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ZERO–FREE REGIONS FOR ANALYTIC FUNCTIONS

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Abstract. In this paper we obtain some interesting zero-free regions for a class of analytic functions which generalise a number of already known results by putting less restrictive conditions on the coefficients of the analytic functions.

1. Introduction and statement of results

The following famous result due to Enestrom and Kakeya [8] is well known in the theory of the location of the zeros of a polynomial.

Theorem 1.1. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0,$$

is a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 \geq 0, \tag{1.1}$$

then $P(z)$ does not vanish in $|z| > 1$.

This is a very elegant result but is equally limited in scope. In the literature [1-7, 9], there already exist some extensions and generalizations of Enestrom–Kakeya Theorem. Aziz and Zargar [5] relaxed the hypothesis in several ways and among other things proved the following generation of Theorem 1.1.

Theorem 1.2. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0,$$

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is a polynomial of degree n, such that for some $k \geq 1$,

$$ka_n \geq a_{n-1} \geq \ldots\ldots.. \geq a_1 \geq a_0 \geq 0,$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k. \tag{1.2}$$

Aziz and Mohammad [2] extended Enestrom-Kakeya Theorem to a class of analytic functions $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$ with its coefficients a_j satisfying a relation analogus to (1.1) and proved.

Theorem 1.3. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$ be analytic in $|z| \leq t$.*

If $a_j > 0$ and $a_{j-1} - ta_j \geq 0$, $j = 1, 2, 3, \cdots$, then $f(z)$ does not vanish in $|z| < t$.

Recently Aziz and Shah [3] relaxed the hypothesis of Theorem 1.3 and established the following result:

Theorem 1.4. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$ be analytic in $|z| \leq t$ such that for some $k \geq 1$,*

$$ka_0 \geq ta_1 \geq t^2 a_2 \geq \cdots,$$

then $f(z)$ does not vanish in

$$\left| z - \frac{k-1}{2k-1} t \right| < \frac{kt}{2k-1}.$$

In this paper we shall prove the following more general result which includes Theorems 1.3 and 1.4 as special cases. These Theorems and many other such results can be established from Theorem 1.5 by a fairly uniform procedure.

Theorem 1.5. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$ be analytic for $|z| \leq 1$. If for some $k \geq 1$,*

$$Max_{|z|=1} \left| (ka_0 - a_1) + (a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots \right| \leq M, \tag{1.3}$$

then $f(z)$ does not vanish in the disk

$$\left| z - \frac{(k\text{-}1)\, |a_0|^2}{M^2 - (k\text{-}1)^2\, |a_0|^2} \right| \leq \frac{M\, |a_0|}{M^2 - |a_0|^2\, (k-1)^2}. \tag{1.4}$$

Applying this result to $f(tz)$, we immediately get the following generalization of Theorem 1.4.

Corollary 1.6. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be analytic in $|z| \leq t$. If for some $k \geq 1$,*

$$Max_{|z|=1} \left| (ka_0 - ta_1) + (a_1 - ta_2)z + (a_2 - ta_3)z^2 + \cdots \right| \leq M,$$

then $f(z)$ does not vanish in the disk

$$\left|z-\frac{(k-1)\left|a_0\right|^2 t}{M^2-(k-1)^2\left|a_0\right|^2}\right| \leq \frac{Mt\left|a_0\right|}{M^2-\left|a_0\right|^2(k-1)^2}. \tag{1.5}$$

Remark 1.7. *Suppose $f(z)$ satisfies the hypothesis of Theorem 1.4. Tthen*

$$\begin{aligned} &Max_{|z|=1}\left|(ka_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)z^2+\cdots\right| \\ &\leq |ka_0-ta_1|+|a_1-ta_2|\,t+|a_2-ta_3|\,t^2+\cdots \\ &=(ka_0-a_1)+(ta_1-t^2a_2)+\cdots \\ &=ka_0=M. \end{aligned}$$

So that,

$$M^2-\left|a_0\right|^2(k-1)^2=\left|a_0\right|^2\left[k^2-(k-1)^2\right]=\left|a_0\right|^2(2k-1).$$

Using this in (1.5), it follows that $f(z)$ does not vanish in the disk

$$\left|z-\frac{(k-1)t}{2k-1}\right| \leq \frac{tk}{2k-1}. \tag{1.6}$$

This is precisely the conclusion of Theorem 1.4.

Next we prove the following result which is a generalization of Theorem 1.3.

Theorem 1.8. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j\,(\neq 0)$ be analytic for $|z|\leq t$. If for some $k\geq 1$,*

$$Max_{|z|=1}\left|(ka_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)z^2+\cdots\right| \leq M,$$

then $f(z)$ does not vanish in the disk

$$|z|<\frac{t\left|a_0\right|}{(k-1)\left|a_0\right|+M}. \tag{1.7}$$

Remark 1.9. *If $f(z)$ satisfies the hypothesis of Theorem 1.3, then clearly*

$$\begin{aligned} &Max_{|z|=t}\left|(ka_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)z^2+\cdots\right| \\ &\leq |ka_0-ta_1|+|a_1-ta_2|\,t+|a_2-ta_3|\,t^2+\cdots \\ &=(ka_0-ta_1)+(ta_1-t^2a_2)+\cdots=ka_0 \end{aligned}$$

and we immediately get the following generalization of Theorem 1.3.

Corollary 1.10. *If $f(z)$ is analytic in $|z|\leq t$ and if for some positive number $k\geq 1$,*

$$ka_0 \geq ta_1 \geq t^2a_2 \geq \cdots,$$

then $f(z)$ does not vanish in

$$|z| \leq \frac{t}{2k-1}.$$

Remark 1.11. *For k =1, Corollary 1.10 reduces to Theorem 1.3. If*

$$H(z) = \left|(ka_0 - ta_1) + (a_1 - ta_2)z + (a_2 - ta_3)z^2 + \cdots\right|,$$

then

$$\begin{aligned} M &= Max_{|z|=t}\,|H(z)| \geq |H(t)| \\ &= \left|(ka_0 - ta_1) + (a_1 - ta_2)t + (a_2 - ta_3)t^2 + \cdots\right| \\ &= k\,|a_0|\,. \end{aligned}$$

Now for $k \geq 1$, we have

$$2\,|a_0| \leq 2k\,|a_0|\,,$$

which implies,

$$|a_0| \leq (k-1)\,|a_0| + k\,|a_0| \leq (k-1)\,|a_0| + M.$$

So that

$$\frac{t\,|a_0|}{(k-1)\,|a_0| + M} \leq t.$$

This gives

$$\min\left\{\frac{t\,|a_0|}{(k-1)\,|a_0| + M}\ ,\ t\right\} = \frac{t\,|a_0|}{(k-1)\,|a_0| + M}.$$

Keeping these observation in view, we see that Theorem 1.8 is a special case of the following more general result for $R = t$.

Theorem 1.12. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$, be analytic for $|z| \leq R$. If for some positive real numbers t and R,*

$$Max_{|z|=R}\,|H(z)| \leq M,$$

where

$$H(z) = \left|(ka_0 - ta_1) + (a_1 - ta_2)z + (a_2 - ta_3)z^2 + \cdots\right|. \tag{1.8}$$

Then $f(z)$ does not vanish in the disk

$$|z| \leq \min\left\{\frac{t\,|a_0|}{(k-1)\,|a_0| + M}\ ,\ R\right\}.$$

If we take $k = 1$, we immediately get the following result;

Corollary 1.13. *If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ $(\neq 0)$, be analytic in $|z| \leq R$ and*

$$Max_{|z|=R}\left|\sum_{j=0}^{\infty}(a_{j-1} - ta_j)z^{j-1}\right| \leq M,$$

then $f(z)$ does not vanish in the disk

$$|z| \leq \min\left\{\frac{t\,|a_0|}{M}\ ,\ R\right\}.$$

Corollary 1.13 was independently proved by Aziz and Shah [4, Cor 3].

2. Proofs of the theorems

Proof of Theorem 1.5. It is obvious that, $\lim_{j\to\infty} a_j = 0$, consider the function.

$$\begin{aligned}F(z) &= (z-1)f(z) = -a_0 + (a_0 - a_1)z + (a_1 - a_2)z^2 + \cdots \\ &= -a_0 - ka_0 z + a_0 z + (ka_0 - a_1)z + (a_1 - a_2)z^2 + \cdots \\ &= -a_0 - a_0 z(k-1) + zH(z),\end{aligned}$$

where

$$H(z) = (ka_0 - a_1) + (a_1 - a_2)z + \cdots .$$

Clearly,

$$\begin{aligned}M &= \mathrm{M}ax_{|z|=1} |H(z)| \\ &\geq |H(1)| = |(ka_0 - a_1) + (a_1 - a_2) + \cdots| = k|a_0| .\end{aligned} \tag{2.1}$$

Now for $|z| < 1$, we have

$$|F(z)| \geq |a_0| |z(k-1) + 1| - |z| |H(z)| , \tag{2.2}$$

where

$$H(z) = (ka_0 - a_1) + (a_1 - a_2)z + \cdots .$$

Since $H(z)$ is analytic for $|z| \leq 1$ and

$$|H(z)| \leq M \quad \text{for} \quad |z| = 1.$$

By maximum modulus principle, it follows that

$$|H(z)| \leq M \quad \text{for} \quad |z| \leq 1.$$

Using this fact in (2.2) we get

$$|F(z)| \geq |a_0| |z(k-1) + 1| - |z| M > 0,$$

if

$$\frac{M|z|}{|a_0|} < |z(k-1) + 1| .$$

Now it can be verified that the region E defined by

$$E = \left\{ z : \frac{M|z|}{|a_0|} < |z(k-1) + 1| \right\}$$

is precisely the disk

$$E = \left\{ z : \left| z - \frac{(k-1)|a_0|^2}{M^2 - (k-1)^2 |a_0|^2} \right| \leq \frac{M|a_0|}{M^2 - (k-1)^2 |a_0|^2} \right\}.$$

It can be easily seen that E is contained in the disk $|z| \le 1$. For if $w \in E$, then clearly

$$\left| w - \frac{(k-1)\,|a_0|^2}{M^2 - (k-1)^2\,|a_0|^2} \right| \le \frac{M\,|a_0|}{M^2 - (k-1)^2\,|a_0|^2}.$$

Which implies

$$|w| \le \frac{|a_0|\,\{(k-1)\,|a_0| + M\}}{M^2 - (k-1)^2\,|a_0|^2} \le 1,$$

if

$$|a_0| < M - (k-1)\,|a_0|$$

or, if

$$k\,|a_0| \le M.$$

Which is true by (2.1). Using these observations we conclude that $F(z)$ does not vanish in the disk defined by (1.4). Since all the zeros of $f(z)$ are also the zeros of $F(z)$, we conclude that $f(z)$ does not vanish in the disk.

$$\left| z - \frac{(k-1)\,|a_0|^2}{M^2 - (k-1)^2\,|a_0|^2} \right| \le \frac{M\,|a_0|}{M^2 - (k-1)^2\,|a_0|^2}.$$

Which completes the proof of Theorem 1.5.

Proof of Theorem 1.12. Consider the function

$$\begin{aligned} F(z) = (z-t)f(z) &= -a_0t + (a_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots \\ &= -a_0t - ka_0z + a_0z + (ka_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots \\ &= -a_0t - a_0z(k-1) + zH(z), \end{aligned}$$

where

$$H(z) = (ka_0 - ta_1) + (a_1 - ta_2)z + \cdots .$$

We first assume that

$$M \ge |a_0| \left\{ \frac{t}{R} - (k-1) \right\}. \tag{2.3}$$

Since $H(z)$ is analytic for $|z| \le R$ and $|H(z)| \le M$ for $|z| = R$ therefore by maximum modulus theorem it follows that $|H(z)| \le M$ for $|z| \le R$. Hence for$|z| \le R$,

$$|F(z)| = |-ta_0 - a_0(k-1)z + zH(z)| \ge |ta_0| - |a_0|\,|k-1|\,|z| - M\,|z|\,.$$

This implies

$$|F(z)| > 0,$$

if

$$|ta_0| - (|a_0|\,(k-1) + M)\,|z| > 0$$

or if, by (2.3),

$$|z| < \frac{|ta_0|}{(|a_0|\,(k-1) + M)} < R.$$

Now assume,

$$M < |a_0| \left(\frac{t}{R} - (k-1) \right),$$

then for $|z| \le R$, we have,

$$|H(z)| \le M$$

and by (2.4),

$$|F(z)| = |-ta_0 - a_0(k-1)z + zH(z)| > t\,|a_0| - (|a_0|\,(k-1) + M)R \ge 0.$$

Thus $|F(z)| > 0$ for $|z| \le R$, this implies $F(z) \ne 0$ for $|z| \le R$. From which it follows that $F(z)$ does not vanish for $|z| \le R$ in this case also. Combining this with (2.3). We conclude that $F(z)$ does not vanish in the disk

$$|z| \le \min \left\{ \frac{t\,|a_0|}{(k-1)\,|a_0| + M}\ ,\ R \right\}$$

and this completes the proof of Theorem 1.12.

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