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EXISTENCE OF MILD SOLUTIONS OF NEUTRAL EVOLUTION INTEGRODIFFERENTIAL EQUATIONS

K. Balachandran[1], N. Annapoorani[2] and J.K. Kim[3]

[1]Department of Mathematics
Bharathiar University, Coimbatore-641 046, India
e-mail: `balachandran_k@lycos.com`

[2]Department of Mathematics
Bharathiar University, Coimbatore-641 046, India
e-mail: `pooranimaths@gmail.com`

[3]Department of Mathematics Education
Kyungnam University, Masan-631 701, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

Abstract. In this paper we establish a set of sufficient conditions for the existence of mild solutions of nonlinear neutral integrodifferential equations in Banach spaces. The results are obtained by using the fixed point theorems. An example is provided to illustrate the theory.

1. Introduction

Neutral differential equations arise in many areas of science and engineering and for this reason these equations have received much attention in the last decades. The theory of neutral integrodifferential equations has been studied by several authors [1, 2, 3, 6, 13]. Grimmer [7] studied the resolvent operators for integral equations in Banach spaces. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of semilinear equations have been discussed by Pazy [15]. Ntouyas and Tsamatos [14] studied the global existence of solutions for functional integrodifferential equations of neutral type by using the Schaefer fixed point theorem.

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The purpose of this paper is to prove the existence of mild solutions for neutral evolution integrodifferential equations of the form

$$\frac{d}{dt}\Big[x(t) - g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big)\Big]$$
$$= A(t)x(t) + \int_0^t B(t,s)x(s)ds + f\Big(t, x_t, \int_0^t h(t,s,x_s)ds\Big), \qquad (1.1)$$

for all $t \in J = [0, b]$ and

$$x_0 \;=\; \phi, \text{ on } [-r, 0] \qquad (1.2)$$

where $A(t)$ and $B(t,s)$ are closed linear operators on a Banach space X with a dense domain $D(A)$ which is independent of t, $h : \Delta \times C \to X, k : \Delta \times C \to X, f : J \times C \times X \to X$ and $g : J \times C \times X \to X$ are continuous functions and $\Delta = \{(t,s) : 0 \le s < t \le b\}$. Denote $C = C([-r, 0] : X)$, the Banach space of all continuous functions $\phi : [-r, 0] \to X$ with the norm $\|\phi\| = sup\,\{|\phi(\theta)| : -r \le \theta \le 0\}$. Also for $x \in C([-r, b] : X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$. Related work in this type of equations can also be found in [9, 10].

2. Preliminaries

We shall make the following assumptions:

(1) $A(t)$ generates a strongly continuous semigroup of evolution operators.
(2) Suppose Y is a Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $B(t,s)$ are in the set $L(Y, X)$, of bounded linear operators from Y to X, for $0 \le t \le b$ and $0 \le s \le t \le b$,respectively. $A(t)$ and $B(t,s)$ are closed linear operators, it follows that $A(t)$ and $B(t,s)$ are continuous on $0 \le t \le b$ and $0 \le s \le t \le b$, respectively, into $L(Y, X)$.

Definition 2.1. A resolvent operator for (1.1)-(1.2) is a bounded operator valued function $R(t,s) \in L(X), 0 \le s \le t \le b$, the space of bounded linear operators on X, having the following properties.

(i) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, (the identity operator on X) $0 \le s \le b$. $\|R(t,s)\| \le Me^{\beta(t-s)}$ $t, s \in J$ and M, β are constants.
(ii) $R(t,s)Y \subset Y, R(t,s)$ is strongly continuous in s and t on Y.

(iii) For each $x \in D(A)$, $R(t,s)x$ is strongly continuously differentiable in t and s and

$$\frac{\partial R(t,s)}{\partial t}x = A(t)R(t,s)x + \int_s^t B(t,r)R(r,s)xdr,$$

$$\frac{\partial R(t,s)}{\partial s}x = -R(t,s)A(s)x - \int_s^t R(t,r)B(r,s)xdr,$$

with $\dfrac{\partial R(t,s)}{\partial t}x$ and $\dfrac{\partial R(t,s)}{\partial s}x$ being strongly continuous on $0 \le s \le t \le b$.

Definition 2.2. A solution $x \in C([-r,b]:X)$ is called a mild solution of the problem (1.1)-(1.2) if the following holds: $x_0 = \phi$ on $[-r,0]$ and for $s \in [0,t]$, the function

$$A(s)R(t,s)g(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau)$$

is integrable and for $t \in [0,b]$ the integral equation

$$\begin{aligned} x(t) \;=\; & R(t,0)[\phi(0) - g(0,\phi,0)] + g\Big(t,x_t,\int_0^t k(t,s,x_s)ds\Big) \\ & + \int_0^t R(t,s)A(s)g\Big(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau\Big)ds \\ & + \int_0^t R(t,s)\int_0^s B(s,\theta)g\Big(\theta,x_\theta,\int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds \\ & + \int_0^t R(t,s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds \end{aligned}$$

is satisfied.

Schaefer's Theorem [16] Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator , that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta(F) = \{x \in E; x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Further we assume the following hypotheses:

(H1) There exists a resolvent operator $R(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exist constants $M_i > 0, i = 1,2,3$, such that
   (i) $|R(t,s)| \le M_1$,
   (ii) $|A(t)g(t,x,y)| \le M_2$ and
   (iii) $|B(t,s)g(t,x,y)| \le M_3$.

(H2) For each $s \in J, x \in C$ the function $k(., s, x) : J \to X$ is completely continuous, the function $k(., ., x) : \Delta \to X$ is strongly measurable and $\{t \to k(t, s, x_s)\}$ is equicontinuous in $C([0, b] : X)$.

(H3) For each $t, s \in \Delta$, the function $h(t, s, .) : C \to X$ is continuous and for each $x \in C$, the function $h(., ., x) : \Delta \to X$ is strongly measurable.

(H4) For each $t \in J, f(t, ., .) : C \times X \to X$ is continuous and for each $(x, y) \in C \times X$, the function $f(., x, y) : J \to X$ is strongly measurable.

(H5) There exists an integrable function $p : J \to [0, \infty)$, such that

$$|f(t, x, y)| \leq p(t)\Omega_0(\|x\| + |y|), t \in J, x \in C, y \in X,$$

where $\Omega_0 : [0, \infty) \to (0, \infty)$ is nondecreasing function.

(H6) The function $g : J \times C \times X \to X$ is completely continuous and for any bounded set D in $C([-r, b] : X)$, the set $\{t \to g(t, x_t, \int_0^t k(t, s, x_s)ds) : x \in D\}$ is equicontinuous in $C([0, b] : X)$ and there exist constants $c_1 \in (0, 1)$, $c_2 > 0$ and $c_3 \geq 0$, such that

$$|g(t, \phi, y)| \leq c_1 \|\phi\| + c_2 |y| + c_3, \quad \text{for all} \quad t \in J, \phi \in C, y \in X.$$

(H7) There exist integrable functions $m_i : J \to [0, \infty)$ and constants $r_i, i = 0, 1$, such that
 (i) $|k(t, s, x)| \leq r_0 m_0(s)\Omega_1(\|x\|), \quad x \in C,$
 (ii) $|h(t, s, x)| \leq r_1 m_1(s)\Omega_2(\|x\|), \quad x \in C,$
 where $\Omega_i : [0, \infty) \to (0, \infty), i = 1, 2$ are continuous nondecreasing functions.

(H8) $\displaystyle\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega_1(s) + \Omega_2(s)}$
 where $c = \frac{1}{1-c_1}\Big\{M_1(1 + c_1) \|\phi\| + (1 + M_1)c_3 + M_1 M_2 b + M_1 M_3 b^2\Big\}$
 and $\hat{m}(t) = max\Big\{\frac{c_2}{1-c_1} r_0 m_0(t), \frac{M_1 p(t)}{1-c_1}, r_1 m_1(t)\Big\}$

3. Existence Results

Theorem 3.1. *Assume that hypotheses (H1)-(H8) hold. Then the problem (1.1)-(1.2) admits a mild solution on* $[-r, b]$.

Proof. Consider the space $C_b = C([-r, b] : X)$, with the norm $\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\}$. Define $C_b^0 = \{x \in C_b : x_0 = 0\}$. To prove the existence of mild solutions of (1.1)-(1.2), consider the following operator equation

$$x(t) = \lambda F x(t), \quad 0 < \lambda < 1, \tag{3.1}$$

where $F : C_b^0 \to C_b^0$ is defined as

$$\begin{aligned}(Fx)(t) &= R(t,0)[\phi(0) - g(0,\phi,0)] + g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big) \\ &\quad + \int_0^t R(t,s)A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)ds \\ &\quad + \int_0^t R(t,s)\int_0^s B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds \\ &\quad + \int_0^t R(t,s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds.\end{aligned}$$

From (3.1), we have

$$\begin{aligned}|x(t)| &= |\lambda Fx(t)| \le |Fx(t)| \\ &\le M_1(1+c_1)\|\phi\| + (1+M_1)c_3 + M_1M_2b \\ &\quad + M_1M_3b^2 + c_1\|x_t\| + c_2r_0\int_0^t m_0(s)\Omega_1(\|x_s\|)ds \\ &\quad + M_1\int_0^t p(s)\Omega_0\Big(\|x_s\| + r_1\int_0^s m_1(\tau)\Omega_2(\|x_\tau\|)d\tau\Big)ds.\end{aligned}$$

Consider the function μ defined by

$$\mu(t) = sup\{|x(s)| : -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,b]$, by the previous inequality, we have

$$\begin{aligned}\mu(t) &\le M_1(1+c_1)\|\phi\| + (1+M_1)c_3 + M_1M_2b \\ &\quad + M_1M_3b^2 + c_1\mu(t) + c_2r_0\int_0^{t^*} m_0(s)\Omega_1(\mu(s))ds \\ &\quad + M_1\int_0^{t^*} p(s)\Omega_0\Big(\mu(s) + r_1\int_0^s m_1(\tau)\Omega_2(\mu(\tau))d\tau\Big)ds. \\ \mu(t) &\le M_1(1+c_1)\|\phi\| + (1+M_1)c_3 + M_1M_2b \\ &\quad + M_1M_3b^2 + c_1\mu(t) + c_2r_0\int_0^{t} m_0(s)\Omega_1(\mu(s))ds \\ &\quad + M_1\int_0^{t} p(s)\Omega_0\Big(\mu(s) + r_1\int_0^s m_1(\tau)\Omega_2(\mu(\tau))d\tau\Big)ds.\end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \geq 1$. Hence

$$\begin{aligned}\mu(t) \quad &\leq \quad \frac{1}{1-c_1}\Big\{M_1(1+c_1)\|\phi\| + (1+M_1)c_3 \\ &\quad + M_1M_2b + M_1M_3b^2 + c_2r_0\int_0^t m_0(s)\Omega_1(\mu(s))ds \\ &\quad + M_1\int_0^t p(s)\Omega_0\Big(\mu(s) + r_1\int_0^s m_1(\tau)\Omega_2(\mu(\tau))d\tau\Big)ds\Big\}.\end{aligned}$$

Let us denote the right hand side of the above inequality as $v(t)$. Then,

$$c = v(0) = \frac{1}{1-c_1}\Big\{M_1(1+c_1)\|\phi\| + (1+M_1)c_3 + M_1M_2b + M_1M_3b^2\Big\},$$

and $\mu(t) \leq v(t), t \in J$,

$$\begin{aligned}&v'(t) \\ &= \frac{1}{1-c_1}\left\{c_2r_0m_0(t)\Omega_1(\mu(t)) + M_1p(t)\Omega_0\Big(\mu(t) + r_1\int_0^t m_1(s)\Omega_2(\mu(s))ds\Big)\right\} \\ &\leq \frac{1}{1-c_1}\left\{c_2r_0m_0(t)\Omega_1(v(t)) + M_1p(t)\Omega_0\Big(v(t) + r_1\int_0^t m_1(s)\Omega_2(v(s))ds\Big)\right\}.\end{aligned}$$

Let $w(t) = v(t) + r_1\int_0^t m_1(s)\Omega_2(v(s))ds$. Then $w(0) = v(0), v(t) \leq w(t)$ and

$$\begin{aligned}w'(t) \quad &= \quad v'(t) + r_1m_1(t)\Omega_2(v(t)) \\ &\leq \quad \frac{1}{1-c_1}\{c_2r_0m_0(t)\Omega_1(w(t)) + M_1p(t)\Omega_0(w(t))\} + r_1m_1(t)\Omega_2(w(t)) \\ &\leq \quad \hat{m}(t)\{\Omega_0(w(t)) + \Omega_1(w(t)) + \Omega_2(w(t))\}.\end{aligned}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega_1(s) + \Omega_2(s)} \leq \int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega_1(s) + \Omega_2(s)}.$$

This inequality implies that $w(t)$ must be bounded by some positive constant L on $[0, b]$. Consequently, $\|x\|_1 \leq L$, where L depends on b and on the functions $m_i, i = 0, 1$ and $\Omega_i, i = 0, 1, 2$.

Next we shall prove that the operator F is a completely continuous operator. Let $B_q = \{x \in C_b^0 : \|x\|_1 \leq q\}$ for some $q \geq 1$. We first show that the set $\{Fx : x \in B_q\}$ is equicontinuous. Let $x \in B_q$ and $t_1, t_2 \in J$. Then, if

$0 < t_1 < t_2 \le b$, we have

$$
\begin{aligned}
&|(Fx)(t_1) - (Fx)(t_2)| \\
&\le |R(t_1,0) - R(t_2,0)||\phi(0) - g(0,\phi,0)| \\
&\quad + \Big| g\Big(t_1, x_{t_1}, \int_0^{t_1} k(t_1,s,x_s)ds\Big) - g\Big(t_2, x_{t_2}, \int_0^{t_2} k(t_2,s,x_s)ds\Big)\Big| \\
&\quad + \Big| \int_0^{t_1} [R(t_1,s) - R(t_2,s)]A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)ds\Big| \\
&\quad + \Big| \int_{t_1}^{t_2} R(t_2,s)A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)ds\Big| \\
&\quad + \Big| \int_0^{t_1} [R(t_1,s) - R(t_2,s)] \int_0^s B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds\Big| \\
&\quad + \Big| \int_{t_1}^{t_2} R(t_2,s) \int_0^s B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds\Big| \\
&\quad + \Big| \int_0^{t_1} [R(t_1,s) - R(t_2,s)]f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big| \\
&\quad + \Big| \int_{t_1}^{t_2} R(t_2,s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big| \\
&\le |R(t_1,0) - R(t_2,0)||\phi(0) - g(0,\phi,0)| \\
&\quad + \Big| g\Big(t_1, x_{t_1}, \int_0^{t_1} k(t_1,s,x_s)ds\Big) - g\Big(t_2, x_{t_2}, \int_0^{t_2} k(t_2,s,x_s)ds\Big)\Big| \\
&\quad + M_2 \int_0^{t_1} |R(t_1,s) - R(t_2,s)|ds + M_2 \int_{t_1}^{t_2} |R(t_2,s)|ds \\
&\quad + M_3 b \int_0^{t_1} |R(t_1,s) - R(t_2,s)|ds + M_3 b \int_{t_1}^{t_2} |R(t_2,s)|ds \\
&\quad + \int_0^{t_1} |R(t_1,s) - R(t_2,s)|p(s)\Omega_0(q')ds + \int_{t_1}^{t_2} |R(t_2,s)|p(s)\Omega_0(q')ds
\end{aligned}
$$

where $q' = q + qr_1 \int_0^b m_1(s)ds$. The right hand side of the above inequality is independent of $x \in B_q$ and tends to zero as $t_2 - t_1 \to 0$, since g is completely continuous and by (H1), $R(t,s)$ for $t > s$ is continuous in the uniform operator topology. Thus the set $\{Fx : x \in B_q\}$ is equicontinuous.

We consider here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ and $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_q is uniformly bounded. Next we show that $\overline{FB_q}$ is compact. Since we have shown FB_q to be an equicontinuous collection, it suffices, by the Arzela-Ascoli theorem, to show that F maps B_q into a precompact set in X.

Let $0 < t \leq s \leq b$ be fixed and let ϵ, a real number satisfying $0 < \epsilon < t$. For $x \in B_q$, we define

$$\begin{aligned}(F_\epsilon x)(t) &= R(t,0)[\phi(0) - g(0,\phi,0)] + g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big)\\ &\quad + \int_0^{t-\epsilon} R(t,s)A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)ds\\ &\quad + \int_0^{t-\epsilon} R(t,s)\int_0^s B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds\\ &\quad + \int_0^{t-\epsilon} R(t,s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds.\end{aligned}$$

Since $R(t,s)$ is a compact operator, the set $y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_q\}$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in B_q$, we have

$$\begin{aligned}&|(Fx)(t) - (F_\epsilon x)(t)|\\ &\leq \int_{t-\epsilon}^t |R(t,s)|\Big|A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)\Big|ds\\ &\quad + \int_{t-\epsilon}^t |R(t,s)|\int_0^s \Big|B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)\Big|d\theta ds\\ &\quad + \int_{t-\epsilon}^t |R(t,s)|\Big|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big|ds\\ &\leq M_2\int_{t-\epsilon}^t |R(t,s)|ds + M_3 b\int_{t-\epsilon}^t |R(t,s)|ds\\ &\quad + \int_{t-\epsilon}^t |R(t,s)|p(s)\Omega_0(q')ds.\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_q\}$. Hence the set $\{(Fx)(t) : x \in B_q\}$ is precompact in X.

It remains to show that $F : C_b^0 \to C_b^0$ is continuous. Let $\{x_n\}_{n=0}^\infty$ be a converging sequence in C_b^0 to x. Then there is an integer r such that $\|x_n(t)\| \leq r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$.
By assumptions (H2)-(H4) and (H8),

$$\begin{aligned}f\Big(t, x_{n_t}, \int_0^t h(t,s,x_{n_s})ds\Big) &\to f\Big(t, x_t, \int_0^t h(t,s,x_s)ds\Big),\\ g\Big(t, x_{n_t}, \int_0^t k(t,s,x_{n_s})ds\Big) &\to g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big),\end{aligned}$$

as $n \to \infty$ for each $t \in J$. Moreover, by virtue of (H4), (H6) and (H7), we obtain

$$\Big|f\Big(t, x_{n_t}, \int_0^t h(t,s,x_{n_s})ds\Big) - f(t, x_t, \int_0^t h(t,s,x_s)ds\Big)\Big| \leq 2p(t)\Omega_0(q')$$

and

$$\begin{aligned}&\Big|g\Big(t, x_{n_t}, \int_0^t k(t,s,x_{n_s})ds\Big) - g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big)\Big|\\ &\leq 2c_2 r_0 \int_0^t m_0(s)\Omega_1(r)ds + 2c_1 r + 2c_3.\end{aligned}$$

Since g is completely continuous, we have by dominated convergence theorem

$$\begin{aligned}&\|Fx_n - Fx\|\\ &\leq \Big|g\Big(t, x_{n_t}, \int_0^t k(t,s,x_{n_s})ds\Big) - g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big)\Big|\\ &\quad + \int_0^t |R(t,s)|\Big|A(s)\Big[g\Big(s, x_{n_s}, \int_0^s k(s,\tau,x_{n_\tau})d\tau\Big) - g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)\Big]\Big|ds\\ &\quad + \int_0^t |R(t,s)| \int_0^s \Big|B(s,\theta)\Big[g\Big(\theta, x_{n_\theta}, \int_0^\theta k(\theta,\tau,x_{n_\tau})d\tau\Big)\\ &\quad - g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)\Big]\Big|d\theta ds + \int_0^t |R(t,s)|\Big|f\Big(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau\Big)\\ &\quad - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big|ds \quad \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus F is continuous. Hence F is completely continuous. Finally, the set

$$\zeta(F) = \big\{x \in C_b^0 : x = \lambda Fx, \text{ for some } 0 < \lambda < 1\big\}$$

is bounded as we proved in the first step. By Schaefer's theorem, the operator F has a fixed point in C_b^0. Hence (1.1)-(1.2) has a mild solution on $[-r, b]$. □

Krasnoselskii Theorem [4]**:** Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be two operators such that

(i) $Px + Qy \in S$ whenever $x, y \in S$;
(ii) P is a contraction mapping;
(iii) Q is compact and continuous.

Then there exists $z \in S$ such that $z = Pz + Qz$.

For using the Krasnoselskii fixed point theorem we need the following hypotheses:

(H9) There exist constants $M_4 > 0$ and $M_5 > 0$ such that
  (i) $|g(t,x,y)| \leq M_4$,

(ii) $|f(t,x,y)| \le M_5$,
for $t \in J,\ x \in C$ and $y \in X$.

(H10) $M_1[\|\phi\| + M_4] + M_4 + M_1M_2b + M_1M_3b^2 + M_1M_5b \le q$.

(H11) The function g is continuous and there exist constants $N_i > 0 (i = 1,2,3)$ such that
(i) $|g(t,x_1,y_1) - g(t,x_2,y_2)| \le N_1[\|x_1 - x_2\| + |y_1 - y_2|]$,
(ii) $|A(t)[g(t,x_1,y_1) - g(t,x_2,y_2)]| \le N_2[\|x_1 - x_2\| + |y_1 - y_2|]$,
(iii) $|B(s,t)[g(t,x_1,y_1) - g(t,x_2,y_2)]| \le N_3[\|x_1 - x_2\| + |y_1 - y_2|]$,
for $t \in J,\ x_1,x_2 \in C$ and $y_1,y_2 \in X$.

(H12) There exists a constant $L_1 > 0$ such that

$$\Big|\int_0^t [k(t,s,x) - k(t,s,y)]ds\Big| \le L_1\|x-y\|, \quad t,s \in J \quad \text{and} \quad x,y \in C.$$

(H13) $[N_1 + M_1N_2b + M_1N_3b^2][1 + L_1] < 1$.

Theorem 3.2. *Assume that the hypotheses (H1), (H3)-(H5), (H7)(ii) and (H9)-(H13) hold. Then the problem (1.1)-(1.2) has a mild solution.*

Proof. Define the operators P and Q on B_q as

$$\begin{aligned} Px(t) &= R(t,0)[\phi(0) - g(0,\phi,0)] + g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big) \\ &\quad + \int_0^t R(t,s)A(s)g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big)ds \\ &\quad + \int_0^t R(t,s)\int_0^s B(s,\theta)g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big)d\theta ds, \\ Qx(t) &= \int_0^t R(t,s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds. \end{aligned}$$

Let $x, y \in B_q$. Then we have

$$\begin{aligned} |Px(t) + Qy(t)| &\le |R(t,0)[\phi(0) - g(0,\phi,0)]| + \Big|g(t,x_t,\int_0^t k(t,s,x_s)ds)\Big| \\ &\quad + \int_0^t |R(t,s)|\Big|A(s)g(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau)\Big|ds \\ &\quad + \int_0^t |R(t,s)|\int_0^s \Big|B(s,\theta)g(\theta,x_\theta,\int_0^\theta k(\theta,\tau,x_\tau)d\tau)\Big|d\theta ds \\ &\quad + \int_0^s |R(t,s)|\Big|f(s,y_s,\int_0^s h(s,\tau,y_\tau)d\tau)\Big|ds \\ &\le M_1[\|\phi\| + M_4] + M_4 + M_1M_2b + M_1M_3b^2 + M_1M_5b \\ &\le q. \end{aligned}$$

Therefore $Px + Qy \in B_q$. Further

$$
\begin{aligned}
&|Px(t) - Py(t)| \\
&\leq \Big|g\Big(t, x_t, \int_0^t k(t,s,x_s)ds\Big) - g\Big(t, y_t, \int_0^t k(t,s,y_s)ds\Big)\Big| \\
&\quad + \int_0^t |R(t,s)|\Big|A(s)\Big[g\Big(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau\Big) \\
&\quad - g\Big(s, y_s, \int_0^s k(s,\tau,y_\tau)d\tau\Big)\Big]\Big|ds \\
&\quad + \int_0^t |R(t,s)| \int_0^s \Big|B(s,\theta)\Big[g\Big(\theta, x_\theta, \int_0^\theta k(\theta,\tau,x_\tau)d\tau\Big) \\
&\quad - g\Big(\theta, y_\theta, \int_0^\theta k(\theta,\tau,y_\tau)d\tau\Big)\Big]\Big|d\theta ds \\
&\leq N_1\Big[\|x_t - y_t\| + \Big|\int_0^t [k(t,s,x_s) - k(t,s,y_s)]ds\Big|\Big] \\
&\quad + M_1N_2 \int_0^t \Big[\|x_s - y_s\| + \Big|\int_0^s [k(s,\tau,x_\tau) - k(s,\tau,y_\tau)]d\tau\Big|\Big]ds \\
&\quad + M_1N_3 \int_0^t \int_0^s \Big[\|x_\theta - y_\theta\| + \Big|\int_0^\theta [k(\theta,\tau,x_\tau) - k(\theta,\tau,y_\tau)]d\tau\Big|\Big]d\theta ds \\
&\leq [N_1 + M_1N_2b + M_1N_3b^2][1 + L_1]\|x - y\|_1.
\end{aligned}
$$

By hypotheses (H13), P is a contraction.

The equicontinuity of $(Qu)(t)$ is already proved in Theorem 3.1 and so $Q(B_q)$ is relatively compact. By the Arzela-Ascoli Theorem, Q is compact. Hence by the Krasnoselskii Theorem there exists a mild solution to the problem (1.1)-(1.2). □

4. Example

Consider the following equation which arises in the study of heat conduction in materials with memory [8, 11, 12]

$$
\begin{aligned}
C\theta''(x,t) + \beta(0)\theta'(x,t) &= \alpha(0)\Delta\theta(x,t) - \int_{-\infty}^t \beta'(t-s)\theta'(x,s)ds \\
&\quad + \int_{-\infty}^t \alpha'(t-s)\Delta\theta(x,s)ds + r'(x,t). \qquad (4.1)
\end{aligned}
$$

Let us assume that Ω is a bounded open connected subset of R^3 with C^∞ boundary. Also, assume that α and β are in $C^2([0,\infty), R)$ with $C = \alpha(0) =$

$\beta(0) = 1$. If we suppose the dependence on $x \in \Omega$ and assume that $\theta(x,t)$ is known for $t \leq 0$ we may pose the problem as

$$\theta''(t) + \theta'(t) = \Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + r(t)$$

where Δ is the Laplacian on Ω with boundary condition $\theta|_\Gamma = 0$ and the initial conditions are

$$\theta(x,0) = \theta_0(x) \in H^2(\Omega) \cap H_0^1(\Omega), \quad \theta'(x,0) = \theta_0'(x) \in H_0^1(\Omega).$$

We rewrite this as

$$\begin{aligned} \begin{pmatrix} \theta \\ \phi \end{pmatrix}' &= \begin{pmatrix} 0, & I \\ \Delta, & -I \end{pmatrix} \begin{pmatrix} \theta(s) \\ \phi(s) \end{pmatrix} ds + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}, \\ \begin{pmatrix} \theta(0) \\ \phi(0) \end{pmatrix} &= \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix}. \end{aligned}$$

Letting $\omega = (\theta, \phi)^*$ this can be written as

$$\omega'(t) = A\omega(t) + \int_0^t B(t-s)A\omega(s)du + f(t), \quad \omega(0) \in D(A) \subset H.$$

Here H is the space $H_0^1(\Omega) \oplus H^\circ(\Omega)$ with inner product

$$\langle(\theta_1, \phi_1), (\theta_2, \phi_2)\rangle = \int_\Omega (\nabla\theta_1 \nabla\theta_2 + \theta_1\theta_2)dx,$$

and A has domain $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \oplus H_0^1(\Omega)$. It follows from [5] that A generates a semigroup $T(t)$ on H with $\|T(t)\| \leq Me^{-\gamma t}$ for some $\gamma > 0$. Further the resolvent operator $R(t)$ exists and it follows from Theorem 4.1 in [7] that the resolvent operator satisfies the condition $\|R(t)\| \leq Me^{-\mu t}$ for some $\mu > 0$.

By introducing the neutral term in the above model (4.1) we can obtain the following neutral integrodifferential equation

$$\begin{aligned} &\frac{\partial}{\partial t}[\theta(x,t) + \int_\infty^t k(s-t,\eta,x)\theta(s,x)d\eta ds] \\ &= \Delta\theta(x,t) - \int_{-\infty}^t \beta(t-s)\theta(x,s)ds + \int_0^t a(t-s)\Delta\theta(x,s)ds + f(x,t,\theta) \end{aligned}$$

and

$$\theta(t,0) = \theta(t,1) = 0; \quad \theta(\tau,x) = \theta_0(\tau,x), \quad \tau \leq 0$$

for $t \in [0,b]$ and $x \in [0,1]$. By the similar analysis above the equation generates a semigroup and a resolvent operator $R(t)$ which satisfies the hypothesis (H1). Moreover imposing appropriate conditions on the functions a, k, β and

f and applying the Theorem 3.1 we can see that the neutral integrodifferential equation has at least one mild solution.

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