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GENERALIZED UNIVERSAL THEOREM AND ALEKSANDROVE ISOMETRIC EXTENSION PROBLEM

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Abstract. In this paper, Figiel's lemma that concerns the isometric embedding from a Banach space X into the continuous space $C(L)$ are extended. That is a form of extension of universal theorem. Besides we obtain some isometric extending mappings which satisfy Aleksandrov's, i.e., some mapping that preserve one distance can be extend to isometry in whole space.

1. Introduction

In 1932, Mazur-Ulam ([1]) established the following problem: *if X, Y are normed linear spaces and $f : X \to Y$ is an surjective isometric mapping, then f is an affine mapping.* The "surjection" is a necessary condition. As a matter of fact, there is an example ([2]) which shows that the above result may not be true without this condition. When is this affinity result true without the "surjectio"? Many authors today investigate the cases not restricted to this surjection condition.

In 1968, T. Figiel ([3, 4]) extended Mazur-Ulam's theorem and proved the following theorem.

Theorem 1.1. *Suppose that X, Y are both real Banach spaces. If $F : X \to Y$ is an isometry mapping and $F(0) = 0$, then there exists a continuous linear mapping $f : \overline{spanF(X)} \mapsto X$, which satisfies $f \circ F = i_X$, and that f is unique and $||f|_{span\,F(x)}|| = 1$.*

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Since 1983, T. Rassias, B. Mielnik, Ma Yumei etc., ([5-11]) gave a series results about how to let a (DOPP) mapping between metric spaces to be an isometry. "One Distance Preserving Property"(denoted by DOPP): *Suppose that* (X, d), (Y, d) *are two metric linear spaces. The mapping* $f : X \to Y$ *is said to have if* $d(x, y) = 1$ *for any* x, $y \in X$, *then* $d(f(x), f(y)) = 1$.

Many authors today, under certain additional conditions, keep investigating analogous topics.

In this paper we work as follows: In section 2, we establish an extension of Figiel's lemma and obtain a generalized Universal theorem. In section 3, we are concerned how to extend the (DOPP) mapping to an isometry.

2. Extension of universal theorem

Universal theorem ([4]) is very important in Banach space:
(1) Any separable Banach space X can be isometrically embedded to $C[0, 1]$.
(2) Any Banach space X can be isometrically embedded to $C(K)$. (Here K is the unit ball of X^*, which is w^*- compact set.)

We know the following two properties.

Property 2.1. ([4]) *Suppose that* X, Y *are two real Banach spaces. Let* a *is one of smooth points of the spheres* $\{x \in X, ||x|| = ||a||\}$, *and* $F : X \to Y$ *is an isometry mapping, such that* $F(0) = 0$, *and besides there exists* $f \in Y^*, ||f|| = 1$, *such that for any* $r \in \mathbb{R}$, $f(F(ra)) = r||a||$, *then* $f \circ F = f_a$, *where* f_a *is a support function at* a.

Property 2.2. ([4]) *Suppose that* Y *is a real normed linear space,* $\mathbb{R}$ *is a real space, and* $F : \mathbb{R} \to Y$ *is an isometric embedding. Then there exists* $f \in Y^*, ||f|| = 1$ *such that* $f \circ F = id_{\mathbb{R}}$. *Here,* $id_{\mathbb{R}}$ *is an identity operator on* $\mathbb{R}$.

Now,we have the extension of Property 2.2, as follows:

Theorem 2.3. *Suppose that* X, Y *are two real Banach spaces,* $F : X \to Y$ *is an isometry mapping and* $F(0) = 0$. *Then,*
i) There exists $f \in Y^*, ||f|| = 1$, *such that* $f \circ F \in S(X^*)$.
ii) $f \circ F = id_{\mathbb{R}}$ *as* $X = \mathbb{R}$.
iii) There exists a compact L_X^Y*(*w^**-topology), such that* $U : X \mapsto C(L_X^Y)$, $U(x) = F(x)|_{L_X}$ *is an linear isometric mapping.*

Remark 2.4. ([4]) If $X = Y$, then (2) in universal theorem is obvious by Theorem 2.3.

The proof of Theorem 2.3 need Mazur's Theorem ([5]): *"Suppose that A is a solid closed convex set of a separable Banach space. Then the smooth points set $sm(A)$ of A, is the residual set of a bounded $\partial(A)$, and $sm(A)$ is dense in $\partial(A)$."*

Thanks to R. Villa ([12]) we give proof of Theorem 2.3.

Proof. **Case 1:** Suppose X is a separable space. Let $a \in X$ be the smooth point of $\{x \in X, ||x|| = ||a||\}$, and f_a be the support function at a, i.e. $||f_a|| = 1$, $f_a(a) = ||a||$. Then for $\forall\ n \in \mathbb{N}$, by Hahn-Banach Theorem, there exists $g_n \in Y^*, ||g_n|| = 1$, so that

$$g_n(F(na) - F(-na)) = ||F(na) - F(-(na))|| = 2n||a||,$$

and for $\forall r, |r| \le n$, we obtain

$$\begin{aligned} 2n||a|| &\le |g_n(F(na) - F(ra))| + |g_n(F(ra) - F(-na))| \\ &\le ||F(na) - F(ra)|| + ||F(ra) - F(-na)|| \\ &= |n - r| \cdot ||a|| + |n + r| \cdot ||a|| \\ &= 2n||a||. \end{aligned}$$

This implies that

$$(n - r)||a|| = |g_n(F(na) - F(ra))|. \tag{2.1}$$

If $r = 0$, then $|g_n(F(na))| = n||a||$. Let us take $\varepsilon_n \in \{\{-1, 1\}\}$ such that

$$\varepsilon_n g_n(F(na)) = n||a||. \tag{2.2}$$

Moreover, for any $|r| \le n$, we claim that

$$\varepsilon_n g_n(F(ra)) = r||a||. \tag{2.3}$$

Indeed,

$$\begin{aligned} \varepsilon_n(g_n(F(na)) - g_n(F(ra))) &= n||a|| - \varepsilon_n g_n F(ra) \\ &\ge n||a|| - |g_n F(ra)| \\ &\ge (n - |r|)||a|| \\ &\ge 0. \end{aligned}$$

From (2.1),

$$\varepsilon_n(g_n(F(na)) - g_n(F(ra))) = |(g_n(F(na)) - g_n(F(ra)))| = (n - r)||a||$$

and from (2.2) we show that (2.3) is true. Because $B(Y^*)$ is w^*-compact, there is $g_a \in B(Y^*)$ such that

$$g_a(F(ra)) = r||a|| \quad (\forall r \in \mathbb{R}). \tag{2.4}$$

By Property 2.1, we can show that $g_a \circ F = f_a$ and we denote that $f = g_a$. Thus complete the proof of i) under the condition of X is separable.

For ii), we can let $a = 1$, then by (2.4), $g_1 \circ F(r) = r$, this also extend Property 2.2.

Next, we will prove that iii). Let

$$L_X^Y = \{g \in B(Y^*),\ g \circ F \in B(X^*)\}. \tag{2.5}$$

Then L_X^Y is w^*-compact. Let

$$U : X \to C(L_X^Y),\ \ U(x) = F(x)|_{L_X^Y}\ \ (x \in X). \tag{2.6}$$

Obviously, $||U(x)|| \le ||F(x)|| = ||x||$ and U is a continuous linear mapping on X. In fact, for any $x, y \in X, \alpha, \beta \in \mathbb{R}, g \in L_X^Y$,

$$\begin{aligned} U(\alpha x + \beta y)(g) &= F(\alpha x + \beta y)(g) \\ &= g \circ F(\alpha x + \beta y) \\ &= \alpha g \circ F(x) + \beta g \circ F(y) \\ &= \alpha U(x)(g) + \beta U(y)(g). \end{aligned}$$

Since X is separable, then there exists $g_a \in S(Y^*)$ such that $|g_a(F(a))| = \|a\|$ for any smooth point $a \in X$. Hence $\|U(a)\| = \|a\|$ and again Mazur's theorem implies that the smooth points set of X is dense. Thus

$$||U(x)|| = ||x||. \tag{2.7}$$

This completes the proof of iii) under the condition that f X is separable.

Case 2: If X is not separable, let $\Xi = \{X_\gamma : X_\gamma (\subset X)\ is\ separable\}$. Again

$$U_\gamma : X_\gamma \to C(L_{X_\gamma}^Y),\ \ U_\gamma(x) = F(x)|_{L_{X_\gamma}^Y}\ \ (x \in X_\gamma).$$

If $X', X'' \subset X$ are two separable subspaces, then $\overline{X' + X''}$ is also separable, and

$$L_{\overline{X'+X''}} \subset L_{X'} \cap L_{X''}.$$

Then the w^*-closed set family $\{L_{X_\gamma}^Y : X_\gamma \subset X,\quad X_\gamma$ is separable $\}$ have finite intersection property. Because that $B(Y^*)$ is w^*- compact, $L_X^Y = \{g \in B(Y^*),\ g \circ F \in B(X^*)\}$, so

$$\bigcap \{L_{X_\gamma}^Y : X_\gamma \subset X, X_\gamma\ is\ separable\} \neq \emptyset$$

and

$$L_X^Y = \bigcap \{L_{X_\gamma}^Y : X_\gamma \subset X, X_\gamma\ is\ separable\}.$$

In fact, if $g \in L_X^Y$, then $g \circ F \in X^*$, so $g \circ F \in X_\gamma^*$ for any subsets $X_\gamma \subset X$.

Conversely, if $g \in L_{X_\gamma}^Y$ for any γ, then $g \circ F$ is a continuous linear functional on any X_γ, thus for any $x_1, x_2 \in X$, we set $X_1 = \overline{span\{x_1, x_2\}}$. We easily prove that $g \circ F$ is a linear functional on X_1, furthermore, $g \circ F \in X^*$.

Next, we claim that

$$||F(x)|_{L_X^Y}|| = ||x||,\ \forall x \in X.$$

In fact, let $x \in X$, and set

$$\Xi' = \{X_\gamma : x \in X_\gamma, X_\gamma \ is\ separable\}.$$

Then Ξ' is a direction set which is non-empty, by (2.6) and (2.7) above Case 1

$$||U_{X_\gamma}(x)|| = ||x||.$$

For any $X_\gamma \in \Xi'$ there exists $g_\gamma \in L_{X_\gamma}^Y$ such that

$$|g_\gamma \circ F(x)| = ||x||.$$

So $\{g_\gamma : X_\gamma \in \Xi'\}$ is a net in $B(Y^*)$. Then there is a w^*- limit $g_0 \in B(Y^*)$. Hence for any V which is a w^*- neighborhood of g_0, there exists $X_\beta \in \Xi$, such that $X_\beta \supseteq X_\gamma$ and $g_\beta \in V$.

Now, we will prove that $g_0 \in L_X^Y$. In fact, for any $X_\gamma \in \Xi$, $g_0 \in L_{X_\gamma}^Y$. Otherwise, assume that $\widetilde{X_0} \in \Xi$, but $g_0 \notin L_{\widetilde{X_0}}^Y$. Because $L_{\widetilde{X_0}}^Y$ is w^*-closed set, then we can find a w^*-neighborhood V_0 of g_0 such that

$$V_0 \cap L_{\widetilde{X_0}}^Y = \emptyset.$$

Since g_0 is a w^*-limit of $\{g_\gamma : X_\gamma \in \Xi'\}$, then there exists $X_\gamma \in \Xi'$ such that $X_\gamma \supseteq \widetilde{X_0}$ and $g_\gamma \in V_0$, therefore $g_\gamma \in L_{X_\gamma}^Y \subset L_{\widetilde{X_0}}^Y$, which implies that $g_0 \in L_{\widetilde{X_0}}^Y$, leading to a contradiction. According to (2.6), (2.7), and $|g_0 \circ F(x)| = ||x||$, then $||F(x)|_{L_X^Y}|| = ||x||$, U is an isometry and $||g_0 \circ F|| = 1$. This completes the proof. □

3. The (Dopp) mapping

Yang ([13]) give that: *Let X, Y be normed linear spaces, B be a bounded set of X, and $0 \in intB$. If $f : X \to Y$ satisfies the Lipschitz condition, and $f|_{\partial B}$ (the boundary of B) is an isometry, then f is an isometry on B. Furthermore f can be extended to a positive homogeneous isometry $\widetilde{f}$ from X to Y.*

We instead some conditions and obtain the following results:

Theorem 3.1. *Let X and Y be real normed vector spaces such that $B_r(X)$ is a closed ball containing an original point 0, and its radius is r. If $f : B_r(X) \to Y, (2 \geq r > 1)$ is a Distance One Preserving Property (DOPP) satisfying the Lipschitz condition with $k = 1$, then*

(1) $f|_{B_{\frac{r-1}{2}}}$ is an isometry on $B_{\frac{r-1}{2}}(X)$.

(2) $f|_{B_{\frac{r-1}{2}}}$ *can be extended to a positive homogeneous isometry from* X *to* Y*.*

Proof. If $x, y \in B_{\frac{r-1}{2}}(X)$, then by the Lipschitz condition we have that

$$||f(x) - f(y)|| \leq ||x - y||.$$

Assume that $||f(x) - f(y)|| < ||x - y||$, let

$$z = x + \frac{1}{||x - y||}(y - x).$$

Then $||z|| \leq ||x|| + 1 \leq \frac{r-1}{2} + 1 = \frac{r+1}{2} < r$, so $z \in B_r(X)$ and $||y - x|| \leq r - 1 \leq 1$, $\quad ||z - x|| = 1, ||z - y|| = 1 - ||y - x|| \leq 1$. So by Lipschitz condition on $B_r(X)$, we have that

$$\begin{aligned} 1 = ||f(z) - f(x)|| &\leq ||f(y) - f(z)|| + ||f(x) - f(y)|| \\ &< 1 - ||x - y|| + ||x - y|| \\ &= 1. \end{aligned}$$

This contradicts hence that $||f(x) - f(y)|| = ||x - y||$ and f is an isometry from $B_{\frac{r-1}{2}}(X) \to Y$.

Let

$$\tilde{f}(x) = \begin{cases} \frac{||x||}{\frac{r-1}{2}} f(\frac{x}{||x||} \cdot (\frac{r-1}{2})), & x \neq 0 \\ 0, & x = 0 \end{cases}. \tag{3.1}$$

Then for $\lambda > 0$, by (2.8) we have that

$$\tilde{f}(\lambda x) = \frac{||\lambda x||}{\frac{(r-1)}{2}} f(\frac{\lambda x}{||\lambda x||} \cdot \frac{(r-1)}{2}) = \lambda \tilde{f}(x). \tag{3.2}$$

Thus, for any $x, y \in X$, there exists $\lambda > 0$ with $\lambda x, \lambda y \in B_{r-1}(X)$ such that

$$||\tilde{f}(\lambda x) - \tilde{f}(\lambda y)|| = ||\lambda x - \lambda y||.$$

(2.9) implying that

$$||\tilde{f}(x) - \tilde{f}(y)|| = ||x - y||.$$

□

Corollary 3.2. *Let* X, Y *be two normed spaces and let the mapping* $f : X \to Y$ *satisfy the (DOPP), if* $||f(x) - f(y)|| \leq ||x - y||$ *for* $x, y \in X$ *with* $||x - y|| \leq 1$*, then* f *is an isometry on* $B_{\frac{1}{2}}(X)$*. Furthermore* $f|_{B_{\frac{1}{2}}}(X)$ *can be extended to a positive homogeneous isometry* $\widetilde{f}$ *from* X *to* Y*.*

Proof. At first, we prove that if $||x-y|| \leq 1$, then $||f(x)-f(y)|| = ||x-y||$. Assume that

$$||f(x)-f(y)|| < ||x-y||.$$

Set

$$z = x + \frac{1}{||y-x||}(y-x).$$

Clearly,

$$||z-x|| = 1 \text{ and } ||z-y|| = 1 - ||x-y|| \leq 1.$$

It follows that

$$||f(z)-f(x)|| = 1 \text{ and } ||f(z)-f(y)|| \leq 1 - ||x-y||.$$

Besides,

$$\begin{aligned} ||f(z)-f(x)|| &\leq ||f(z)-f(y)|| + ||f(y)-f(x)|| \\ &< 1 - ||x-y|| + ||x-y|| \\ &= 1, \end{aligned}$$

which is a contradiction $||f(z)-f(x)|| = 1$. Hence we have

$$||f(x)-f(y)|| = ||x-y||.$$

Second, it easy to see that $||x-y|| \leq 1$ for all $x, y \in B_{\frac{1}{2}}(X)$. Then f is an isometry on $B_{\frac{1}{2}}(X)$, and f can be extended to a positive homogeneous isometry $\widetilde{f}$ from X to Y by Theorem 3.1. □

Benz ([15]) gave the following result. *Let X, Y be real normed linear spaces such that $dimX > 2$ and Y is strictly convex. Suppose that $p > 0$ is a fixed real number and that $N > 1$ is a fixed integer. Finally, let $f : X \to Y$ be a mapping such that.for all $x, y \in X$ $||x-y|| = p \Rightarrow ||f(x)-f(y)|| \leq p$, and $||x-y|| = Np \Rightarrow ||f(x)-f(y)|| \geq Np$. Then f is an affine isometry.*

In this section we do not assume the above condition of "strictly convex and $dimX \geq 2$," and show the following refined theorem.

Theorem 3.3. *Let X, Y be two normed spaces, and the mapping f : $X \to Y$ satisfying the (DOPP), $||f(x)-f(y)|| \leq ||x-y||$ for $x, y \in X$ with $||x-y|| \leq 1$ and distance n preserving property($n > 0$ is any integer, i.e. $||f(x)-f(y)|| = n$ with $||x-y|| = n$). Then f is an isometry from X to Y.*

Proof. First, we prove that if $||x-y|| \leq 1$, then $||f(x)-f(y)|| = ||x-y||$. Assume that

$$||f(x)-f(y)|| < ||x-y||.$$

Set

$$z = x + \frac{1}{||y-x||}(y-x).$$

Clearly,

$$||z-x|| = 1 \; and \; ||z-y|| = 1 - ||x-y|| \leq 1.$$

Due to the (DOPP), it follows that

$$\begin{aligned} 1 &= ||f(z) - f(x)|| \\ &\leq ||f(z) - f(y)|| + ||f(x) - f(y)|| \\ &< 1 - ||x-y|| + ||x-y|| \\ &= 1. \end{aligned}$$

Hence $||f(x) - f(y)|| = ||x-y||$.

Second, for any $x, y \in X$ with $||x-y|| > 1$, there exists an integer n with $n < ||x-y|| \leq n+1$,

$$z = x + n\frac{1}{||y-x||}(y-x).$$

Then $||z-x|| = n, ||z-y|| = ||x-y|| - n < 1$, since f satisfies (DOPP). It follows that $||f(z) - f(x)|| = n$ and $||f(z) - f(y)|| = ||z-y||$ by the above proof. This implies that

$$||f(z) - f(x)|| + ||f(z) - f(y)|| = n + ||x-y|| - n = ||x-y||,$$

then

$$||f(x) - f(y)|| \leq ||x-y||.$$

Next we'll prove that

$$||f(x) - f(y)|| = ||x-y||. \tag{3.3}$$

Suppose that

$$||f(x) - f(y)|| < ||x-y||.$$

Let

$$z_1 = x + (n+1)\frac{1}{||y-x||}(y-x).$$

Thus $||z_1 - x|| = n+1, \; ||z_1 - y|| = n+1 - ||x-y|| \leq 1$ and

$$\begin{aligned} n+1 &= ||f(z_1) - f(x)|| \\ &\leq ||f(z_1) - f(y)|| + ||f(y) - f(x)|| \\ &< n+1 - ||x-y|| + ||x-y|| \\ &= n+1. \end{aligned}$$

This is a contradiction and we complete the proof of (3.3)

$$||f(x) - f(y)|| = ||x-y||.$$

□

Remark 3.4. The above Theorems ture for Lipschitz condition $||f(x) - f(y)|| \leq ||x - y||$ for any $x, y \in X$.

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