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EXTENDING THE APPLICABILITY OF A SECANT–TYPE METHOD OF ORDER TWO USING RECURRENT FUNCTIONS

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Abstract. We approximate a locally unique solution of an equation in a Banach space setting by using a quadratically convergent Secant–type method studied in [4]–[6]. Using our new idea of recurrent functions, we extend the applicability of this method, and also simlify the existing sufficient convergence conditions for this method [4]–[6].

Numerical examples, where the method is compared favorably to the Secant and Newton's methods are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic

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systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

In [4]–[6], we studied the Secant–type method (STM)

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}\ F(x_n), \quad (n \geq 0) \\ A_n &= [2\ x_n - x_{n-1}, x_{n-1}; F],\ (x_{-1}, x_0 \in \mathcal{D}) \end{aligned}$$

to generate a sequence $\{x_n\}$ converging to $x^\star$.

Here, $[x, y; F] \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes a divided difference of order one at the points $(x, y) \in \mathcal{D}^2$, satisfying (see [4], [11]):

$$[x, y; F]\ (x - y) = F(x) - F(y), \quad \text{for all } x \neq y. \tag{1.2}$$

The (STM) has geometrical interpretation similar to the Secant method (SM):

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1}\ F(x_n), \quad x_{-1}, x_0 \in \mathcal{D}, \quad (n \geq 0)$$

in the scalar case [4]–[6]. The (STM) serves as an alternative to the usage of Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n), \quad x_0 \in \mathcal{D}, \quad (n \geq 0)$$

in the case of solving nondifferentiable operator equations. Note also that even if the analytical representation of $F'(x_n)$ exists, it may be too expensive or impossible to compute the inverse $F'(x_n)^{-1}$ $(n \geq 0)$ at each step.

The quadratic convergence of (STM) was shown in [4]–[6]. We also note though that the efficiency indices of (STM), (SM), (NM) are $\sqrt{2} = 1.414213562$, $\dfrac{1+\sqrt{5}}{2} = 1.618033989$, and $\sqrt{2}$, respectively.

(STM) have been used by other reseachers. Potra [12] used a three point method and divided differences of order one as an alternative to Ulm's method [4] using divided differences of order two. Potra's method is of convergence $1.839\cdots$ (STM) of order between 1.618033989 and $1.839\cdots$ or less than two can also be found in the works of Hernández, et al. [9], [10]. As already noted above we studied the local as well as the semilocal convergence of (STM) in [4]–[6] (see, also [2], [3]). Due to its importance, we use our new idea or recurrent functions to provide a new semilocal convergence, which is expanding the applicability of the (STM) by on the one hand simplifying existing hypotheses or replacing them by weaker ones.

Numerical examples where the method is compared favorably to the Secant and Newton's methods are also provided in this study.

2. Semilocal convergence analysis of (NTM)

It is convenient for us to define certain numbers, parameters, and polynomials.

Let $\alpha > 0$, $\beta > 0$, and $\gamma \geq 0$ be given constants. Define numbers d, δ_i, $i = 1, \cdots, 8$ by

$$d = 1 - \beta\ \gamma^2, \tag{2.1}$$

δ_0 to be the minimal non–negative zero of polynomial P_0 given by

$$\begin{aligned} P_0(t) &= \beta\ d^2\ t^2 + 3\ \alpha\ d\ (1 - 2\ \alpha\ t)\ t + 2\ \alpha\ (1 - 2\ \alpha\ t)^2\ t \\ &\quad -(1 - 2\ \alpha\ t)\ d^2, \end{aligned} \tag{2.2}$$

$$\delta_1 = \frac{d}{2\ \alpha}, \tag{2.3}$$

$$\delta_2 = \frac{(\sqrt{9\ \alpha^2 + 4\ \beta\ d} - 5\ \alpha)\ d}{2\ (\beta\ d - 4\ \alpha^2)}, \tag{2.4}$$

$$\delta_3 = \frac{2 + 3\ d - \sqrt{(2 - d)\ (2 + 7\ d)}}{8\ \alpha}, \tag{2.5}$$

$$\delta_4 = \min\{\delta_i,\ i = 0, 1, 2, 3\}, \tag{2.6}$$

$$\delta_5 = \frac{d}{5\ \alpha}, \tag{2.7}$$

$$\delta_6 = \min\{\delta_0,\ \delta_1,\ \delta_3,\ \delta_5\}, \tag{2.8}$$

$$\delta_7 = \frac{5\ \alpha + \sqrt{9\ \alpha^2 + 4\ \beta\ d}}{4\ \alpha^2 - \beta\ d}, \tag{2.9}$$

$$\delta_8 = \min\{\delta_0,\ \delta_1,\ \delta_3,\ \delta_7\}, \tag{2.10}$$

and polynomials P_1, P_2 by

$$P_1(t) = (\beta\ d - 4\ \alpha^2)\ t^2 + 5\ \alpha\ d\ t - d^2, \tag{2.11}$$

$$P_2(t) = 4\ \alpha^2\ t^2 - \alpha\ (2 + 3\ d)\ t + d^2. \tag{2.12}$$

Consider Conditions $(\mathcal{C}_i)$, $i = 1, 2, 3$:

$(\mathcal{C}_1)$

$$0 < d < 1, \quad \beta\ d - 4\ \alpha^2 > 0, \quad \delta \in [0, \delta_4];$$

$(\mathcal{C}_2)$

$$0 < d < 1, \quad \beta\ d - 4\ \alpha^2 = 0, \quad \delta \in [0, \delta_6];$$

$(\mathcal{C}_3)$

$$0 < d < 1, \quad \beta\ d - 4\ \alpha^2 < 0, \quad \delta \in [0, \delta_8].$$

It is simple algebra to show that under Conditions $(\mathcal{C}_i)$, the following hold

$$\delta_4 \geq 0, \tag{2.13}$$

$$\delta_6 \geq 0, \tag{2.14}$$

$$\delta_8 \geq 0, \tag{2.15}$$

and

$$P_0(\delta) \leq 0, \tag{2.16}$$

$$P_1(\delta) \leq 0, \tag{2.17}$$

$$P_2(\delta) \geq 0, \tag{2.18}$$

$$P_0(\delta_0) = 0, \tag{2.19}$$

$$P_1(\delta_2) = 0, \tag{2.20}$$

$$P_2(\delta_3) = 0, \tag{2.21}$$

in all three cases.

Let us also define polynomial g by:

$$g(t) := g_\delta(t) = 2\ \alpha\ t^2 + (\alpha + \beta\ \delta)\ t - (\alpha + \beta\ \delta), \tag{2.22}$$

and parameter $s_\infty := s_\infty(\delta)$

$$s_\infty = 1 - \frac{2\ \alpha\ \delta}{d}. \tag{2.23}$$

Note that

$$q = \frac{\sqrt{(\alpha + \beta\ \delta)^2 + 8\ \alpha\ (\alpha + \beta\ \delta)} - (\alpha + \beta\ \delta)}{4\ \alpha}, \tag{2.24}$$

is the only non–negative zero of polynomial g.

It is then simple algebra to show under any of Conditions $(\mathcal{C}_i)$, $i = 1, 2, 3$:

$$\frac{\lambda}{2} = \max \left\{ \frac{\alpha\ \delta + \beta\ \gamma^2}{d - 2\ \alpha\ \delta}, \frac{\alpha\ s_\infty\ \delta + \beta\ \delta^2}{d - 2\ \alpha\ (1 + s_\infty)\ \delta}, q \right\} \leq s_\infty. \tag{2.25}$$

We need the following result on majorizing sequences for (STM):

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ be given constants, and δ a non–negative parameter. Define scalar iteration $\{t_n\}$ $(n \geq -1)$ by*

$$\begin{aligned} & t_{-1} = 0, \quad t_0 = \gamma, \quad t_1 = \gamma + \delta, \\ & t_{n+2} = t_{n+1} + \frac{\alpha\ (t_{n+1} - t_n) + \beta\ (t_n - t_{n-1})^2}{d - 2\ \alpha\ (t_{n+1} - \gamma)}\ (t_{n+1} - t_n). \end{aligned} \tag{2.26}$$

Then, under Conditions $(\mathcal{C}_1)$ or $(\mathcal{C}_2)$ or $(\mathcal{C}_3)$, sequence $\{t_n\}$ $(n \geq -1)$ is non–decreasing, bounded above by

$$t^{\star\star} = \gamma + \frac{2\ \delta}{2 - \lambda}, \tag{2.27}$$

and converges to its unique least upper bound

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.28}$$

Moreover the following estimates hold for all $n \geq 0$:

$$t_{n+1} - t_n \leq \frac{\lambda}{2}\ (t_n - t_{n-1}) \leq \left(\frac{\lambda}{2}\right)^n\ \delta, \tag{2.29}$$

and

$$t^\star - t_n \leq \frac{2\ \delta}{2 - \lambda}\ \left(\frac{\lambda}{2}\right)^n, \tag{2.30}$$

where, λ is given in (2.25).

Proof. We shall show using induction on the integer m:

$$t_{m+2} - t_{m+1} \leq \frac{\lambda}{2}\ (t_{m+1} - t_m), \tag{2.31}$$

and

$$d - 2\ \alpha\ (t_{m+1} - \gamma) > 0. \tag{2.32}$$

Estimates (2.31), and (2.32) hold for $m = 0, 1$, by (2.25), and (2.26). Let us assume they hold for all $k \leq m$, $(m \geq 1)$. We then obtain:

$$t_{k+1} - t_k \leq \left(\frac{\lambda}{2}\right)^k\ \delta, \tag{2.33}$$

and

$$
\begin{aligned}
t_{k+1} &\leq t_k + \left(\frac{\lambda}{2}\right)^k \delta \\
&\leq t_{k-1} + \left(\frac{\lambda}{2}\right)^{k-1} \delta + \left(\frac{\lambda}{2}\right)^k \delta \\
&\leq t_1 + \left(1 + \frac{\lambda}{2} + \cdots + \left(\frac{\lambda}{2}\right)^k\right) \delta \\
&= \gamma + \frac{1 - \left(\frac{\lambda}{2}\right)^{k+1}}{1 - \frac{\lambda}{2}} \delta \\
&\leq \gamma + \frac{2\ \delta}{2 - \lambda} = t^{\star\star}.
\end{aligned} \tag{2.34}
$$

In view of (2.33), and (2.34), estimates (2.31), and (2.32) certainly hold if

$$
0 \leq \frac{\alpha \left(\frac{\lambda}{2}\right)^k \delta + \beta \left(\frac{\lambda}{2}\right)^{2\,(k-1)} \delta^2}{d - 2\ \alpha\ \dfrac{1 - \left(\frac{\lambda}{2}\right)^{k+1}}{1 - \frac{\lambda}{2}}\ \delta} \leq \frac{\lambda}{2}, \tag{2.35}
$$

or

$$
0 \leq \frac{\alpha \left(\frac{\lambda}{2}\right)^k \delta + \beta \left(\frac{\lambda}{2}\right)^{k} \delta^2}{d - 2\ \alpha\ \dfrac{1 - \left(\frac{\lambda}{2}\right)^{k+1}}{1 - \frac{\lambda}{2}}\ \delta} \leq \frac{\lambda}{2}. \tag{2.36}
$$

It is convenient for us to set $s = \dfrac{\lambda}{2}$, which together with (2.35) leads to showing:

$$
f_k(s) = (\alpha + \beta\ \delta)\ \delta\ s^{k-1} + 2\ \alpha\ (1 + s + s^2 + \cdots + s^k)\ \delta - d \leq 0 \quad (k \geq 2). \tag{2.37}
$$

We have $f_k(0) = 2\ \alpha\ \delta - d \leq 0$, $f_k(s) > 0$ for sufficiently large $s > 0$, and $f_k'(s) \geq 0$. Hence, each f_k has a unique nonnegative zero s_k, by the intermediate value theorem.

We need to find a relationship between two consecutive polynomials f_k:

$$\begin{aligned} f_{k+1}(s) &= (\alpha+\beta\delta)\delta s^k + 2\alpha((1+s+s^2+\cdots+s^k)+s^{k+1})\delta - d \\ &= f_k(s) + (\alpha+\beta\delta)\delta s^k - (\alpha+\beta\delta)\delta s^{k-1} + 2\alpha s^{k+1}\delta \\ &= f_k(s) + g(s)\ s^{k-1}\ \delta, \end{aligned} \tag{2.38}$$

where, polynomial g is given in (2.22).

We must show estimate (2.36) holds for all $s \in [0, s_k]$. If there exists $k \geq 0$, such that $s_{k+1} \geq q$, then using (2.38), we get:

$$f_{k+1}(s_{k+1}) = f_k(s_{k+1}) + g(s_{k+1})\ s^{k-1}\ \delta,$$

or

$$f_k(s_{k+1}) \leq 0, \tag{2.39}$$

since $f_{k+1}(s_{k+1}) = 0$, and $g(s_{k+1})\ s^{k-1}\ \delta \geq 0$.

We can certainly choose the last of the s_k's denoted by s_∞ (obtained from (2.35) by letting $k \longrightarrow \infty$, and given in (2.23)), to be s_{k+1}. Hence, we get

$$s_{k+1} \leq s_k \qquad (k \geq 2), \tag{2.40}$$

since by the intermediate value theorem applied to function f_k on $[0, s_{k+1}]$, s_k exists and satisfies (2.40).

Hence, sequence $\{s_m\}$ is non–increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound $s^\star$ satisfying $s^\star \geq s_\infty$. Then, estimate (2.37) certainly holds on $[0, s_k]$ provided that

$$q \leq s_\infty, \tag{2.41}$$

which holds by (2.25). That completes the induction for (2.31), (2.32), and (2.29).

Moreover, sequence $\{t_n\}$ is increasing, bounded above by $t^{\star\star}$, and as such it converges to its unique least upper bound $t^\star$. Finally, estimate (2.30) follows from (2.29) by using standard majorization techniques [4], [11]. That completes the proof of Lemma 2.1. □

We can show the main semilocal convergence result for (STM).

Theorem 2.2. *Let F be a nonlinear operator defined on a subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.*

Assume:

F has divided differences $[x, y; F]$ and $[x, y, z; F]$ on $\mathcal{D}_0 \subseteq \mathcal{D}$; there exist points x_{-1}, x_0 in $\mathcal{D}_0$, such that $A_0 = [2\,x_0 - x_{-1}, x_{-1}; F]$ is invertible on $\mathcal{D}_0$;

$$\text{for all } x, y \in \mathcal{D}_0 \Longrightarrow 2\ y - x \in \mathcal{D}_0; \tag{2.42}$$

there exist constants α, β, γ, *such that*

$$\parallel A_0^{-1}\ ([x,y;F]-[u,v;F])\parallel\leq\alpha\ \Big(\parallel x-u\parallel+\parallel y-v\parallel\Big), \tag{2.43}$$

$$\parallel A_0^{-1}\ ([y,x,y;F]-[2\,y-x,x,y;F])\parallel\leq\beta\ \parallel x-y\parallel \tag{2.44}$$

for all $x,y,u,v\in\mathcal{D}_0$;

$$\parallel x_0-x_{-1}\parallel\leq\gamma; \tag{2.45}$$

any one of the following Conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ *hold such that:*

$$\parallel A_0^{-1}\ F(x_0)\parallel\leq\delta\leq\overline{\delta}, \tag{2.46}$$

where, $\overline{\delta}$ *is* δ_4 *or* δ_6 *or* δ_8, *respectively; and*

$$\overline{U}(x_0,t^\star)=\{x\in\mathcal{X},\ \parallel x-x_0\parallel\leq t^\star\}\subseteq\mathcal{D}_0. \tag{2.47}$$

Then, sequence $\{x_n\}$ $(n\geq 0)$ *generated by (STM) is well defined, remains in* $\overline{U}(x_0,t^\star)$ *for all* $n\geq 0$, *and converges to a solution* $x^\star$ *of equation* $F(x)=0$ *in* $\overline{U}(x_0,t^\star)$. *Moreover, the following estimates hold for all* $n\geq 0$:

$$\parallel x_{n+1}-x_n\parallel\leq t_{n+1}-t_n, \tag{2.48}$$

and

$$\parallel x_n-x^\star\parallel\leq t^\star-t_n, \tag{2.49}$$

where, sequence $\{t_n\}$ $(n\geq 0)$ *is given in (2.26). Furthermore, if* $\mathcal{D}_0$ *is a convex set, and*

$$2\ \alpha\ (\gamma+2\ t^\star)<1, \tag{2.50}$$

then $x^\star$ *is the unique solution of equation (1.1) in* $\overline{U}(x_0,t^\star)$.

Proof. We shall show (2.48) holds for all k. Using (2.26), (2.34), (2.42), (2.45), and (2.46), we deduce $x_{-1},x_1\in\overline{U}(x_0,t^\star)$. Let us assume (2.48) holds for all $n\leq k$, and $x_k\in\overline{U}(x_0,t^\star)$. In view of (2.43), (2.44), and the induction hypotheses, we obtain in turn:

$$\begin{aligned}
&\parallel A_0^{-1}\ (A_{k+1}-A_0)]\parallel\\
=\ &\parallel A_0^{-1}\Big([2\,x_0-x_{-1},x_{-1};F]-[x_0,x_{-1};F]+[x_0,x_{-1};F]\\
&-[x_0,x_0;F]+[x_0,x_0;F]-[x_{k+1},x_0;F]+[x_{k+1},x_0;F]\\
&-[x_{k+1},x_k;F]+[x_{k+1},x_k;F]-[2\,x_{k+1}-x_k,x_k;F]\Big)\parallel\\
=\ &\parallel A_0^{-1}\Big(([2\,x_0-x_{-1},x_{-1},x_0;F]-[x_0,x_{-1},x_0;F])\,(x_0-x_{-1})\\
&+([x_0,x_0;F]-[x_{k+1},x_0;F])+([x_{k+1},x_0;F]-[x_{k+1},x_k;F])\\
&+([x_{k+1},x_k;F]-[2\,x_{k+1}-x_k,x_k;F])\Big)\parallel\\
\leq\ &\beta\gamma^2+\alpha(\parallel x_{k+1}-x_0\parallel+\parallel x_k-x_0\parallel+\parallel x_{k+1}-x_k\parallel)\\
\leq\ &\beta\ \gamma^2+2\ \alpha\ (t_{k+1}-\gamma)<1\quad\text{(by (2.32))}.
\end{aligned} \tag{2.51}$$

It follows from (2.51), and the Banach lemma on invertible operators [4], [11] that A_{k+1}^{-1} exists, and

$$\begin{aligned} &\| A_{k+1}^{-1} A_0 \| \\ \leq\ & \Big(1-\beta\gamma^2-\alpha(\| x_{k+1}-x_0 \| + \| x_k - x_0 \| + \| x_{k+1}-x_k \|)\Big)^{-1} \\ \leq\ & (d-2\alpha(t_{k+1}-\gamma))^{-1}. \end{aligned} \tag{2.52}$$

We can also obtain in turn:

$$\begin{aligned} &\| A_0^{-1}([x_{k+1},x_k;F]-A_k) \| \\ &=\| A_0^{-1}\Big([x_{k+1},x_k;F]-[x_k,x_k;F]+[x_k,x_k;F] \\ &\quad -[x_k,x_{k-1};F]+[x_k,x_{k-1};F]-[2\,x_k-x_{k-1},x_{k-1};F]\Big) \| \\ &=\| A_0^{-1}\Big(([x_{k+1},x_k;F]-[x_k,x_k;F]) \\ &\quad +([x_k,x_{k-1},x_k;F]-[2\,x_k-x_{k-1},x_{k-1},x_k;F])\ (x_k-x_{k-1})\Big) \| \\ &\leq \alpha\ \ \| x_{k+1}-x_k \| +\beta\ \ \| x_k-x_{k-1} \|^2 \\ &\leq \alpha\ (t_{k+1}-t_k)+\beta\ (t_k-t_{k-1})^2. \end{aligned} \tag{2.53}$$

Using (STM), (2.26), (2.52), and (2.53), we get:

$$\begin{aligned} &\| x_{k+2}-x_{k+1} \| \\ &=\| (A_{k+1}^{-1}\ A_0)\ (A_0^{-1}\ F(x_{k+1})) \| \\ &\leq\| A_{k+1}^{-1}\ A_0 \| \ \| A_0^{-1}\ (F(x_{k+1})-F(x_k)-A_k\ (x_{k+1}-x_k)) \| \\ &\leq\| A_{k+1}^{-1}\ A_0 \| \ \| A_0^{-1}\ ([x_{k+1},x_k;F]-A_k) \| \ \| x_{k+1}-x_k \| \\ &\leq t_{k+2}-t_{k+1}, \end{aligned} \tag{2.54}$$

which together with

$$\begin{aligned} \| x_{k+2}-x_0 \| &\leq \sum_{i=0}^{k+1} \| x_{i+1}-x_i \| \\ &\leq \sum_{i=0}^{k+1}(t_{i+1}-t_i) \leq t^\star \end{aligned} \tag{2.55}$$

complete the induction. Hence, sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0,t^\star)$ (since $\overline{U}(x_0,t^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (2.55), we obtain $F(x^\star)=0$. Estimate (2.49) follows from (2.48) by using standard majorization techniques [4], [11].

Finally to show uniqueness, let $y^\star \in \overline{U}(x_0, t^\star)$ be a solution of equation $F(y^\star) = 0$. Define linear operator $\mathcal{M}$ by:

$$\mathcal{M} = \int_0^1 [y^\star + \theta\ (x^\star - y^\star), y^\star + \theta\ (x^\star - y^\star); F]\ d\theta. \tag{2.56}$$

We have in turn by (2.43), and (2.50):

$$\begin{aligned}
&\| A_0^{-1}\ (A_0 - \mathcal{M}) \| \\
&\leq \alpha\ \int_0^1 \Big(\| 2\,x_0 - x_{-1} - y^\star - \theta\ (x^\star - y^\star) \| \\
&\quad + \| x_{-1} - y^\star - \theta\ (x^\star - y^\star) \| \Big)\ d\theta \\
&\leq \alpha\ \Big(\| x_0 - x_{-1} \| + \| x_0 - y^\star \| + \frac{\| x_0 - y^\star \| + \| x_0 - x^\star \|}{2} \\
&\quad + \| x_0 - x_{-1} \| + \| x_0 - y^\star \| + \frac{\| x_0 - x^\star \| + \| x_0 - y^\star \|}{2} \Big) \\
&\leq 2\ \alpha\ (\gamma + 2\ t^\star) < 1.
\end{aligned} \tag{2.57}$$

It follows by (2.57), and the Banach lemma on invertible operators, that $\mathcal{M}$ is invertible. In view of the identity

$$F(x^\star) - F(y^\star) = \mathcal{M}\ (x^\star - y^\star), \tag{2.58}$$

we deduce $x^\star = y^\star$. That completes the proof of Theorem 2.2. □

Remark 2.3. (a) Delicate condition (2.42) is automatically satisfied if $\mathcal{D}_0 = \mathcal{D} = \mathcal{X}$. Otherwise, as shown in [4]–[6] simply replace $\mathcal{D}_0$ by $\mathcal{D}_1 = U(x_0, 3\,t^\star)$.

(b) The sufficient convergence conditions given in [5] are clearly more complicated than the ones in Theorem 2.2, where the conditions in [4], [6] require, e.g., $2\ \beta\ \gamma^2 < 1$ compared to $\beta\ \gamma^2 < 1$ (here).

3. Applications

Application 3.1. *Let us define quadratic polynomial operator on $\mathcal{X}$ by*

$$P(x) = B\ x^2 + L\ x + w, \tag{3.1}$$

where, B is a bilinear operator on $\mathcal{L}(\mathcal{X} \times \mathcal{X}, \mathcal{Y})$, $L \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and w is a fixed element in $\mathcal{X}$; see, [1], [7], [8].

Define divided difference $[x, y; P]$ satisfying (1.2) by

$$P(2y - x) - P(x) = 2\ [2\,y - x, x; P]\ (y - x), \quad \text{for all } x, y \in \mathcal{D},\ (x \neq y). \tag{3.2}$$

It can then easily be seen that (STM) coincides with (NM), and is given by

$$x_{n+1} = x_n - (2\ B\,x_n + L)^{-1}\ P(x_n). \tag{3.3}$$

Let us provide a numerical example for equation $P(x) = 0$ to show that (STM) is faster than (SM).

Example 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D}_0 = \mathcal{D} = (.4, 1.5)$, and define polynomial P on $\mathcal{D}_0$ by

$$P(x) = x^2 - 6\,x + 5. \tag{3.4}$$

Then (STM) becomes (NM) (3.3), and is given by

$$x_{n+1} = \frac{x_n^2 - 5}{2\,(x_n - 3)}, \tag{3.5}$$

whereas the (SM) is:

$$x_{n+1} = \frac{x_{n-1}\;x_n - 5}{x_{n-1} + x_n - 6}. \tag{3.6}$$

Choose $x_{-1} = .6$, and $x_0 = .7$. Then, we have the results:

Comparison table

n	(STM)=(NM)	(SM)
1	.980434783	.96875
2	.999905228	.997835498
3	.999999998	.99998323
4	$1=x^\star$	.99999991
5		1

Another example is suggested involving Chandrasekhar quadratic integral equations appearing in radiative transfer [1], [4], [7], [11].

Example 3.3. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$ with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "Cubic" integral equation:

$$u(s) = \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{3.7}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (3.7) arise in the kinetic theory of gasses [6]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator P on $\mathcal{D}$ by

$$P(x)(s) = -x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{3.8}$$

for all $s \in [0,1]$, then every zero of P satisfies equation (3.7).

Finally, we suggest an example involving a more general nonlinear equation.

Example 3.4. Let $G(x,t,x(t))$ be a continuous function of its arguments, which is sufficiently many times differentiable with respect to x. It can easily be seen that if operator F is given by

$$F(x(s)) = x(s) - \int_0^1 G(s,t,x(t))\; dt,$$

then, the divided difference A_n appearing in (STM) can be defined by

$$A_n(s,t) = \frac{G(s,t,2\,x_n(t) - x_{n-1}(t)) - G(s,t,x_{n-1}(t))}{2\;(x_n(t) - x_{n-1}(t))}$$

provided that if for $t = t_m$, we get $x_n(t) = x_{n-1}(t)$, then, the above function equals $G'(s,t_m,x_n(t_m))$. Note that this way $A_n(s,t)$ is continuous for all $t \in [0,1]$.

Conclusion

Using our new idea of recurrent functions, we provided a semilocal convergence analysis for (STM) in order to approximate a locally unique solution of an equation in a Banach space, and extend the applicability of this method. Numerical examples further validating the results are also provided in this study, where the (STM) is compared favorably to the Secant and Newton's methods.

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