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A NOTE ON THE IMPROVEMENT OF THE ERROR BOUNDS FOR A CERTAIN CLASS OF OPERATORS

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Abstract. We use Newton's method to approximate a locally unique solution of an equation in a Banach space setting. We show that our error bounds on the distances involved can be tighter than before [6]–[12], under new sufficient convergence conditions, provided that the Fréchet–derivative of the operator involved is p–Hölder continuous, where $p \in (0, 1]$.

Numerical examples validating our result are given at the end of this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator such that F' is a p–Hölder continuous operator, $p \in (0, 1]$, defined on an open, convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake

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of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$ (for some suitable operator Q), where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We study the convergence of Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n), \quad (n \geq 0) \quad (x_0 \in \mathcal{D}). \tag{1.2}$$

A survey of sufficient conditions for the local as well as the semilocal convergence of Newton–type methods as well as an error analysis for such methods can be found in [1]–[4], [8], and the references there. This note is a continuation of our work in [5], where, we projected that the order of convergence $(1 + p)$ not obtained in [5] can be achieved under new sufficient convergence conditions.

2. Preliminaries and background

Let $x_0 \in \mathcal{D}$ be such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ the space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$. Assume F' satisfies a center–Hölder condition

$$\| F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \| \leq \ell_0\ \ \| x - x_0 \|^p, \qquad \ell_0 > 0, \tag{2.1}$$

and a Hölder condition

$$\| F'(x_0)^{-1}\ (F'(x) - F'(y)) \| \leq \ell\ \ \| x - y \|^p, \qquad \ell > 0, \tag{2.2}$$

for all $x, y \in U(x_0, R) = \{x \in \mathcal{X}\ : \| x - x_0 \| < R,\ R > 0\} \subseteq \mathcal{D}$.

Note that in general

$$\ell_0 \leq \ell \tag{2.3}$$

holds, and that $\dfrac{\ell}{\ell_0}$ can be arbitrarily large [4].

We refer the reader to [3], [4], [7], [9] for a historical background, and the attempts made to provide new semilocal convergence results, and tighter error bounds on the distances $\| x_{n+1} - x_n \|$, $\| x_n - x^\star \|$, $(n \geq 0)$.

Recently, a new sufficient condition was given in [7], which improves earlier sufficient convergence conditions [1], [3], [6]–[8], [10]–[12], but not necessarily the error bounds.

Let

$$c_0 = \frac{\ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}}{2\ \ell}, \tag{2.4}$$

and

$$h_0(t) = \left(1 - \frac{1}{t}\right)^p \frac{1+p}{\left((\ell_0\ (1+p))^{\frac{1}{1-p}} + (\ell\ t\ (t-1))^{\frac{1}{1-p}}\right)^{1-p}}. \tag{2.5}$$

Then, the condition is for $b \geq \| F'(x_0) \|^{-1}$:

$$\eta^p\ b \leq h_0(c_0). \tag{2.6}$$

We show that our new sufficient convergence conditions can hold in cases (2.6) is violated. In order for us to achieve this task, we note that the results in [7] can be given in affine invariant form by simply replacing F by $F'(x_0)^{-1}\ F$. Therefore, the condition (2.6) becomes

$$\eta^p \leq h_0(c_0), \tag{2.7}$$

and the corresponding error estimates are:

$$0 \leq s_n - s_{n-1} \leq s_n \left(\frac{c_0 - 1}{c_0^n - 1}\right), \qquad (n \geq 1) \tag{2.8}$$

where, the sequence $\{s_n\}$ $(n \geq 0)$ is given by:

$$\begin{aligned} &s_0 = 0, \quad s_1 = \eta, \\ &s_{n+2} = s_{n+1} + \frac{\ell\ (s_{n+1} - s_n)^{1+p}}{(1+p)\ (1 - \ell_0\ s_{n+1}^p)}, \quad (n \geq 0). \end{aligned} \tag{2.9}$$

3. Semilocal convergence analysis for Newton's method

We can show the following lemma on majorizing sequences for Newton's method (1.2).

Lemma 3.1. *Assume there exist constants $\ell_0 \geq 0$, $\ell \geq 0$, $\eta \geq 0$, and $p \in (0, 1]$, such that:*

$$q = \alpha\ \eta \leq \frac{1}{1+p}, \tag{3.1}$$

where,

$$\alpha = \frac{1}{1+p}\left\{\frac{1}{\ell_0}\left(1-\frac{2\ \ell}{\ell+\sqrt{\ell^2+4\ (1+p)^p\ p^{1-p}\ \ell_0\ \ell}}\right)\right\}^{-\frac{1}{p}}. \tag{3.2}$$

Then, scalar sequence $\{t_n\}$ $(n \geq 0)$ given by

$$\begin{aligned} &t_0 = 0, \quad t_1 = \eta, \\ &t_{n+1} = t_n + \frac{\ell_1\ (t_n - t_{n-1})^{1+p}}{(1+p)\ (1-\ell_0\ t_n^p)}, \quad (n \geq 1). \end{aligned} \tag{3.3}$$

is well defined, nondecreasing, bounded above by

$$t_{\star\star} = \frac{(1+p)\ \eta}{1+p-\delta}, \tag{3.4}$$

and converges to its unique least upper bound $t_\star$, such that $t_\star \in [0, t_{\star\star}]$, where,

$$\ell_1 = \begin{cases} \ell_0 & if \quad n = 1 \\ \ell & if \quad n \geq 2 \end{cases}, \tag{3.5}$$

and

$$\delta = \frac{2\ (1+p)\ \ell}{\ell+\sqrt{\ell^2+4\ (1+p)^p\ p^{1-p}\ \ell_0\ \ell}} < 1+p, \tag{3.6}$$

Moreover the following estimates hold for all $n \geq 1$:

$$\ell_0\ t_\star^\lambda < 1, \tag{3.7}$$

$$0 \leq t_{n+1} - t_n \leq \frac{\delta}{1+p}\ (t_n - t_{n-1}) \leq \left(\frac{\delta}{1+p}\right)^n \eta, \tag{3.8}$$

$$t_{n+1} - t_n \leq \left(\frac{\delta}{1+p}\right)^n ((1+p)\ q)^{(1+p)^n - 1}\eta, \tag{3.9}$$

and

$$t_\star - t_n \leq \left(\frac{\delta}{1+p}\right)^n \frac{((1+p)\ q)^{(1+p)^n-1}\ \eta}{1-((1+p)\ q)^{(1+p)^n}}. \tag{3.10}$$

Proof. The proof as very similar to the one in Lemma 3.1 in [5] is omitted. □

However, we will cover the cases $\ell_0 = 0$, and $\ell = 0$. If $\ell_0 = 0$, then (3.7) holds trivially. In this case, for $\ell > 0$, an induction argument shows

$$t_{k+1} - t_k = \frac{1}{a}\ (a\ (t_k - t_{k-1}))^{(1+p)^k},$$

where,

$$a = \left(\frac{\ell}{1+p}\right)^{\frac{1}{p}}.$$

That is we have:

$$t_{k+1} = t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) = \frac{1}{a} \sum_{m=0}^{k} (a\ \eta)^{(1+p)^m},$$

and

$$t_\star = \lim_{k \longrightarrow \infty} t_k = \frac{1}{a} \sum_{m=0}^{\infty} (a\ \eta)^{(1+p)^m}.$$

Clearly, this serie converges, if $a\ \eta < 1$, and is bounded by the number:

$$\frac{1}{a} \sum_{m=0}^{\infty} (a\ \eta)^{(1+p)^m} = \frac{1}{a}\ \frac{1}{1 - (a\ \eta)^{1+p}}.$$

If $\ell = 0$, and $0 \leq \ell_0 \leq \ell$, we deduce: $\ell_0 = 0$, $t_\star = t_k = \eta$, $(k \geq 1)$.

We can show the main semilocal convergence theorem for Newton's method (1.2):

Theorem 3.2. *Let $F : \mathcal{D} \subseteq\ X \longrightarrow \mathcal{Y}$ be a Fréchet–differentiable operator. Assume:*
there exist a point $x_0 \in \mathcal{D}$ and parameters $\ell_0 > 0$, $\ell > 0$, $p \in (0, 1]$, $R > 0$, and $\eta \geq \| F'(x_0)^{-1}\ F(x_0) \| > 0$, such that: conditions (2.1), (2.2), hypotheses of Lemma 3.1 hold,

and

$$\overline{U}(x_0, t_\star) \subseteq U(x_0, R).$$

Then, $\{x_n\}$ $(n \geq 0)$ generated by Newton's method (1.2) is well defined, remains in $\overline{U}(x_0, t_\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t_\star)$ of equation $F(x) = 0$.

Moreover, the following estimates bounds hold for all $n \geq 0$:

$$\| x_{n+2} - x_{n+1} \| \leq \frac{\ell\ \| x_{n+1} - x_n \|^{1+p}}{(1+p)\ [1 - \ell_0\ \| x_{n+1} - x_0 \|^p]} \leq t_{n+2} - t_{n+1},$$

and

$$\| x_n - x^\star \| \leq t_\star - t_n,$$

where, iteration $\{t_n\}$ $(n \geq 0)$, and point $t_\star$ are given in Lemma 3.1. Furthermore, if there exists $R > t_\star$ such that

$$R_0 \leq R,$$

and

$$\ell_0 \int_0^1 [\theta t_\star + (1-\theta) R]^p\ d\theta \leq 1,$$

the solution $x^\star$ is unique in $U(x_0, R_0)$.

Proof. The proof can be found in [5, Theorem 3.3] (see, also Theorem 5 in [3]). Simply replace Lemma 3.1 of [5] by Lemma 3.1 in this proof. □

4. Special cases and applications

Case 1. (Lipschitz case: $p = 1$).

Application 4.1. *It is simple algebra to show that all hypotheses of Lemma 3.1 hold true provided that*

$$h_A = \alpha\ \eta \le \frac{1}{2}, \tag{4.1}$$

where,

$$\alpha = \frac{1}{8}\ \left(\ell + 4\ \ell_0 + \sqrt{\ell^2 + 8\ \ell\ \ell_0}\right). \tag{4.2}$$

The famous for its simplicity and clarity Newton–Kantorovich hypothesis for solving nonlinear equations is given by:

$$h_K = \ell\ \eta \le \frac{1}{2}. \tag{4.3}$$

It then follows from (4.1), (4.2), and (4.3):

$$h_K \le \frac{1}{2} \Longrightarrow h_A \le \frac{1}{2} \tag{4.4}$$

but not necessarily vice verca unless if $\ell_0 = \ell$.

Note also that our ratio of convergence "$2\ h_A$" is smaller that "$2\ h_K$" (see (3.9) and (3.10) for $p = 1$).

Example 4.2. Define the scalar function F by $F(x) = c_0\ x + c_1 + c_2\ \sin e^{c_3\ x}$, $x_0 = 0$, where c_i, $i = 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\dfrac{\ell}{\ell_0}$ can be arbitrarily large. That is (4.1) may be satisfied but not (4.3).

Example 4.3. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x : |x - x_0| \le 1 - \beta\}$, $\beta \in \left[0, \dfrac{1}{2}\right)$, and define function F on U_0 by

$$F(x) = x^3 - \beta. \tag{4.5}$$

Using hypotheses of Theorem 3.2, we get:

$$\eta = \frac{1}{3}\ (1 - \beta), \quad \ell_0 = 3 - \beta, \quad \text{and} \quad \ell = 2\ \ (2 - \beta).$$

The Kantorovich condition (4.3) is violated, since

$$\frac{4}{3}\ (1-\beta)\ (2-\beta) > 1 \quad \text{for all} \quad \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (1.2) converges to $x^\star = \sqrt[3]{\beta}$, starting at $x_0 = 1$. However, our condition (4.1) is true for all $\beta \in I = \left[.450339002, \frac{1}{2}\right)$.

Hence, the conclusions of our Theorem 3.2 can apply to solve equation (4.5) for all $\beta \in I$.

Example 4.4. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$ with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{4.6}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (4.6) arise in the kinetic theory of gasses [4]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{4.7}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (4.6).

We have the estimates

$$\max_{0 \le s \le 1} \left| \int \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therefore, if we set $\xi = \| F'(u_0)^{-1} \|$, then it follows from hypotheses of Theorem 3.2 that

$$\eta = \xi\,(|\lambda|\,\ln 2 + 1 - \theta),$$

$$\ell = 2\,\xi\,(|\lambda|\,\ln 2 + 3\,(2-\theta)) \quad \text{and} \quad \ell_0 = \xi\,(2\,|\lambda|\,\ln 2 + 3\,(3-\theta)).$$

It follows from Theorem 3.2 that if condition (4.1) holds, then problem (4.6) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (4.3). Note also that $\ell_0 < \ell$ for all $\theta \in [0,1]$.

Example 4.5. Consider the following nonlinear boundary value problem [4]

$$\begin{cases} u'' = -u^3 - \gamma\ u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)\ (u^3(t) + \gamma\ u^2(t))\ dt \tag{4.8}$$

where, Q is the Green function:

$$Q(s,t) = \begin{cases} t\ (1-s), & t \leq s \\ s\ (1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \leq s \leq 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, with norm

$$\| x \| = \max_{0 \leq s \leq 1} |x(s)|.$$

Then problem (4.8) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\ (x^3(t) + \gamma\ x^2(t))\ dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\ \gamma < 5$, then

$$\begin{aligned} \| I - F'(u_0) \| &\leq \frac{3\ \| u_0 \|^2 + 2\ \gamma\ \| u_0 \|}{8} = \frac{3 + 2\ \gamma}{8}, \\ \| F'(u_0)^{-1} \| &\leq \frac{1}{1 - \dfrac{3 + 2\ \gamma}{8}} = \frac{8}{5 - 2\ \gamma}, \\ \| F(u_0) \| &\leq \frac{\| u_0 \|^3 + \gamma\ \| u_0 \|^2}{8} = \frac{1 + \gamma}{8}, \\ \| F(u_0)^{-1}\ F(u_0) \| &\leq \frac{1 + \gamma}{5 - 2\ \gamma}. \end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$, we have:

$$[(F'(x) - F'(y))v]\,(s) = -\int_0^1 Q(s,t)\ (3\ x^2(t) - 3\ y^2(t) + 2\ \gamma\ (x(t) - y(t)))\ v(t)\ dt.$$

Consequently,

$$\begin{aligned}\| F'(x) - F'(y) \| &\leq \frac{\| x - y \| \ (2\,\gamma + 3\,(\| x \| + \| y \|))}{8} \\ &\leq \frac{\| x - y \| \ (2\,\gamma + 6\,R + 6\ \| u_0 \|)}{8} \\ &= \frac{\gamma + 6\,R + 3}{4}\ \| x - y \|, \\ \| F'(x) - F'(u_0) \| &\leq \frac{\| x - u_0 \| \ (2\,\gamma + 3\,(\| x \| + \| u_0 \|))}{8} \\ &\leq \frac{\| x - u_0 \| \ (2\,\gamma + 3\,R + 6\ \| u_0 \|)}{8} \\ &= \frac{2\,\gamma + 3\,R + 6}{8}\ \| x - u_0 \|\,. \end{aligned}$$

Therefore, conditions of Theorem 3.2 hold with

$$\eta = \frac{1+\gamma}{5 - 2\,\gamma}, \quad \ell = \frac{\gamma + 6\,R + 3}{4}, \quad \ell_0 = \frac{2\,\gamma + 3\,R + 6}{8}.$$

Note also that $\ell_0 < \ell$.

Case 2. (Hölder Case: $p \neq 1$**).**

Application 4.6. *Let* $\ell = 1$, $\ell_0 = p = \frac{1}{2}$, *and* $\eta = .2$. *Then, we have*

$$\delta = 1.13084151, \qquad \alpha = 2.75172827, \quad q = .550345654 < .6666.$$

Hence, hypotheses of Lemma 3.1, and (2.7) are satisfied. Then, we obtain:

$$t_0 = 0, \qquad t_1 = .2, \qquad t_2 = .276801908, \qquad t_3 = .296056509,$$
$$t_4 = .298503386, \qquad t_5 = .298614405, \qquad t_6 = .298615478,$$
$$c_0 = 1.326445825, \qquad \delta_0 = .444, \qquad h(c_0) = .859253735.$$

Then, we get the following comparison table:

Comparison table

	Actual	*ours*	[7]
n	$t_n - t_{n-1}$ *(2.9)*	*(3.8)*	*(2.8)*
2	*.076801908*	*.124470742*	*.118980595*
3	*.019254601*	*.035971114*	*.072458013*
4	*.002447296*	*.009445671*	*.46497804*
5	*.000111019*	*.001151911*	*.03138214*
6	*.000001073*	*.000030298*	*.021922049*

The table justifies the claims made in this study.

So far, we showed error bound (3.9) can be finer than (2.8). However, the sufficient convergence condition (3.1) is stronger than (2.7) for $p \in (0,1)$, unless if $\ell_0 = 0$. Indeed, first note that

$$\delta = \frac{1+p}{c_0}, \tag{4.9}$$

and (3.1) can be written as

$$\eta^p \leq h_1(c_0) = \frac{1}{\ell_0}\left(1 - \frac{1}{c_0}\right). \tag{4.10}$$

We shall show

$$h_1(c_0) < h_0(c_0). \tag{4.11}$$

Instead of showing (4.11), we can show

$$\frac{1}{\ell_0}\left(1-\frac{1}{c_0}\right) \leq \left(1-\frac{1}{c_0}\right)^p \frac{1+p}{\left((\ell_0\ (1+p))^{\frac{1}{1-p}} + (\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}}\right)^{1-p}}$$

or

$$\left\{\left(1-\frac{1}{c_0}\right)\left((\ell_0\ (1+p))^{\frac{1}{1-p}} + (\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}}\right)\right\}^{1-p} < \ell_0\ (1+p)$$

or

$$\left(1-\frac{1}{c_0}\right)\left((\ell_0\ (1+p))^{\frac{1}{1-p}} + (\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}}\right) \geq (\ell_0\ (1+p))^{\frac{1}{1-p}}$$

or

$$(\ell_0\ (1+p))^{\frac{1}{1-p}} + (\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}} - \frac{1}{c_0}\ (\ell_0\ (1+p))^{\frac{1}{1-p}} - \frac{1}{c_0}\ (\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}} < (\ell_0\ (1+p))^{\frac{1}{1-p}}$$

or

$$\left(1-\frac{1}{c_0}\right)(\ell\ c_0\ (c_0-1))^{\frac{1}{1-p}} < \frac{1}{c_0}\ (\ell_0\ (1+p))^{\frac{1}{1-p}}$$

or

$$\left(1-\frac{1}{c_0}\right)^{1-p} \ell\ c_0\ (c_0-1) < \left(\frac{1}{c_0}\right)^{1-p} \ell_0\ (1+p)$$

or

$$(c_0-1)^{1-p}\ \ell\ c_0\ (c_0-1) < \ell_0\ (1+p)$$

or

$$(c_0-1)^{2-p}\ \ell\ c_0 < \ell_0\ (1+p). \tag{4.12}$$

But, we have by (2.4):

$$\begin{aligned} c_0 - 1 &= \frac{\ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}}{2\ \ell} - 1 \\ &= \frac{2\ \ell_0\ (1+p)^p\ p^{1-p}}{\ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}}. \end{aligned} \tag{4.13}$$

It then follows from (4.12), and (4.13) that we must show:

$$\left(\frac{2\ell_0(1+p)^p p^{1-p}}{\ell + \sqrt{\ell^2 + 4\ell_0\ell(1+p)^p p^{1-p}}}\right)^{2-p} \frac{\ell + \sqrt{\ell^2 + 4\ell_0\ell(1+p)^p p^{1-p}}}{2} < \ell_0(1+p)$$

or

$$\left(2\ell_0\ (1+p)^p\ p^{1-p}\right)^{2-p} < 2\ \ell_0\ (1+p)\ \left(\ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}\right)^{1-p}$$

or

$$2^{1-p}\ \ell_0^{1-p}\ (1+p)^{-(1-p)^2}\ p^{(1-p)(2-p)} < \left(\ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}\right)^{1-p}$$

or

$$2\ \ell_0\ (1+p)^{-(1-p)}\ p^{2-p} < \ell + \sqrt{\ell^2 + 4\ \ell_0\ \ell\ (1+p)^p\ p^{1-p}}$$

or

$$2\ \ell_0\ (1+p)^{-(1-p)}\ p^{2-p} < \ell\ \left(1 + \sqrt{1 + 4\ \frac{\ell_0}{\ell}\ (1+p)^p\ p^{1-p}}\right)$$

or

$$2\ \frac{\ell_0}{\ell}\ (1+p)^{-(1-p)}\ p^{2-p} < 1 + \sqrt{1 + 4\ \frac{\ell_0}{\ell}\ (1+p)^p\ p^{1-p}}. \tag{4.14}$$

But, we have:

$$2\ \frac{\ell_0}{\ell}\ (1+p)^{-(1-p)}\ p^{2-p} = 2\ \frac{\ell_0}{\ell}\ \left(\frac{p}{1+p}\right)^{1-p}\ p < 2,$$

whereas

$$1 + \sqrt{1 + 4\ \frac{\ell_0}{\ell}\ (1+p)^p\ p^{1-p}} > 2,$$

which shows (4.11).

In case $\ell_0 = 0$, (2.7) becomes

$$\eta \leq 0, \tag{4.15}$$

whereas from the proof of Lemma 3.1, we have:

$$\frac{L\ \eta^p}{1+p} \leq 1. \tag{4.16}$$

Conclusion

Case $p = 1$**.** Our condition (4.1) is weaker than (4.3) [4], [9], and the ratio of convergence $2\ q_A$ is smaller than $2\ q_K$ for $\ell_0 < \ell$.

If $\ell_0 = \ell$, conditions (4.1), and (4.3) coincide.

Case $p \in (0, 1)$**.** Condition (3.1) is stronger than (2.7) if $\ell_0 \neq 0$, but the error bounds (3.9) can be finer than (2.8) [7]. If $\ell_0 = 0$, condition (4.15) cannot apply for $\eta \neq 0$, but (4.16) does apply.

Hence, in practice we will always use our results for $p = 1$, and a combination of these results if $p \in (0, 1)$. For example, if (3.1) and (2.7) hold, we use error bounds (3.9). If (2.7) only holds, then we use bounds which x_{N_0} (replacing x_0) satisfies (3.1), after which we use bounds (3.9) (for $\ell_0 \neq 0$). Finally, if $\ell_0 = 0$, and (4.16) holds, we use the bounds introduced in the proof of Lemma 3.1.

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