$$x_{n+1} = W(x_n, Tx_n; \theta_n), \text{ where } \theta_n \in [0,1], (n = 0, 1, 2, \cdots). \tag{2.1}$$

If there exists a real number $c \in [0, 1)$ such that

$$d(x_{n+2}, x_{n+1}) \leq cd(x_{n+1}, x_n), \ (n = 0, 1, 2, \cdots), \tag{2.2}$$

then $\{x_n\}$ converges to a point $x^* \in E$.
In a similar manner, if there exists a comparison function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(x_{n+2}, x_{n+1}) \leq \phi(d(x_{n+1}, x_n)), \ (n = 0, 1, 2, \cdots), \tag{2.3}$$

then $\{x_n\}$ converges to a point $x^* \in E$. For details we refer to Beg [2].

If in (2.1), $\theta_n = \alpha_n \in [0, 1]$, we have Mann iteration. If in (2.1), $\theta_n = \lambda \in [0, 1]$,we obtain Schaefer's iteration. Also (2.1) reduces to the Kransnoselskii's iteration if $\theta_n = \frac{1}{2}$.

Theorem 2.1. *Let (X, d, W) be a complete convex metric space, E a nonempty closed convex subset of X and $T\colon E \to E$ is a $(k, L)-$Lipschitzian mapping. Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a comparison function such that, for arbitrary $x \in E$, there exists $u \in E$ with*
(i) $d(Tu, u) \leq \phi(d(Tx, x))$;
(ii) $d(u, x) \leq bd(Tx, x)$, $b > 0$.
Then, T has a fixed point in E.

Proof. Let $x_0 \in E$ be an arbitrary point. Consider a sequence $\{x_n\}_{n=0}^{\infty} \subset E$. By conditions (i) and (ii), we have

$$d(Tx_{n+1}, x_{n+1}) \leq \phi(d(Tx_n, x_n)), \ n = 0, 1, 2, \cdots, \tag{2.4}$$

and

$$d(x_{n+1}, x_n) \leq bd(Tx_n, x_n), \ b > 0, \ n = 0, 1, 2, \cdots. \tag{2.5}$$

We obtain by induction in (2.4) that

$$\begin{aligned} d(Tx_{n+1}, x_{n+1}) &\leq \phi(d(Tx_n, x_n)) \leq \phi^2(d(Tx_{n-1}, x_{n-1})) \\ &\leq \cdots \leq \phi^{n+1}(d(Tx_0, x_0)). \end{aligned} \tag{2.6}$$

Using (2.6) in (2.5) gives

$$d(x_{n+1}, x_n) \leq b\phi^n(d(Tx_0, x_0)) \to \ 0 \text{ as n} \to \infty. \tag{2.7}$$

Therefore $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$. By (1.3), (2.6) and triangle inequality, we have

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x_n) + d(x_n, x^*) \\ &\leq Ld(Tx_n, x_n) + kd(x_n, x^*) + d(Tx_n, x_n) + d(x_n, x^*) \\ &= (1+L)d(Tx_n, x_n) + (1+k)d(x_n, x^*) \\ &\leq (1+L)\phi^n(d(Tx_0, x_0)) + (1+k)d(x_n, x^*) \\ &\to 0 \text{ as n} \to \infty. \end{aligned} \tag{2.8}$$

It follows from (2.8) that $Tx^* = x^*$. Hence, x^* is a fixed point of T. $\square$

Theorem 2.1 can be extended to the next result under the assumption of two metrics ρ and d such that (X, ρ) is complete and $T\colon E \to E$ is $(k, L)-$Lipschitzian with respect to d.

Theorem 2.2. *Let X be a nonempty set, $E \subset X$, ρ and d are two metrics on X and $T\colon E \to E$ be a mapping. Suppose that:*
(H_1) there exists a comparison function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for arbitrary $x \in E$, there exists $u \in E$ with (i) $d(Tu, u) \leq \phi(d(Tx, x))$; (ii) $d(u, x) \leq bd(Tx, x),\ b > 0$;
(H_2) there exists a real numbers $A \geq 0,\ R \geq 0$ such that, for arbitrary $x \in E$, there exists $u \in E$ with;
(i) $\rho(u, x) \leq Ad(u, x)$; (ii) $\rho(Tx, x) \leq Rd(Tx, x)$;
(H_3) (X, ρ, W) is a complete convex metric space and E is a closed convex subset of (X, ρ, W);
(H_4) $T\colon (E, \rho) \to (E, \rho)$ is continuous;
(H_5) $T\colon (E, d) \to (E, d)$ is $(k, L)-$Lipschitzian.
Then, T has a fixed point in E.

Proof. Let $x_0 \in E$ be an arbitrary point. By using conditions (H_1)(i) and (H_1)(ii), we obtain as in Theorem 2.1 that

$$d(x_{n+1}, x_n) \leq b\phi^n(d(Tx_0, x_0)) \to 0 \text{ as n} \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence in (E, d). By condition (H_2)(i), we have that

$$\begin{aligned} \rho(x_{n+1}, x_n) &\leq Ad(x_{n+1}, x_n) \\ &\leq Ab\phi^n(d(Tx_0, x_0)) \to 0 \text{ as n} \to \infty, \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in (E, ρ). By (H_3), (X, ρ, W) is a complete convex metric space. Therefore, (E, ρ, W) is a complete subspace of complete convex metric space (X, ρ, W). Thus, there exists $x^* \in E$ such that

$\lim_{n\to\infty} x_n = x^*$. Using (H$_2$)(ii) and (H$_4$), we obtain that

$$\begin{aligned} \rho(Tx^*, x^*) &\le \rho(Tx^*, Tx_n) + \rho(Tx_n, x_n) + \rho(x_n, x^*) \\ &\le \rho(Tx^*, Tx_n) + Rd(Tx_n, x_n) + \rho(x_n, x^*) \\ &\le \rho(Tx^*, Tx_n) + R\phi^n(d(Tx_0, x_0)) + \rho(x_n, x^*) \\ &\to 0 \text{ as n} \to \infty, \end{aligned} \tag{2.9}$$

It follows from (2.9) that $Tx^* = x^*$. Hence, x^* is a fixed point of T. □

Remark 2.1. Theorem 2.2 is a generalization/ extension of Beg [3, Theorem 2.1]. Condition (H$_5$) provides us with the fact that $T\colon (E, d) \to (E, d)$ needs not be continuous.

Theorem 2.3. *Let (X, d, W) be a complete convex metric space, E a nonempty closed convex subset of X and $T\colon E \to E$ be a $(k, L)-$Lipschitzian involution. For $x_0 \in E$, let $\{x_n\}$ defined by (1.6) be the Mann iterative process, with $\alpha_n \in [0, 1]$. If $\alpha_n < \frac{1-(k+L)L}{(k+L)(1-L)}$, $L(k + L) < 1$, then, T has a fixed point in E.*

Proof. For any $x \in E$, let $u = W(x, Tx, \alpha_n)$. Then,

$$\begin{aligned} d(u, x) &= d(W(x, Tx, \alpha_n), x) \\ &\le (1 - \alpha_n)d(x, x) + \alpha_n d(x, Tx) \\ &= \alpha_n d(Tx, x). \end{aligned} \tag{2.10}$$

Also,

$$\begin{aligned} d(u, Tu) &= d(W(x, Tx, \alpha_n), Tu) \\ &\le (1 - \alpha_n)d(x, Tu) + \alpha_n d(Tx, Tu) \\ &= (1 - \alpha_n)d(T^2x, Tu) + \alpha_n d(Tx, Tu) \\ &\le (1 - \alpha_n)[Ld(Tx, T^2x) + kd(Tx, u)] \\ &\quad +\alpha_n[Ld(x, Tx) + kd(x, u)] \\ &= L(1 - \alpha_n)d(Tx, T^2x) + k(1 - \alpha_n)d(Tx, u) \\ &\quad +\alpha_n Ld(x, Tx) + k\alpha_n d(x, u) \\ &\le [\ (1 - \alpha_n)(k + L)L + \alpha_n L + k\alpha_n^2 \\ &\quad +k\alpha_n(1 - \alpha_n)\ ]d(Tx, x) \\ &= (k + L)\,[L + (1 - L)\alpha_n]\,d(Tx, x) \\ &= \beta d(Tx, x), \end{aligned} \tag{2.11}$$

where $\beta = (k + L)\,[L + (1 - L)\alpha_n]$ and $0 \le \beta < 1$ since $\alpha_n < \frac{1-(k+L)L}{(k+L)(1-L)}$, $(k + L)L < 1$. For arbitrary $x_0 \in E$, we define the Mann iterative sequence $\{x_n\} \subset E$ by

$$x_{n+1} = W(x_n, Tx_n, \alpha_n),\ (n = 0, 1, 2, \cdots).$$

By using (2.11), we have as in Theorem 2.1 that

$$\begin{aligned} d(Tx_{n+1}, x_{n+1}) &\leq \beta d(Tx_n, x_n) \leq \beta^2 d(Tx_{n-1}, x_{n-1}) \\ &\leq \cdots \leq \beta^{n+1} d(Tx_0, x_0). \end{aligned} \tag{2.12}$$

Using (2.12) in (2.10) gives

$$d(x_{n+1}, x_n) \leq \alpha_n \beta^n d(Tx_0, x_0) \rightarrow\ 0 \text{ as n} \rightarrow \infty. \tag{2.13}$$

From (2.13), we have that $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $x^* \in E$ such that $\lim\limits_{n\to\infty} x_n = x^*$. By (1.3), (2.12) and triangle inequality, we have that

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x_n) + d(x_n, x^*) \\ &\leq (1+L)d(Tx_n, x_n) + (1+k)d(x_n, x^*) \\ &\leq (1+L)\beta^n d(Tx_0, x_0) + (1+k)d(x_n, x^*) \\ &\rightarrow\ 0 \text{ as n} \rightarrow \infty, \end{aligned}$$

from which it follows that $Tx^* = x^*$. Hence, x^* is a fixed point of T. □

Remark 2.2. Theorems 3.1 and 3.2 of are generalizations/extensions of Beg [3, Theorems 3.1 and 3.2] as well as a result of Goebel [6].

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