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EXISTENCE AND STABILITY OF SOLUTIONS OF NEUTRAL HYBRID STOCHASTIC INFINITE DELAY DIFFERENTIAL EQUATIONS WITH POISSON JUMPS

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Abstract. In this paper, we prove the existence, uniqueness and stability of solutions of neutral stochastic infinite delay differential equations with Poisson jumps and Markovian switching in the phase space $BC\left((-\infty, 0]; \Re^d\right)$ which is the family of bounded continuous $\Re^d$-valued functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$ under non-Lipschitz condition and weakened linear growth condition. The solutions are constructed by the successive approximation method. Also we prove the continuous dependence of solutions on the initial value.

1. Introduction

Stochastic delay differential equations have many applications in economics, biology, medicine and finance. Many of the processes either natural or manual involve time delays and they are dependent on the impact of the past. Indeed, stochastic delay differential equations as the stochastic models appear frequently in applied research and the related study has received considerable

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attention. Stability of stochastic differential equations is essential to avoid possible explosion of solutions. Mao [1] discussed stochastic functional differential equations under uniform Lipschitz condition and linear growth condition on the coefficients. Liu and Xia [2] investigated the exponential stability in mean square of neutral stochastic functional differential equations. In contrast to these theoretical studies there appear several papers on numerical stability of stochastic differential equations see [3, 4, 5] and references therein.

The Poisson jumps become very popular in recent years, because it is extensively used to model many of the phenomena arising in the areas such as economics, finance, physics, biology, medicine and other sciences. For example, a system jumps from a "normal state" to a "bad state", the strength of system is of random. It is natural and necessary to include jumps term in the stochastic delay differential equations. Fixed point method was first introduced by Liu et al.[6] to study the stability problem of neutral stochastic delay differential equations with Poisson jumps. Another important kind of stochastic functional differential equations is the neutral stochastic functional differential equations with finite delay which could be used in chemical engineering and aeroelasticity [7]. Wei and Wang [8] and Zhou and Xue [9] established the existence and uniqueness of solutions to the neutral stochastic functional differential equations with infinite delay. Ayoola and Gbolagade ivestigated the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations [16]. In [10, 11], the authors have studied the existence, uniqueness and stability of solutions to neutral stochastic functional differential equations with infinite delay under non-Lipschitz condition and weakened linear growth condition.

The hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. One of the important classes of the hybrid systems is the stochastic delay differential equations(SDDEs) with Markovian switching.

Motivated by the above work [10, 11], in this paper, we prove the existence, uniqueness and stability of solutions of neutral stochastic infinite delay differential equations with Poisson jumps and Markovian switching (NSIDDEPM) in the phase space $BC\left((-\infty, 0]; \Re^d\right)$ under non-Lipschitz condition and weakened linear growth condition. The solution is constructed by the successive approximation technique. Furthermore, we prove the continuous dependence of solutions on the initial value.

2. Preliminaries and Notations

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ (where $t_0 > 0$) satisfying the usual conditions, that is, the filtration is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $\left\{ W(t) = (W_1(t), W_2(t), \cdots, W_m(t))^T, t \geq 0 \right\}$ denote a standard m-dimensional Wiener process defined on the probability space. Let $BC\left((-\infty, 0]; R^d\right)$ denotes the family of bounded continuous $\Re^d$-value functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\|$=$sup_{-\infty<\theta\leq 0}$ $|\varphi(\theta)|$, where $|.|$ is the Euclidean norm in $\Re^d$, i.e., $|x| = \sqrt{x^T x}\ \left(x \in \Re^d\right)$. If A is a vector or matrix, its trace norm is defined by $|A| = \sqrt{trace\,(A^T A)}$. Denote by $\mathcal{M}^2\left((-\infty, 0]\,; \Re^d\right)$ the family of all $\mathcal{F}_{t_0}$-measurable, $\Re^d$-valued process $\psi(t) = \psi(t, \omega)$, $t \in (-\infty, 0]$ such that $E \int_{-\infty}^{0} |\psi(t)|^2 dt < \infty$.

Let $\{v\,(dt, du)\,, t \in \Re_+, u \in \Re\}$ be a centered Poisson random measure with parameter $\pi\,(du)\,dt$. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\left\{r\,(t+\Delta) = j | r\,(t) = i\right\} = \begin{cases} \gamma_{ij}\Delta + o\,(\Delta) & if\ i \neq j, \\ 1 + \gamma_{ii}\Delta + o\,(\Delta) & if\ i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(.)$ is independent of the Brownian motion $w(.)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\Re_+$. As for $r(t)$, the following lemma is satisfied (see[14]).

Lemma 2.1. *Given $h > 0$, then $\left\{r_n^h = r\,(nh)\,, n = 0, 1, 2, ...\right\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P\,(h) = (P_{ij}\,(h))_{N\times N} = e^{h\Gamma}.$$

Since the γ_{ij} are independent of x, the paths of Markov chain r can be generated independent of x and, in fact, before computing x. It is well known that $r(t)$ is ergodic Markov chain and that there is a sequence $\{\tau_k\}_{k\geq 0}$ of stopping times such that $t_0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_k \to \infty$ and $r(t)$ is a constant on every interval $[\tau_k, \tau_{k+1})$, i.e., for every $k \geq 0$, $r(t) = r(\tau_k)$ on $t \in [\tau_k, \tau_{k+1})$ (see[15]).

Consider the following d-dimensional neutral stochastic infinite delay differential equations with Poisson jumps and Markovian switching of the form

$$\begin{aligned} & d\left[X(t) - G\left(t, X_t, r(t)\right)\right] \\ = \; & \left(t, X_t, r(t)\right) dt + g\left(t, X_t, r(t)\right) dW(t) \\ & + \int_{-\infty}^{+\infty} h\left(t, X_t, u\right) \tilde{v}\left(dt, du\right), \quad t_0 \leq t \leq T, \end{aligned} \tag{2.1}$$

where $X_t = \{X\left(t+\theta\right) : -\infty < \theta \leq 0\}$ can be regarded as a $BC\left((-\infty, 0]; \Re^d\right)$-value stochastic process and

$$f : [t_0, T] \times BC\left((-\infty, 0]; \Re^d\right) \times S \to \Re^d,$$

$$g : [t_0, T] \times BC\left((-\infty, 0]; \Re^d\right) \times S \to \Re^{d \times m},$$

$$h : [t_0, T] \times BC\left((-\infty, 0]; \Re^d\right) \times \Re \to \Re^d,$$

and

$$G : [t_0, T] \times BC\left((-\infty, 0]; \Re^d\right) \times S \to \Re^d$$

be Borel measurable.

Let $\tilde{v}\left(dt, du\right) = v\left(dt, du\right) - \pi(du)dt$ be a compensated Poisson random measure which is independent of $\{W(t)\}$ and we assume that $\int_{-\infty}^{+\infty} \pi(du) < \infty$. The initial value of (2.1) is given by

$$X_{t_0} = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\} \tag{2.2}$$

is $\mathcal{F}_{t_0}$-measurable, $BC\left((-\infty, 0]; \Re^d\right)$-value random variable such that $\xi \in \mathcal{M}^2\left((-\infty, 0]\,; \Re^d\right)$.

Definition 2.2. $\Re^d$-value stochastic process $X(t)$ defined on $-\infty < t \leq T$ is called the solution of NSIDDEPM (2.1) with initial value (2.2), if

(i) $X(t)$ is continuous and for all $t_0 \leq t \leq T$, $X(t)$ is $\mathcal{F}_t$-adapted.

(ii) $\{f\left(t, X_t, r(t)\right)\} \in \mathcal{L}^1\left([t_0, T]\,; \Re^d\right)$, $\{g\left(t, X_t, r(t)\right)\} \in \mathcal{L}^2\left([t_0, T]\,; \Re^{d \times m}\right)$, and $\{h\left(t, X_t, u\right)\} \in \mathcal{L}^3\left([t_0, T] \times \Re; \Re^d\right)$.

(iii) $X_{t_0} = \xi$, for each $t_0 \leq t \leq T$,

$$\begin{aligned} X(t) \;=\; & \xi(0) + G\left(t, X_t, r(t)\right) - G\left(t_0, \xi, r(t_0)\right) + \int_{t_0}^{t} f\left(s, X_s, r(s)\right) ds \\ & + \int_{t_0}^{t} g\left(s, X_s, r(s)\right) dW(s) + \int_{t_0}^{t} \int_{-\infty}^{+\infty} h\left(s, X_s, u\right) \tilde{v}\left(ds, du\right) \quad \text{a.s.} \end{aligned}$$

Definition 2.3. $X(t)$ is called as a unique solution, if any other solution $\bar{X}(t)$ is distinguishable with $X(t)$, that is

$$P\left(X(t) = \bar{X}(t), \text{ for all} - \infty < t \leq T\right) = 1.$$

To establish the existence of solution of (2.1) with initial vaue (2.2), we assume the following conditions:

(H1): For all $\varphi, \psi \in BC\left((-\infty, 0]; \Re^d\right)$, $t \in [t_0, T]$ and $i \in S$, it follows

$$|f(t,\varphi,i) - f(t,\psi,i)|^2 \vee |g(t,\varphi,i) - g(t,\psi,i)|^2 \vee \int_{-\infty}^{+\infty} |h(t,\varphi,u) - h(t,\psi,u)|^2 \pi(du) \leq \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(.)$ is a concave nondecreasing function from $\Re_+$ to $\Re_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\kappa(u)} = \infty$.

(H2): For all $t \in [t_0, T]$, it follows that $f(t,0,i), g(t,0,i), h(t,0,u) \in \mathcal{L}^2$ and $i \in S$, such that

$$|f(t,0,i)|^2 \vee |g(t,0,i)|^2 \vee \int_{-\infty}^{+\infty} |h(t,0,u)|^2 \pi(du) \leq K,$$

where $K > 0$ is a constant.

(H3): For any $\varphi, \psi \in BC\left((-\infty, 0]; \Re^d\right)$, $t \in [t_0, T]$ and $i \in S$, there exists a positive number K_0 such that $K_0 < \frac{1}{10}$ such that

$$|G(t,\varphi,i) - G(t,\psi,i)|^2 \leq K_0 \|\varphi - \psi\|^2.$$

Remark 2.4. The importance of the above types of conditions are discussed in [10].

In order to obtain the uniqueness of solutions, we use the Bihari inequality proved in [12].

Lemma 2.5. *(Bihari inequality) Let $T > 0$ and $u_0 > 0$, $u(t)$, $v(t)$ be continuous functions on $[0,T]$. Let $\kappa : \Re_+ \to \Re_+$ be a concave continuous nondecreasing function such that $\kappa(r) > 0$ for $r > 0$. If*

$$u\,(t) \leq u_0 + \int_0^t v(s) \kappa\,(u(s))\, ds \quad \textit{for all} \quad 0 \leq t \leq T,$$

then

$$u\,(t) \leq G^{-1}\left(G(u_0) + \int_0^t v(s) ds\right)$$

for all such $t \in [0,T]$ that

$$G(u_0) + \int_0^t v(s) ds \in Dom\left(G^{-1}\right)$$

where $G(r)=\int_1^r \frac{ds}{\kappa(s)}$, $r\geq 0$ *and* G^{-1} *is the inverse function of* G. *In particular, if, moreover,* $u_0=0$ *and* $\int_{0^+}\frac{ds}{\kappa(s)}=\infty$, *then* $u(t)=0$ *for all* $0\leq t\leq T$.

For establishing the stability of solutions, we give a generalization of Bihari inequality which appeared in [[13],Lemma 3.2] and its corollary.

Lemma 2.6. *Let the assumptions of Lemma 2.5 hold. If*

$$u(t)\leq u_0+\int_t^T v(s)\kappa\left(u(s)\right)ds \quad \text{for all} \quad 0\leq t\leq T,$$

then

$$u(t)\leq G^{-1}\left(G(u_0)+\int_t^T v(s)ds\right)$$

for all such $t\in[0,T]$ *that*

$$G(u_0)+\int_t^T v(s)ds\in Dom\left(G^{-1}\right)$$

where $G(r)=\int_1^r \frac{ds}{\kappa(s)}$, $r\geq 0$ *and* G^{-1} *is the inverse function of* G.

Corollary 2.7. *Let the assumptions of Lemma 2.5 hold and* $v(t)\geq 0$ *for* $t\in[0,T]$. *If for all* $\epsilon>0$, *there exists* $t_1\geq 0$ *such that for* $0\leq u_0<\epsilon$, $\int_{t_1}^T v(s)ds\leq\int_{u_0}^{\epsilon}\frac{ds}{\kappa(s)}$ *holds. Then for every* $t\in[t_1,T]$, *the estimate* $u(t)\leq\epsilon$ *holds.*

3. Existence and Uniqueness of Solutions

Now we prove the existence and uniqueness theorem to the equation (2.1) with initial value (2.2) under the above non-Lipschitz condition and the weakened linear growth condition.

Theorem 3.1. *Assume that* $(H1)$, $(H2)$ *and* $(H3)$ *hold. Then, there exists a unique solution to NSIDDEPM (2.1) with initial value (2.2).*

To prove the above theorem, we first consider Eq.(2.1) on $t\in[t_0,\tau_1\wedge T]$. The solution of Eq.(2.1) on $t\in[t_0,\tau_1\wedge T]$ is denoted by $X^{\tau_1}(t)$. Now we have to prove several lemmas. For that we need the following definition.

Definition 3.2. Let $(X^{\tau_1})^0(t) = \xi(0)$ and $(X^{\tau_1})^n_{t_0} = \xi$, $(n = 1, 2, ...)$. Then the Picard sequence is

$$\begin{aligned} & (X^{\tau_1})^n(t) - G\left(t, (X^{\tau_1})^n_t, i_0\right) \\ = \ & \xi(0) - G\left(t_0, (X^{\tau_1})^n_{t_0}, i_0\right) \\ & + \int_{t_0}^t f\left(s, (X^{\tau_1})^{n-1}_s, i_0\right) ds + \int_{t_0}^t g\left(s, (X^{\tau_1})^{n-1}_s, i_0\right) dW(s) \\ & + \int_{t_0}^t \int_{-\infty}^{+\infty} h\left(s, (X^{\tau_1})^{n-1}_s, u\right) \tilde{v}\left(ds, du\right), \quad t_0 \le t \le \tau_1 \wedge T. \end{aligned} \tag{3.1}$$

Lemma 3.3. *Under the assumptions of Theorem 3.1 , for all* $t \in (-\infty, \tau_1 \wedge T]$, $n \ge 1$,

$$E\left|(X^{\tau_1})^n(t)\right|^2 \le C_1, \tag{3.2}$$

where C_1 *is a positive constant.*

Proof. Obviously, $(X^{\tau_1})^0(t) \in \mathcal{M}^2\left((-\infty, \tau_1 \wedge T]; \Re^d\right)$.
By induction, $(X^{\tau_1})^n(t) \in \mathcal{M}^2\left((-\infty, \tau_1 \wedge T]; \Re^d\right)$. In fact, from (3.1) and $r(t) = i_0$ for $t \in [t_0, \tau_1 \wedge T]$, we have

$$\begin{aligned} & E\left[\sup_{t_0 \le s \le t} \left|(X^{\tau_1})^n(s)\right|^2\right] \\ \le \ & 5E\left[\sup_{t_0 \le s \le t} \left|G(s, (X^{\tau_1})^n_s, i_0) - G(t_0, (X^{\tau_1})^n_{t_0}, i_0)\right|^2\right] \\ & +5E\left|\xi(0)\right|^2 + 5E\left|\int_{t_0}^t f\left(s, (X^{\tau_1})^{n-1}_s, i_0\right) ds\right|^2 \\ & +5E\left|\int_{t_0}^t g\left(s, (X^{\tau_1})^{n-1}_s, i_0\right) dW(s)\right|^2 \\ & +5E\left|\int_{t_0}^t \int_{-\infty}^{+\infty} h\left(s, (X^{\tau_1})^{n-1}_s, u\right) \tilde{v}\left(ds, du\right)\right|^2. \end{aligned} \tag{3.3}$$

From Hölder inequality, we get

$$\begin{aligned} & E\left[\sup_{t_0 \le s \le t} \left|(X^{\tau_1})^n(s)\right|^2\right] \\ \le \ & 5E\left[\sup_{t_0 \le s \le t} \left|G(s, (X^{\tau_1})^n_s, i_0) - G(t_0, (X^{\tau_1})^n_{t_0}, i_0)\right|^2\right] + 5E\left|\xi(0)\right|^2 \\ & +5\left((\tau_1 \wedge T) - t_0\right) E\int_{t_0}^t \left|f\left(s, (X^{\tau_1})^{n-1}_s, i_0\right) - f\left(s, 0, i_0\right) + f\left(s, 0, i_0\right)\right|^2 ds \end{aligned}$$

$$
\begin{aligned}
&+5E\int_{t_0}^{t}\left|g\left(s,(X^{\tau_1})_s^{n-1},i_0\right)-g\left(s,0,i_0\right)+g\left(s,0,i_0\right)\right|^2 ds\\
&+5E\int_{t_0}^{t}\int_{-\infty}^{+\infty}\left|h\left(s,(X^{\tau_1})_s^{n-1},u\right)-h\left(s,0,u\right)+h\left(s,0,u\right)\right|^2\pi\left(du\right)ds\\
\leq\ & 10K_0E\left[\sup_{t_0\leq s\leq t}\|(X^{\tau_1})_s^n\|^2\right]+\left(10K_0+5\right)E\left\|\xi\right\|^2\\
&+10\left(\left(\tau_1\wedge T\right)-t_0\right)E\int_{t_0}^{t}\left(\left|f\left(s,(X^{\tau_1})_s^{n-1},i_0\right)-f\left(s,0,i_0\right)\right|^2+\left|f\left(s,0,i_0\right)\right|^2\right)ds\\
&+10E\int_{t_0}^{t}\left(\left|g\left(s,(X^{\tau_1})_s^{n-1},i_0\right)-g\left(s,0,i_0\right)\right|^2+\left|g\left(s,0,i_0\right)\right|^2\right)ds\\
&+10E\int_{t_0}^{t}\int_{-\infty}^{+\infty}\left(\left|h\left(s,(X^{\tau_1})_s^{n-1},u\right)-h\left(s,0,u\right)\right|^2+\left|h\left(s,0,u\right)\right|^2\right)\pi\left(du\right)ds.\\
\leq\ & 10K_0E\left[\sup_{t_0\leq s\leq t}\|(X^{\tau_1})_s^n\|^2\right]+\left(10K_0+5\right)E\left\|\xi\right\|^2\\
&+10\left(\left(\tau_1\wedge T\right)-t_0+2\right)\left(\left(\tau_1\wedge T\right)-t_0\right)K\\
&+10\left(\left(\tau_1\wedge T\right)-t_0+2\right)E\int_{t_0}^{t}\kappa\left(\left\|(X^{\tau_1})_s^{n-1}\right\|^2\right)ds. \qquad (3.4)
\end{aligned}
$$

For

$$
E\left[\sup_{t_0\leq s\leq t}\left\|(X^{\tau_1})_s^{n-1}\right\|^2\right]\leq E\left[\sup_{t_0\leq s\leq t}\left|(X^{\tau_1})^{n-1}(s)\right|^2\right]+E\left\|\xi\right\|^2.
$$

We obtain

$$
\begin{aligned}
&E\left[\sup_{t_0\leq s\leq t}|(X^{\tau_1})^n(s)|^2\right]\\
&\leq 10K_0E\left[\sup_{t_0\leq s\leq t}|(X^{\tau_1})^n(s)|^2\right]+\left(20K_0+5\right)E\left\|\xi\right\|^2\\
&\quad+10\left(\left(\tau_1\wedge T\right)-t_0+2\right)\left(T-t_0\right)K\\
&\quad+10\left(\left(\tau_1\wedge T\right)-t_0+2\right)E\int_{t_0}^{t}\kappa\left(\left\|(X^{\tau_1})_s^{n-1}\right\|^2\right)ds. \qquad (3.5)
\end{aligned}
$$

Given that $\kappa(.)$ is concave and $\kappa(0)=0$, we can find a pair of positive constants a and b such that

$$
\kappa(u)\leq a+bu,\quad \text{for all}\quad u\geq 0.
$$

So, we have

$$
\left(1-10K_0\right)E\left[\sup_{t_0\leq s\leq t}|(X^{\tau_1})^n(s)|^2\right]\leq C+10\left(\left(\tau_1\wedge T\right)-t_0+2\right)bE\int_{t_0}^{t}\left\|(X^{\tau_1})_s^{n-1}\right\|^2ds.
$$

Note that

$$\begin{aligned}&\max_{1\le n\le k} E\left|(X^{\tau_1})^{n-1}(s)\right|^2\\&=\max\left\{E\left|\xi(0)\right|^2, E\left|(X^{\tau_1})^1(s)\right|^2, ..., E\left|(X^{\tau_1})^{k-1}(s)\right|^2\right\}\\&\le\max\left\{E\left|\xi(0)\right|^2, E\left|(X^{\tau_1})^1(s)\right|^2, ..., E\left|(X^{\tau_1})^{k-1}(s)\right|^2, E\left|(X^{\tau_1})^{k}(s)\right|^2\right\}\\&\le E\left\|\xi\right\|^2+\max_{1\le n\le k} E\left|(X^{\tau_1})^{n}(s)\right|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&\max_{1\le n\le k} E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n}(s)\right|^2\right]\\&\le C+\frac{10\left((\tau_1\wedge T)-t_0+2\right)b}{(1-10K_0)}\int_{t_0}^{\tau_1\wedge T}\max_{1\le n\le k} E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n}(s)\right|^2\right]dt.\end{aligned}$$

From Grownwall's inequality and since k is arbitrary, we have

$$\begin{aligned}E\left|(X^{\tau_1})^{n}(t)\right|^2\le C\exp\left(\frac{10\left((\tau_1\wedge T)-t_0+2\right)b}{(1-10K_0)}\left((\tau_1\wedge T)-t_0\right)\right),\\ t_0\le t\le\tau_1\wedge T,\quad n\ge 1.\end{aligned}$$

So, the proof is completed with $C_1=C\exp\left(\frac{10((\tau_1\wedge T)-t_0+2)b}{(1-10K_0)}\left((\tau_1\wedge T)-t_0\right)\right)$. □

Lemma 3.4. *Under the assumptions of Theorem 3.1, there exists a positive constant C_2 such that*

$$\begin{aligned}&E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n+m}(s)-(X^{\tau_1})^{n}(s)\right|^2\right]\\&\le C_2\int_{t_0}^{t}\kappa\left(E\left(\sup_{t_0\le r\le s}\left|(X^{\tau_1})^{n+m-1}(r)-(X^{\tau_1})^{n-1}(r)\right|^2\right)\right)ds \quad (3.6)\end{aligned}$$

for all $t_0\le t\le\tau_1\wedge T$, $n,m\ge 1$.

Proof. From (3.1), we have

$$
\begin{aligned}
&(X^{\tau_1})^{n+m}(t) - (X^{\tau_1})^{n}(t)\\
&= G\left(t, (X^{\tau_1})_t^{n+m}, i_0\right) - G\left(t, (X^{\tau_1})_t^{n}, i_0\right)\\
&\quad + \int_{t_0}^{t} \left[f\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - f\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right] ds\\
&\quad + \int_{t_0}^{t} \left[g\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - g\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right] dW(s)\\
&\quad + \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left[h\left(s, (X^{\tau_1})_s^{n+m-1}, u\right) - h\left(s, (X^{\tau_1})_s^{n-1}, u\right)\right] \tilde{v}\left(ds, du\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
&E\left|(X^{\tau_1})^{n+m}(t) - (X^{\tau_1})^{n}(t)\right|^2\\
&\leq 4E\left|G\left(t, (X^{\tau_1})_t^{n+m}, i_0\right) - G\left(t, (X^{\tau_1})_t^{n}, i_0\right)\right|^2\\
&\quad +4E\left|\int_{t_0}^{t} \left[f\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - f\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right] ds\right|^2\\
&\quad +4E\left|\int_{t_0}^{t} \left[g\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - g\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right] dW(s)\right|^2\\
&\quad +4E\left|\int_{t_0}^{t}\int_{-\infty}^{+\infty} \left[h\left(s, (X^{\tau_1})_s^{n+m-1}, u\right) - h\left(s, (X^{\tau_1})_s^{n-1}, u\right)\right] \tilde{v}\left(ds, du\right)\right|^2\\
&\leq 4EK_0\left|(X^{\tau_1})^{n+m}(t) - (X^{\tau_1})^{n}(t)\right|^2\\
&\quad +4\left(t-t_0\right)E\int_{t_0}^{t} \left|f\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - f\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right|^2 ds\\
&\quad +4E\int_{t_0}^{t} \left|g\left(s, (X^{\tau_1})_s^{n+m-1}, i_0\right) - g\left(s, (X^{\tau_1})_s^{n-1}, i_0\right)\right|^2 ds\\
&\quad +4E\int_{t_0}^{t}\int_{-\infty}^{+\infty} \left|h\left(s, (X^{\tau_1})_s^{n+m-1}, u\right) - h\left(s, (X^{\tau_1})_s^{n-1}, u\right)\right|^2 \pi\left(du\right) ds.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&E\left[\sup_{t_0\leq s\leq t}\left|(X^{\tau_1})^{n+m}(s) - (X^{\tau_1})^{n}(s)\right|^2\right]\\
&\leq \frac{4\left(\left(\tau_1\wedge T\right) - t_0 + 2\right)}{\left(1-4K_0\right)} E\int_{t_0}^{t} \kappa\left(\left\|(X^{\tau_1})_s^{n+m-1} - (X^{\tau_1})_s^{n-1}\right\|^2\right) ds.
\end{aligned}
$$

From Jensen's inequality, we get

$$E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n+m}(s)-(X^{\tau_1})^{n}(s)\right|^2\right]$$
$$\le\frac{4\left((\tau_1\wedge T)-t_0+2\right)}{(1-4K_0)}\int_{t_0}^{t}\kappa\left(E\sup_{t_0\le r\le s}\left|(X^{\tau_1})^{n+m-1}(r)-(X^{\tau_1})^{n-1}(r)\right|^2\right)ds.$$

If we choose $C_2=\frac{4((\tau_1\wedge T)-t_0+2)}{(1-4K_0)}$, we can show that the lemma holds. □

Lemma 3.5. *Under the assumptions of Theorem 3.1 , there exists a positive constant C_3 such that*

$$E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n+m}(s)-(X^{\tau_1})^{n}(s)\right|^2\right]\le C_3\,(t-t_0)\tag{3.7}$$

for all $t_0\le t\le\tau_1\wedge T$, $n,m\ge 1$.

Define

$$\varphi_1(t)=C_3\,(t-t_0)\,,$$

$$\varphi_{n+1}(t)=C_2\int_{t_0}^{t}\kappa\left(\varphi_n(s)\right)ds,\quad n\ge 1,$$

$$\varphi_{n,m}(t)=E\left(\sup_{t_0\le r\le t}\left|(X^{\tau_1})^{n+m}(r)-(X^{\tau_1})^{n}(r)\right|^2\right),\quad n,m\ge 1.$$

Choose $T_1\in[t_0,\tau_1\wedge T)$ such that $C_2\kappa(C_3(t-t_0))\le C_3$ for all $t_0\le t\le T_1$.

Lemma 3.6. *There exists a positive $t_0\le T_1<\tau_1\wedge T$ such that for all $n,m\ge 1$,*

$$0\le\varphi_{n,m}(t)\le\varphi_n(t)\le\varphi_{n-1}(t)\le...\le\varphi_1(t)\tag{3.8}$$

for all $t_0\le t\le T_1$.

The proofs of Lemmas 3.5 and 3.6 are similar to Lemmas 9 and 10 of [10] and hence they are omitted.

Proof of Theorem 3.1.
Uniqueness: Let $X^{\tau_1}(t)$ and $\bar{X}^{\tau_1}(t)$ be two solutions of (2.1) on $[t_0,\tau_1\wedge T]$.

Then we have

$$\begin{aligned} E\left|X^{\tau_1}(t)-\bar{X}^{\tau_1}(t)\right|^2 &= E\bigg|G\left(t,X^{\tau_1}(t),i_0\right)-G\left(t,\bar{X}^{\tau_1}(t),i_0\right) \\ &\quad+\int_{t_0}^{t}\left[f\left(s,X_s^{\tau_1},i_0\right)-f\left(s,\bar{X}_s^{\tau_1},i_0\right)\right]ds \\ &\quad+\int_{t_0}^{t}\left[g\left(s,X_s^{\tau_1},i_0\right)-g\left(s,\bar{X}_s^{\tau_1},i_0\right)\right]dW(s) \\ &\quad+\int_{t_0}^{t}\int_{-\infty}^{+\infty}\left[h\left(s,X_s^{\tau_1},u\right)-h\left(s,\bar{X}_s^{\tau_1},u\right)\right]\tilde{v}\left(ds,du\right)\bigg|^2. \end{aligned}$$

So, we derive

$$\begin{aligned} &E\left[\sup_{t_0\le s\le t}\left|X^{\tau_1}(s)-\bar{X}^{\tau_1}(s)\right|^2\right] \\ &\le 4K_0E\left\|X_t^{\tau_1}-\bar{X}_t^{\tau_1}\right\|^2+4\left((\tau_1\wedge T)-t_0+2\right)E\int_{t_0}^{t}\kappa\left(\left\|X_s^{\tau_1}-\bar{X}_s^{\tau_1}\right\|^2\right)ds \\ &\le 4K_0E\left[\sup_{t_0\le s\le t}\left|X^{\tau_1}(s)-\bar{X}^{\tau_1}(s)\right|^2\right] \\ &\quad+4\left((\tau_1\wedge T)-t_0+2\right)E\int_{t_0}^{t}\kappa\left(\left\|X_s^{\tau_1}-\bar{X}_s^{\tau_1}\right\|^2\right)ds. \end{aligned}$$

From Jensen's inequality, we get

$$\begin{aligned} &E\left[\sup_{t_0\le s\le t}\left|X^{\tau_1}(s)-\bar{X}^{\tau_1}(s)\right|^2\right] \\ &\le \frac{4\left((\tau_1\wedge T)-t_0+2\right)}{(1-4K_0)}\int_{t_0}^{t}\kappa\left(E\left(\left\|X_s^{\tau_1}-\bar{X}_s^{\tau_1}\right\|^2\right)\right)ds \\ &\le \frac{4\left((\tau_1\wedge T)-t_0+2\right)}{(1-4K_0)}\int_{t_0}^{t}\kappa\left(E\sup_{t_0\le r\le s}\left|X^{\tau_1}(r)-\bar{X}^{\tau_1}(r)\right|^2\right)ds. \end{aligned}$$

Bihari inequality yields

$$E\left[\sup_{t_0\le s\le t}\left|X^{\tau_1}(s)-\bar{X}^{\tau_1}(s)\right|^2\right]=0,\quad t_0\le t\le(\tau_1\wedge T).$$

The above expression means that $X^{\tau_1}(t)=\bar{X}^{\tau_1}(t)$ for all $t_0\le t\le(\tau_1\wedge T)$. Therefore, for all $-\infty<t\le(\tau_1\wedge T)$, $X^{\tau_1}(t)=\bar{X}^{\tau_1}(t)$ a.s.

We now consider Eq.(2.1) on $[\tau_1\wedge T,\tau_2\wedge T]$. Following the same path of the above proof, we can obtain that Eq.(2.1) has a unique solution $X^{\tau_2}(t)$ on $[\tau_1\wedge T,\tau_2\wedge T]$. Repeating this procedure, we get Eq.(2.1) has a unique solution

$X(t)$ on $[0,T]$. This establish the uniqueness.

Existence: We claim

$$E\left[\sup_{t_0\le s\le t}\left|(X^{\tau_1})^{n+m}(s)-(X^{\tau_1})^{n}(s)\right|^2\right]\to 0$$

for all $t_0 \le t \le T_1 < \tau_1 \wedge T$, as $n,m\to\infty$. Note that φ_n is continuous on $[t_0,T_1]$. Note also that for each $n\ge 1$, $\varphi_n(.)$ is decreasing on $[t_0,T_1]$, and for each t, $\varphi_n(t)$ is a decreasing sequence. Therefore, we can define the function $\varphi(t)$ as

$$\varphi(t)=\lim_{n\to\infty}\varphi_n(t)=\lim_{n\to\infty}C_2\int_{t_0}^{t}\kappa\left(\varphi_{n-1}(s)\right)ds=C_2\int_{t_0}^{t}\kappa\left(\varphi(s)\right)ds$$

for all $t_0\le t\le T_1$. Bihari inequality implies that $\varphi(t)=0$ for all $t_0\le t\le T_1$. Now, from Lemma 3.6, we have

$$\varphi_{n,n}(t)\le\sup_{t_0\le t\le T_1}\varphi_n(t)\le\varphi_n(T_1)\to 0$$

as $n\to\infty$. That is $(X^{\tau_1})^n(t)$ is a Cauchy sequence in L^2 on $(-\infty,T_1]$. From Lemma 3.3, we can easily derive

$$E\left|X^{\tau_1}(t)\right|^2\le C,$$

where C is a positive constant.

Using the property of the function to $\kappa(.)$, we can obtain that for all $t_0\le t\le T_1$,

$$E\left|\int_{t_0}^{t}\left[f\left(s,(X^{\tau_1})^n_s,i_0\right)-f\left(s,(X^{\tau_1})_s,i_0\right)\right]ds\right|^2\to 0,\ \text{as } n\to\infty,$$

$$E\left|\int_{t_0}^{t}\left[g\left(s,(X^{\tau_1})^n_s,i_0\right)-g\left(s,(X^{\tau_1})_s,i_0\right)\right]dW(s)\right|^2\to 0,\ \text{as } n\to\infty,$$

$$\left|\int_{t_0}^{t}\int_{-\infty}^{+\infty}\left[h\left(s,(X^{\tau_1})^n_s,u\right)-h\left(s,(X^{\tau_1})_s,u\right)\right]\tilde{v}\left(ds,du\right)\right|^2\to 0,\ \text{as } n\to\infty.$$

For all $t_0\le t\le T_1$, taking limits on both sides of (3.1), we have

$$\begin{aligned}
&\lim_{n\to\infty}\left[X^n(t)-G\left(t,(X^{\tau_1})^n_t,i_0\right)\right]\\
&=\xi(0)-\lim_{n\to\infty}G\left(t_0,(X^{\tau_1})^n_{t_0},i_0\right)+\lim_{n\to\infty}\int_{t_0}^{t}f\left(s,(X^{\tau_1})^{n-1}_s,i_0\right)ds\\
&\quad+\lim_{n\to\infty}\int_{t_0}^{t}g\left(s,(X^{\tau_1})^{n-1}_s,i_0\right)dW(s)\\
&\quad+\lim_{n\to\infty}\int_{t_0}^{t}\int_{-\infty}^{+\infty}h\left(s,(X^{\tau_1})^{n-1}_s,u\right)\tilde{v}\left(ds,du\right).
\end{aligned}$$

That is

$$\begin{aligned}&(X^{\tau_1})(t)\\&= \xi(0) + G\left(t, (X^{\tau_1})_t, i_0\right) - G\left(t_0, \xi, i_0\right) + \int_{t_0}^{t} f\left(s, (X^{\tau_1})_s, i_0\right) ds\\&\quad + \int_{t_0}^{t} g\left(s, (X^{\tau_1})_s, i_0\right) dW(s) + \int_{t_0}^{t}\int_{-\infty}^{+\infty} h\left(s, (X^{\tau_1})_s, u\right) \tilde{v}\left(ds, du\right).\end{aligned}$$

The above expression demonstrates that $X^{\tau_1}(t)$ is one solution of (2.1) with initial value (2.2) on $[t_0, T_1]$. By iteration, the existence of solutions to (2.1) on $[t_0, \tau_1 \wedge T]$ can be obtained. □

We now consider Eq.(2.1) on $[\tau_1 \wedge T, \tau_2 \wedge T]$. Following the same way of the above proof, we can obtain that Eq.(2.1) has a solution $X^{\tau_2}(t)$ on $[\tau_1 \wedge T, \tau_2 \wedge T]$. Repeating this procedure, the existence of solutions $X(t)$ on $[0, T]$ can be obtained.

4. Stability of Solutions

In this section, we study the continuous dependence of solutions on the initial value by means of the Corollary 2.7.

Definition 4.1. A solution $X^{\xi}(t)$ of NSIDDEPM (2.1) with initial value (2.2) is said to be stable in mean square if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$E\left|X^{\xi}(t) - X^{\eta}(t)\right|^2 \le \epsilon, \quad \text{when} \quad E\left\|\xi - \eta\right\|^2 < \delta,$$

where $X^{\eta}(t)$ is another solution of NSIDDEPM (2.1) with initial value η defined in (2.2).

Theorem 4.2. *Assume the hypotheses of Theorem 3.1 are satisfied, then the solution of NSIDDEPM (2.1) is stable in mean square.*

Proof. By assumptions, X^{τ_1} and Y^{τ_1} are two solutions of (2.1) on $[t_0, \tau_1 \wedge T]$ with initial value ξ and η, respectively. We have

$$\begin{aligned}X^{\tau_1}(t) &= \xi(0) + G\left(t, X_t^{\tau_1}, i_0\right) - G\left(t_0, \xi, i_0\right)\\&\quad + \int_{t_0}^{t} f\left(s, X_s^{\tau_1}, i_0\right) ds + \int_{t_0}^{t} g\left(s, X_s^{\tau_1}, i_0\right) dW(s)\\&\quad + \int_{t_0}^{t}\int_{-\infty}^{+\infty} h\left(s, X_s^{\tau_1}, u\right) \tilde{v}\left(ds, du\right), \quad t_0 \le t \le \tau_1 \wedge T, \quad \text{a.s.} \quad (4.1)\end{aligned}$$

and

$$Y^{\tau_1}(t) = \eta(0) + G\left(t, Y_t^{\tau_1}, i_0\right) - G\left(t_0, \eta, i_0\right) + \int_{t_0}^{t} f\left(s, Y_s^{\tau_1}, i_0\right) ds + \int_{t_0}^{t} g\left(s, Y_s^{\tau_1}, i_0\right) dW(s) + \int_{t_0}^{t}\int_{-\infty}^{+\infty} h\left(s, Y_s^{\tau_1}, u\right) \tilde{v}\left(ds, du\right), \quad t_0 \le t \le \tau_1 \wedge T, \quad \text{a.s.} \tag{4.2}$$

Subtracting equation (4.2) from equation (4.1), we get

$$\begin{aligned} & X^{\tau_1}(t) - Y^{\tau_1}(t) \\ & = \xi(0) - \eta(0) + G\left(t, X_t^{\tau_1}, i_0\right) - G\left(t, Y_t^{\tau_1}, i_0\right) \\ & \quad - \left(G\left(t_0, \xi, i_0\right) - G\left(t_0, \eta, i_0\right)\right) + \int_{t_0}^{t} \left(f\left(s, X_s^{\tau_1}, i_0\right) - f\left(s, Y_s^{\tau_1}, i_0\right)\right) ds \\ & \quad + \int_{t_0}^{t} \left(g\left(s, X_s^{\tau_1}, i_0\right) - g\left(s, Y_s^{\tau_1}, i_0\right)\right) dW(s) \\ & \quad + \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left(h\left(s, X_s^{\tau_1}, u\right) - h\left(s, Y_s^{\tau_1}, u\right)\right) \tilde{v}\left(ds, du\right), \ t_0 \le t \le \tau_1 \wedge T, \ \text{a.s.} \end{aligned}$$

So, using the same arguments as Lemma 3.3, we have

$$\begin{aligned} & E\left[\sup_{t_0 \le s \le t} \left|X^{\tau_1}(s) - Y^{\tau_1}(s)\right|^2\right] \\ & \le \frac{10}{1-5K_0} E\left\|\xi - \eta\right\|^2 + \frac{5\left(\left(\tau_1 \wedge T\right) - t_0 + 2\right)}{\left(1-5K_0\right)} E\int_{t_0}^{t} \kappa\left(\left\|X_s^{\tau_1} - Y_s^{\tau_1}\right\|^2\right) ds \\ & \le \frac{10}{1-5K_0} E\left\|\xi - \eta\right\|^2 \\ & \quad + \frac{5\left(\left(\tau_1 \wedge T\right) - t_0 + 2\right)}{\left(1-5K_0\right)} E\int_{t_0}^{t} \kappa\left(E\sup_{t_0 \le r \le s} \left|X^{\tau_1}(r) - Y^{\tau_1}(r)\right|^2\right) ds. \end{aligned}$$

Let $\kappa_1(u) = \frac{5((\tau_1 \wedge T) - t_0 + 2)}{(1-5K_0)}\kappa(u)$, for κ is a concave increasing function from $\Re_+$ to $\Re_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\kappa(u)} = +\infty$. So, $\kappa_1(u)$ is obviously a concave function from $\Re_+$ to $\Re_+$ such that $\kappa_1(0) = 0$, $\kappa(u) \ge \kappa_1(u)$, for any $0 \le u \le 1$ and $\int_{0+} \frac{du}{\kappa_1(u)} = +\infty$. So, for any $\epsilon > 0$, $\epsilon_1 \stackrel{\Delta}{=} \frac{1}{2}\epsilon$, we have $\lim_{s\to 0} \int_s^{\epsilon_1} \frac{du}{\kappa_1(u)} = +\infty$. So, there is a positive constant $\delta < \epsilon_1$ such that $\int_\delta^{\epsilon_1} \frac{du}{\kappa_1(u)} \ge (\tau_1 \wedge T) - t_0$. From Corollary 2.7, let $u_0 = \frac{10}{1-5K_0} E\left\|\xi - \eta\right\|^2$, $u(t) = E\left[\sup_{t_0 \le s \le t} \left|X^{\tau_1}(s) - Y^{\tau_1}(s)\right|^2\right]$, $v(t) = 1$, when $u_0 \le \delta \le \epsilon_1$, we have $\int_{u_0}^{\epsilon_1} \frac{du}{\kappa_1(u)} \ge \int_\delta^{\epsilon_1} \frac{du}{\kappa_1(u)} \ge (\tau_1 \wedge T) - t_0 = \int_{t_0}^{\tau_1 \wedge T} v(s) ds$. So, for any $t \in [t_0, (\tau_1 \wedge T)]$, the estimate $u(t) \leq \epsilon_1$ holds. This implies that the solution of NSIDDEPM (2.1) is stable in mean square on $[t_0, \tau_1 \wedge T]$.

Now we consider Eq.(2.1) on $[\tau_1 \wedge T, \tau_2 \wedge T]$. Following the same way of the above proof, we can obtain that the solution of NSIDDEPM (2.1) is stable in mean square on $[\tau_1 \wedge T, \tau_2 \wedge T]$. Repeating this procedure, the solution of NSIDDEPM (2.1) is stable in mean square on $[t_0, T]$. This completes the proof of the theorem. □

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