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ZYGMUND-TYPE INEQUALITIES FOR POLYNOMIALS

N. A. Rather

P.G. Department of Mathematics, University of Kashmir
Hazratbal, Srinagar-190006, India
e-mail: dr.narather@gmail.com

Abstract. In this paper we consider a problem of investigating the dependence of $\|P(Rz) -\beta P(rz)\|_p$ on $\|P(z)\|_p$ for every real or complex number β with $|\beta| \leq 1$, $R > r \geq 1$, $p > 0$ and present compact generalizations of certain well-known polynomial inequalities.

1. Introduction

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. For $P \in P_n$, define

$$\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{1/p}, \ 1 \leq p < \infty,$$

$$\|P(z)\|_\infty := \underset{|z|=1}{Max} |P(z)| \ \ and \ \ m(P,1) := \underset{|z|=1}{Min} |P(z)| .$$

If $P \in P_n$, then

$$\left\| P'(z) \right\|_p \leq n \|P(z)\|_p , \quad p \geq 1, \tag{1.1}$$

and

$$\|P(Rz)\|_p \leq R^n \|P(z)\|_p , \ \ R > 1, \ \ p > 0. \tag{1.2}$$

Inequality (1.1) was found by Zygmund [17] whereas inequality (1.2) is a simple consequence of a result of Hardy [9](see also [13, Theorem 5.5]). Since inequality (1.1) on p was indeed essential. This question was open for a long time. Finally Arestov [2] proved that (1.1) remains true for $0 < p < 1$ as well was deduced from Riesz's interpolation formula [15] by means of Minkowski's

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inequality, it was not clear, whether the restrictil. For $p = \infty$, the inequality (1.1) is due to Bernstein (see [11, p.531] or [16]) whereas the case $p = \infty$ of inequality (1.2) is a simple consequence of the maximum modulus principle (for reference, see [11, p.442] or [12, vol.I, p.137]). Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\left\|P'(z)\right\|_p \leq n\frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p \geq 0 \tag{1.3}$$

and

$$\|P(Rz)\|_p \leq \frac{\|R^n z+1\|_p}{\|1+z\|_p}\|P(z)\|_p, \quad R>1, \quad p>0. \tag{1.4}$$

Inequality (1.3) is due to De-Bruijn [8](see also [4]) for $p \geq 1$. Rahman and Schmeisser [14] extended it for $0 < p < 1$ whereas the inequality (1.4) was proved by Boas and Rahman for $p \geq 1$ and later it was extended for $0 < p < 1$ by Rahman and Schmeisser[14]. For $p = \infty$, the inequality (1.3) was conjectured by Erdös and later verified by Lax [10] whereas inequality (1.4) was proved by Ankeny and Rivlin [1].

Aziz and Dawood [3] refined Erdös-Lax theorem [10] and a result of Ankeny and Rivilin[1] by showing that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\left\|P'(z)\right\|_\infty \leq \frac{n}{2}\left\{\|P(z)\|_\infty - m(P,1)\right\} \tag{1.5}$$

and

$$\|P(Rz)\|_\infty \leq \frac{R^n+1}{2}\|P(z)\|_\infty - \frac{R^n-1}{2}m(P,1), \ R>1. \tag{1.6}$$

Recently Aziz and Rather [5](see also [6]) investigated the dependence of

$$\|P(Rz)-P(z)\|_p \text{ on } \|P(z)\|_p$$

for $R > 1$, $p \geq 1$. As a compact generalization of inequalities (1.1) and (1.2), they have shown that if $P \in P_n$, then for every $R > 1$ and $p \geq 1$,

$$\|P(Rz)-P(z)\|_p \leq (R^n-1)\|P(z)\|_p. \tag{1.7}$$

In the present paper we consider a more general problem of investigate the dependence of

$$\|P(Rz)-\beta P(rz)\|_p \text{ on } \|P(z)\|_p \text{ and } m(P,1)$$

for every real or complex number β with $|\beta| \leq 1$, $R > r \geq 1$, $p > 0$ and develop a unified method for arriving at these results. we first present the following result.

Theorem 1.1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for every real or complex number β with $|\beta| \le 1$, $R > r \ge 1$ and $|z| \ge 1$,*

$$|P(Rz) - \beta P(rz)| \ge |R^n - \beta r^n||z|^n m(P,1). \tag{1.8}$$

The result is best possible and equality in (1.8) holds for $P(z) = az^n, a \neq 0$.

Remark 1.1. Taking $\beta = 0$ in (1.8), we get for $|z| \ge 1$

$$|P(Rz)| \ge R^n |z|^n m(P,1), \quad R > 1. \tag{1.9}$$

For $\beta = 1$ if we divide the two sides of (1.8) $R - r$ and make $R \to r$, we get for $|z| \ge 1$,

$$|P'(rz)| \ge nr^{n-1}|z|^n m(P,1), \quad r \ge 1. \tag{1.10}$$

Next we present the following interesting result which is a compact generalization of inequalities (1.3), (1.4), (1.5) and (1.6).

Theorem 1.2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all real or complex numbers β, δ with $|\beta| \le 1, |\delta| \le 1$, $R > r \ge 1$ and $p > 0$,*

$$\left\| P(Rz) - \beta P(rz) + \delta \left\{ \frac{|R^n - \beta r^n| - |1 - \beta|}{2} \right\} m(P,1) \right\|_p$$
$$\le \frac{\|(R^n - \beta r^n)z + (1-\beta)\|_p}{\|1 + z\|_p} \|P(z)\|_p . \tag{1.11}$$

The result is best possible and equality in (1.11) *holds for $P(z) = az^n + b, |a| = |b| = 1$.*

For different values of parameters, a variety of interesting results can be easily deduced from Theorem 1.2. Here we mention a few of these. The following corollary immediately follows from Theorem 1.2 by letting $p \to \infty$ in (1.11) and choosing the argument of δ suitably with $|\delta| = 1$.

Corollary 1.1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$,then for every real or complex number β with $|\beta| \le 1$, $R > r \ge 1$ and for $|z| = 1$,*

$$|P(Rz) - \beta P(rz)|$$
$$\le \frac{|R^n - \beta r^n| + |1 - \beta|}{2} \underset{|z|=1}{Max} |P(z)| - \frac{|R^n - \beta r^n| - |1 - \beta|}{2} \underset{|z|=1}{Min} |P(z)| . \tag{1.12}$$

The result is sharp and equality in (1.12) *holds for $P(z) = az^n + b, |a| = |b| = 1$.*

Remark 1.2. For $\beta = 0$, inequality (1.12) reduces to (1.6) and for $\beta = 1$,we get the following interesting result.

Corollary 1.2. *If* $P \in P_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$,*then for* $R > r \geq 1$ and $|z| = 1$,

$$|P(Rz) - P(rz)| \leq \frac{(R^n - r^n)}{2} \left\{ \underset{|z|=1}{Max} |P(z)| - \underset{|z|=1}{Min} |P(z)| \right\} \tag{1.13}$$

The result is sharp and equality in (1.13) *holds for* $P(z) = az^n + b, |a| = |b| = 1$.

Remark 1.3. If we divide the two sides of (1.13) by $R - r$ and let $R \to r$, we get for $|z| = 1$,

$$\left|P'(rz)\right| \leq \frac{n}{2} r^{n-1} \left\{ \underset{|z|=1}{Max} |P(z)| - \underset{|z|=1}{Min} |P(z)| \right\}, \quad r \geq 1. \tag{1.14}$$

For $r = 1$ inequality (1.14) reduces to inequality (1.5). The following result which is a generalization of inequalities (1.3) and (1.4) follows from Theorem 1.2 by setting $\delta = 0$ in inequality (1.14).

Corollary 1.3. *If* $P \in P_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every real or complex number* β *with* $|\beta| \leq 1$, $R > r \geq 1$ *and* $p > 0$,

$$\|P(Rz) - \beta P(rz)\|_p \leq \frac{\|(R^n - \beta r^n)z + (1 - \beta)\|_p}{\|1 + z\|_p} \|P(z)\|_p . \tag{1.15}$$

The result is best possible and equality in (1.15) *holds for* $P(z) = az^n + b, |a| = |b| = 1$.

Remark 1.4. For $\beta = 0$, inequality (1.15) reduces to (1.6). If we divide the two sides of (1.15) by $R - r$ with $\beta = 1$ and let $R \to r$,we get

$$\left\|P'(rz)\right\|_p \leq nr^{n-1} \frac{\|P(z)\|_p}{\|1 + z\|_p}, \quad p \geq 0, r \geq 1. \tag{1.16}$$

The result is sharp.

For $r = 1$ inequality (1.16) reduces to inequality (1.3) due to De Bruijn [8].

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. *If $P \in P_n$ and P(z) has all its zeros in $|z| \leq \rho$* where $\rho \geq 0$, *then for every $R \geq r, rR \geq \rho^2$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+\rho}{r+\rho}\right)^n |P(rz)|. \tag{2.1}$$

Proof. Since all the zeros of P(z) lie in $|z| \leq \rho$, we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right)$$

where $r_j \leq \rho, j = 1, 2, \cdots, n$. Now for $0 \leq \theta < 2\pi$, $R \geq r$ and $rR \geq \rho^2$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| &= \left\{\frac{R^2 + r_j^2 - 2Rr_jCos(\theta-\theta_j)}{r^2 + r_j^2 - 2rr_jCos(\theta-\theta_j)}\right\}^{1/2} \\ &\geq \left\{\frac{R+r_j}{r+r_j}\right\} \geq \left\{\frac{R+\rho}{r+\rho}\right\},\end{aligned}$$

$j = 1, 2, \cdots, n$. Hence

$$\begin{aligned}\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| \\ &\geq \prod_{j=1}^{n}\left(\frac{R+\rho}{r+\rho}\right) = \left(\frac{R+\rho}{r+\rho}\right)^n\end{aligned}$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$, $R \geq r$ and $rR \geq \rho^2$

$$|P(Rz)| \geq \left(\frac{R+\rho}{r+\rho}\right)^n |P(rz)|,$$

which completes the proof of Lemma 2.1. □

Lemma 2.2. *If $P \in P_n$ and P(z) does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$,$R \geq r \geq 1$, and $|z| = 1$,*

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| \tag{2.2}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.The result is sharp and equality in (2.2)*holds for $P(z) = z^n + 1$.*

Proof. For the case $R{=}r$, the result follows by observing that $|P(z)|{\leq}|Q(z)|$ for $|z| \geq 1$. Henceforth, we assume that $R > r$. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number α with $|\alpha| > 1$, the polynomial $f(z) = P(z) - \alpha Q(z)$, where $Q(z) = z^n\overline{P(1/\overline{z})}$, has all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $f(z)$ with $\rho = 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^n |f(re^{i\theta})|. \tag{2.3}$$

Since $f(Re^{i\theta}) \neq 0$ for every $R > r \geq 1, 0 \leq \theta < 2\pi$ and $R + 1 > r + 1$, it follows from (2.3) that

$$|f(Re^{i\theta})| > \left(\frac{r+1}{R+1}\right)^n |f(Re^{i\theta})| \geq |f(re^{i\theta})|$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. This gives

$$|f(rz)| < |f(Rz)| \quad \text{for } |z| = 1 \text{ and } R > r \geq 1.$$

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, we conclude that the polynomial

$$T(z) = f(Rz) - \beta f(rz) = \{P(Rz) - \beta P(rz)\} - \alpha \{Q(Rz) - \beta Q(rz)\} \tag{2.4}$$

has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| > 1$ and $R > r \geq 1$. This implies

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| \tag{2.5}$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.5) is not true, then exist a point $z = w$ with $|w| \geq 1$ such that

$$|P(Rw) - \beta P(rw)| > |Q(Rw) - \beta Q(rw)|.$$

But all the zeros of $Q(z)$ lie in $|z| \leq 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $Q(Rz) - \beta Q(rz)$ lie in $|z| < 1$. Hence $Q(Rw) - \beta Q(rw) \neq 0$ with $|w| \geq 1$. We take

$$\alpha = \frac{P(Rw) - \beta P(rw)}{Q(Rw) - \beta Q(rw)},$$

then α is a well defined real or complex number with $|\alpha| > 1$ and with this choice of α, from (2.4) we obtain $T(w) = 0$, where $|w| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)|$$

for $|z| \geq 1$ and $R > r \geq 1$. This proves Lemma 2.2. □

Lemma 2.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$,$R \geq r \geq 1$, and $|z| = 1$,*

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| - (|R^n - \beta r^n| - |1 - \beta|)m(P, 1)$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof. By hypothesis $P(z)$ has all its zeros in $|z| \geq 1$ and

$$m(P,1) \leq |P(z)| \quad for \quad |z| = 1. \tag{2.6}$$

We show $F(z) = P(z) + \lambda m(P,1)$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. This is obvious if $m(P,1) = 0$, that is, if $P(z)$ has a zero on $|z| = 1$. So we assume all the zeros of $P(z)$ lie in $|z| > 1$, then $m(P,1) > 0$ and by the maximum modulus principle, it follows from (2.6),

$$m(P,1) < |P(z)| \quad for \quad |z| < 1. \tag{2.7}$$

Now if $F(z) = P(z) + \lambda m(P,1) = 0$ for some $z = z_0$ with $|z_0| < 1$,then

$$P(z_0) + \lambda m(P,1) = 0.$$

This implies

$$|P(z_0)| = |\lambda| m(P,1) \leq m(P,1), \quad |z_0| < 1,$$

which is clearly a contradiction to (2.7). Thus the polynomial $F(z)$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. Applying Lemma 2.2 to the polynomial $F(z)$, we get

$$|F(Rz) - \beta F(rz)| \leq |G(Rz) - \beta G(rz)|$$

for $|z| = 1$ and $R \geq r \geq 1$ where $G(z) = z^n \overline{F(1/\overline{z})} = Q(z) + \bar{\lambda} z^n m(P,1)$. Replacing $F(z)$ by $P(z) + \lambda m(P,1)$,we obtain

$$|P(Rz) - \beta P(rz) + \lambda(1-\beta)| \leq \left|Q(Rz) - \beta Q(rz) + \bar{\lambda}(R^n - \beta r^n) z^n\right| \tag{2.8}$$

for $|z| = 1$ and $R \geq r \geq 1$. Now choosing the argument of λ in the right hand side of (2.8) such that

$$\left|Q(Rz) - \beta Q(rz) + \bar{\lambda}(R^n - \beta r^n) z^n\right| = |Q(Rz) - \beta Q(rz)| - |\lambda||R^n - \beta r^n|$$

for $|z| = 1$, which is possible by Theorem 1.1, we get

$$|P(Rz) - \beta P(rz)| - |\lambda||1-\beta| \leq |Q(Rz) - \beta Q(rz)| - |\lambda||R^n - \beta r^n|$$

for $|z| = 1$ and $R \geq r \geq 1$. Equivalently,

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| - (|R^n - \beta r^n| - |1-\beta|) m(P,1)$$

for $|z| = 1$ and $R \geq r \geq 1$. This proves Lemma 2.3. □

Next we describe a result of Arestov.

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in P_n$, we define

$$\Lambda_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in C : |z| \leq 1\}$,
(ii) $P(z)$ has all its zeros in$\{z \in C : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.4. ([2, Theorem 4]) *Let $\phi(x) = \psi(logx)$ where ψ is a convex nondecreasing function on* $\boldsymbol{R}$. *Then for all $P \in P_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi(|\Lambda_\gamma P(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(C(\gamma, n)|P(e^{i\theta})|)d\theta$$

where $C(\gamma, n) = max(|\gamma_0|, |\gamma_n|)$.

In particular, Lemma 2.4 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} (|\Lambda_\gamma P(e^{i\theta})|^p)d\theta\right\}^{1/p} \leq C(\gamma, n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{2.9}$$

We use (2.9) to prove the following interesting result.

Lemma 2.5. *If $P \in P_n$ and P(z) does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1, R > r \geq 1$, $p > 0$ and α* real,

$$\int_0^{2\pi} |(P(Re^{i\theta}) - \beta P(re^{i\theta})) + e^{i\alpha}(R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r))|^p d\theta$$
$$\leq |(R^n - \beta r^n) + e^{i\alpha}(1 - \bar{\beta})|^p \int_0^{2\pi} |P(e^{i\theta}|^p d\theta.$$

Proof. Let $Q(z) = z^n\overline{P(1/\overline{z})}$. Since $P(z)$ does not vanish in $|z| < 1$ by Lemma 2.2, for every real or complex number β with $|\beta| \leq 1$, $R \geq r \geq 1$ and $|z| = 1$,we have

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| = \left|R^n P(z/R) - \overline{\beta} r^n P(z/r)\right|.$$

Now(as in the proof of Lemma 2.2), the polynomial

$$H(z) = Q(Rz) - \beta Q(rz) = R^n z^n \overline{P(1/R\overline{z})} - \beta r^n z^n \overline{P(1/r\overline{z})}$$

has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| \leq 1$ and $R > r$. This gives that the polynomial

$$z^n\overline{H(1/\overline{z})} = R^n P(z/R) - \overline{\beta} r^n P(z/r)$$

has all its zeros in $|z| > 1$. Hence the function

$$f(z) = \frac{P(Rz) - \beta P(rz)}{R^n P(z/R) - \overline{\beta} r^n P(z/r)}$$

is analytic in $|z| \le 1$ and $|f(z)| \le 1$ for $|z| = 1$. Since $f(z)$ is not a constant, it follows by the maximum modulus Principle that

$$|f(z)| < 1 \quad for \ \ |z| < 1,$$

or equivalently,

$$|P(Rz) - \beta P(rz)| < \left|R^n P(z/R) - \overline{\beta} r^n P(z/r)\right| \ \ for \ \ |z| < 1. \tag{2.10}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned}\Lambda_\gamma P(z) &= (P(Rz) - \beta P(rz)) + e^{i\alpha}(R^n P(z/R) - \overline{\beta} r^n P(z/r)) \\ &= ((R^n - \beta r^n) + e^{i\alpha}(1 - \overline{\beta})a_n z^n + \cdots + ((1-\beta) + e^{i\alpha}(R^n - \overline{\beta} r^n))a_0\end{aligned}$$

does not vanish in $|z| < 1$ for every β with $|\beta| \le 1, R > r \ge 1$ and α real. Therefore, Λ_γ is an admissibe operator. Applying (2.9) of Lemma 2.5, the desired result follows immediately for each $p > 0$. This completes the proof of Lemma 2.5. □

We also need the following lemma [5].

Lemma 2.6. *If A, B, C are non-negative real numbers such that* $B + C \le A$, *then for every real number* α,

$$|(A - C)e^{i\alpha} + (B + C)| \le |Ae^{i\alpha} + B|.$$

3. Proofs of theorems

Proof of Theorem 1.1. By hypothesis, all the zeros $P(z)$ lie in $|z| \le 1$ and

$$m(P, 1)|z|^n \le |P(z)| \ \ for \ \ |z| = 1.$$

We first show that the polynomial $G(z) = P(z) - \alpha m(P, 1)z^n$ has all its zeros in $|z| \le 1$ for every real or complex number α with $|\alpha| < 1$. This is obvious if $m(P, 1) = 0$, that if $P(z)$ has a zero on $|z| = 1$. Henceforth, we assume $P(z)$ has all its zeros in $|z| < 1$, then $m(P, 1) > 0$ and it follows by the Rouche's theorem that the polynomial $G(z) = P(z) - \alpha m(P, 1)z^n$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| < 1$. Applying Lemma 1.1 to the polynomial $G(z)$ with $\rho = 1$, we deduce as before,

$$|G(Rz)| > |G(rz)| \ \ for \ \ |z| = 1 \, and \ R > r \ge 1. \tag{3.1}$$

Since all the zeros of $G(Rz)$ lie in $|z| < (1/R) < 1$, by Rouche's Theorem again it follows from (3.1) that all the zeros of polynomial

$$H(z) = G(Rz) - \beta G(rz) = (P(Rz) - \beta P(rz)) - \alpha(R^n - \beta r^n)z^n m(P, 1)) \tag{3.2}$$

lie in $|z| < 1$ for every α, β with $|\alpha| < 1, |\beta \ \le 1$ and $R > r \ge 1$. This gives

$$|P(Rz) - \beta P(rz)| \ge |R^n - \beta r^n||z|^n m(P, 1) \tag{3.3}$$

for $|z| \geq 1$ and $R > r \geq 1$. Because if the inequality (3.3) is not true, then there is point $z = z_0$ with $|z_0| \geq 1$ such that

$$|P(Rz_0) - \beta P(rz_0)| > |R^n - \beta r^n||z_0|^n m(P, 1).$$

We choose

$$\alpha = \frac{P(Rz_0) - \beta P(rz_0)}{(R^n - \beta r^n) z_0^n m(P, 1)},$$

then clearly $|\alpha| < 1$ and with choice of α, from (3.2) we get $H(z_0) = 0$ with $|z_0| \geq 1$. This is clearly a contradiction to the fact that the zeros of $H(z)$ lie in $|z| < 1$.Thus for every real or complex number β with $|\beta| \leq 1$,

$$|P(Rz) - \beta P(rz)| \geq |R^n - \beta r^n||z|^n m(P, 1)$$

for $|z| \geq 1$ and $R > r \geq 1$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore, by Lemma 2.3, we have

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| - (|R^n - \beta r^n| - |1 - \beta|) m(P, 1)$$

for $|z| = 1$, $|\beta| \leq 1$ and $R > r \geq 1$ where $Q(z) = z^n \overline{P(1/\bar{z})}$. Equivalently,

$$\begin{aligned} &|P(Rz) - \beta P(rz)| \\ &\leq \left|R^n P(z/R) - \bar{\beta} r^n P(z/r)\right| - (|R^n - \beta r^n| - |1 - \beta|) m(P, 1) \end{aligned}$$

for $|z| = 1$, $|\beta| \leq 1$ and $R > r \geq 1$. This implies for every real or complex number β with $|\beta| \leq 1$, $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$\begin{aligned} &|P(Re^{i\theta}) - \beta P(re^{i\theta})| + \frac{|R^n - \beta r^n| - |1 - \beta|}{2} m(P, 1) \\ &\leq |R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r)| - \frac{|R^n - \beta r^n| - |1 - \beta|}{2} m(P, 1). \end{aligned} \tag{3.4}$$

Taking

$$A = |R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r)|, \;\; B = |P(Re^{i\theta}) - \beta P(re^{i\theta})|$$

and

$$C = \frac{|R^n - \beta r^n| - |1 - \beta|}{2} m(P, 1)$$

in Lemma 2.6 and noting by (3.4) that

$$(B + C) \leq (A - C) \leq A,$$

we get for every real α,

$$\left|\left\{|R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|-\frac{|R^n-\beta r^n|-|1-\beta|}{2}m(P,1)\right\}e^{i\alpha}\right.$$
$$\left.+\left\{|P(Re^{i\theta})-\beta P(re^{i\theta})|+\frac{|R^n-\beta r^n|-|1-\beta|}{2}m(P,1)\right\}\right|$$
$$\leq\left||R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|e^{i\alpha}+|P(Re^{i\theta})-\beta P(re^{i\theta})|\right|.$$

This implies for each $p>0$,

$$\int_0^{2\pi}|F(\theta)+e^{i\alpha}G(\theta)|^pd\theta$$
$$\leq\int_0^{2\pi}\left||R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|e^{i\alpha}+|P(Re^{i\theta})-\beta P(re^{i\theta})|\right|^pd\theta, \quad (3.5)$$

where

$$F(\theta)=|P(Re^{i\theta})-\beta P(re^{i\theta})|+\left(\frac{|R^n-\beta r^n|-|1-\beta|}{2}\right)m(P,1)$$

and

$$G(\theta)=|R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|-\left(\frac{|R^n-\beta r^n|-|1-\beta|}{2}\right)m(P,1).$$

Integrating both sides of (3.5) with respect to α from 0 to 2π, we get with the help of Lemma 2.5 for each $p>0, R>r\geq 1$ and α real,

$$\int_0^{2\pi}\int_0^{2\pi}|F(\theta)+e^{i\alpha}G(\theta)|^pd\alpha d\theta$$
$$\leq\int_0^{2\pi}\int_0^{2\pi}||R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|e^{i\alpha}+|P(Re^{i\theta})-\beta P(re^{i\theta})||^pd\theta d\alpha$$
$$=\int_0^{2\pi}\left\{\int_0^{2\pi}||R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)|e^{i\alpha}+|P(Re^{i\theta})-\beta P(re^{i\theta})||^pd\alpha\right\}d\theta$$
$$=\int_0^{2\pi}\left\{\int_0^{2\pi}|(R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r))e^{i\alpha}+(P(Re^{i\theta})-\beta P(re^{i\theta})|^pd\alpha\right\}d\theta$$
$$=\int_0^{2\pi}\left\{\int_0^{2\pi}|(R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r))e^{i\alpha}+P(Re^{i\theta})-\beta P(re^{i\theta})|^pd\theta\right\}d\alpha$$
$$\leq\int_0^{2\pi}|(R^n-\beta r^n)+e^{i\alpha}(1-\bar{\beta})|^pd\alpha\int_0^{2\pi}|P(e^{i\theta})|^pd\theta. \quad (3.6)$$

Now for every real α, $t \geq 1$ and $p > 0$, we have

$$\int_0^{2\pi} |t + e^{i\alpha}|^p d\alpha \geq \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha.$$

If $F(\theta) \neq 0$, we take $t = |G(\theta)|/|F(\theta)|$, then by (30) $t \geq 1$ and we get

$$\begin{aligned}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^p d\alpha &= |F(\theta)|^p \int_0^{2\pi} \left|1 + e^{i\alpha}\frac{G(\theta)}{F(\theta)}\right|^p d\alpha \\ &= |F(\theta)|^p \int_0^{2\pi} \left|\frac{G(\theta)}{F(\theta)} + e^{i\alpha}\right|^p d\alpha \\ &= |F(\theta)|^p \int_0^{2\pi} \left|\left|\frac{G(\theta)}{F(\theta)}\right| + e^{i\alpha}\right|^p d\alpha \\ &\geq |F(\theta)|^p \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^p d\alpha.\end{aligned}$$

For $F(\theta) = 0$, this inequality is trivially true. Using this in(3.6), we conclude that for every real or complex number β with $|\beta| \leq 1$, $R > r \geq 1$ and α real,

$$\int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha \int_0^{2\pi} \left\{|P(Re^{i\theta}) - \beta P(re^{i\theta})| + \frac{|R^n - \beta r^n| - |1 - \beta|}{2} m(P,1)\right\}^p d\theta$$
$$\leq \left\{\int_0^{2\pi} |(R^n - \beta r^n) + e^{i\alpha}(1 - \bar{\beta})|^p d\alpha\right\} \left\{\int_0^{2\pi} |P(e^{i\theta}|^p d\theta\right\}.$$

This gives for every real or complex number δ, β with $|\delta| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ and α real,

$$\int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha \int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta}) + \delta \left\{\frac{|R^n - \beta r^n| - |1 - \beta|}{2}\right\} m(P,1)|^p d\theta$$
$$\leq \left\{\int_0^{2\pi} |(R^n - \beta r^n) + e^{i\alpha}(1 - \bar{\beta})|^p d\alpha\right\} \left\{\int_0^{2\pi} |P(e^{i\theta}|^p d\theta\right\}. \tag{3.7}$$

Since

$$\begin{aligned}&\int_0^{2\pi} |(R^n - \beta r^n) + e^{i\alpha}(1 - \bar{\beta})|^p d\alpha \\ &= \int_0^{2\pi} ||R^n - \beta r^n| + e^{i\alpha}|1 - \bar{\beta}||^p d\alpha \\ &= \int_0^{2\pi} ||R^n - \beta r^n| + e^{i\alpha}|1 - \beta||^p d\alpha \\ &= \int_0^{2\pi} ||R^n - \beta r^n|e^{i\alpha} + |1 - \beta||^p d\alpha \\ &= \int_0^{2\pi} |(R^n - \beta r^n)e^{i\alpha} + (1 - \beta)|^p d\alpha,\end{aligned} \tag{3.8}$$

the desired result follows immediately by combining (3.7) and (3.8). This completes the proof of Theorem 1.2.

Remark 3.1. From Theorem 1.1, one can easily deduce that if $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R > r \geq 1$,

$$\|P(Rz) - \beta P(rz)\|_p \geq |R^n - \beta r^n| m(P, 1). \tag{3.9}$$

The result is best possible and equality in (3.9) holds for $P(z) = az^n, a \neq 0$.

References

[1] N.C. Ankeny and T.J. Rivlin, *On a theorm of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.

[2] V.V. Arestov, *On integral inequalities for trigonometric polynimials and their derivatives*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 3–22[in Russian]. English translation; Math. USSR-Izv., **18** (1982), 1–17.

[3] A. Aziz and Q.M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1988), 306–313.

[4] A. Aziz, *A new proof and a generalization of a theorem of De Bruijn*, Proc. Amer. Math. Soc., **106** (1989), 345–350.

[5] A. Aziz and N.A. Rather, *L^p inequalities for polynomials*, Glasnik Matematicki, **32** (1997), 39–43.

[6] A. Aziz and N.A. Rather, *Some compact generalization of Zygmund-type inequalities for polynomials*, Nonlinear studies, **6** (1999), 241–255.

[7] R.P. Boas, Jr. and Q.I. Rahman, *L^p inequalities for polynomials and entire functions*, Arch. Rational Mech. Anal., **11** (1962), 34–39.

[8] N.G. Bruijn, *Inequalities concerning polynomials in the complex domain*, Nederal. Akad. Wetensch. Proc., **50** (1947), 1265–1272.

[9] G.H. Hardy, *The mean value of the modulus of an analytic function*, Proc. London Math. Soc., **14** (1915), 269–277.

[10] P.D. Lax, *Proof of a conjecture of P.Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.

[11] G.V. Milovanovic, D.S. Mitrinovic and Th.M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World scientific Publishing Co., Singapore, (1994).

[12] G. Pólya and G. Szegö, *Aufgaben und lehrsätze aus der Analysis*, Springer-Verlag, Berlin (1925).

[13] Q.I. Rahman and G. Schmeisser, *Les Inequalitués de Markoff et de Bernstein*, Presses Univ. Montréal, Montréal, Quebec (1983).

[14] Q.I. Rahman and G. Schmessier, *L^p inequalities for polynomials*, J. Approx. Theory, **53** (1988), 26–32.

[15] M. Riesz, *Formula d'interpolation pour la dérivée d'un polynome trigonométrique*, C.R.Acad. Sci, Paris, **158** (1914), 1152–1254.

[16] A.C. Schaffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc., **47** (1941), 565–579.

[17] A. Zygmund, *A remark on conjugate series*, Proc. London Math. Soc., **34** (1932), 292–400.