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VECTOR VARIATIONAL-LIKE INEQUALITIES WITH PSEUDO SEMI-MONOTONE MAPPINGS

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Abstract. In this paper, concepts of pseudo semi-monotone and η-pseudo semi-monotone mappings are introduced. These concepts are applied to prove the solvability for the class of vector variational like-inequalities by using Kakutani-Fan-Glicksberg's fixed point theorem. The results presented in this paper generalize some known results for vector variational inequalities in recent years. almost everywhere.

1. INTRODUCTION

Vector variational inequalities were initially introduced and considered by Giannessi [5] in a finite-dimensional Euclidean space in 1980. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on vector variational inequalities have been investigated in abstract spaces, *e.g.* [3], [4], [10]. For the past years, vector variational inequalities and their generalizations have been studied and applied in various directions; For details, we refer to [2]–[4], [6]– [16] and references therein. It is known that monotonicity and the compactness operators are two important concepts in nonlinear functional analysis and its applications. It was Browder [1] who first combined the compactness and accretion of operators and posed the concept of a semi-accretive operator. Recently, motivated by this idea, Chen [2] posed the concept of semi-monotone operator, which combines

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the compactness and monotonicity of an operator and applied it to the study of variational inequalities. Recently in 2005, Zheng [16] introduced vector semi-monotone mapping, which generalizes the concept of semi monotonicity and he studied several existence results for the vector variational inequality problem related to concept of vector semi-monotone mapping. Very recently in 2010, Khan [8] introduced the generalized vector variational inequality-type problem and the generalized vector complementarity-type problem in the setting of topological vector spaces and proved that the solution sets of both problems are equivalent to each other under some suitable conditions.

Motivated and inspired by Chen [2], Khan [8] and Zheng [16], in this paper, we introduced concept of η-pseudo semi-monotone mapping and investigated the solvability of vector variational-like inequalities involving η-pseudo semi-monotone mapping by means of the Kakutani-Fan-Glicksberg fixed-point theorem. The results presented in this paper extend and unify corresponding results of Chen [2], Zheng [16] and enrich the theory of variational inequalities.

2. PRELIMINARIES

Throughout the paper unless otherwise specified, let X and Y be two real Banach spaces, $K \subset X$ be a nonempty closed and convex subset of X. Recall that $P \subset Y$, is said to be a closed convex cone, if P is closed and $P + P = P$, $\lambda P \subset P$ for all $\lambda > 0$. In addition, if $P \neq Y$, then P is called a proper closed convex cone. A closed, convex cone is pointed if $P \cap (-P) = \{0\}$.

The partial order $\leq_P$ in Y, induced by the pointed cone P, is defined by declaring $x \leq_P y$ if and only if $y - x \in P$ for all x, y in Y. An ordered Banach space is a pair (Y, P) with the partial order induced by P. The weak order $\leq_{\operatorname{int} P}$ in an ordered Banach space (Y, P) with $\operatorname{int} P \neq \emptyset$ is defined as $x \leq_{\operatorname{int} P} y$ if and only if $y - x \in \operatorname{int} P$ for all x, y in Y, where $\operatorname{int} P$ denotes the interior of P. Let $L(X, Y)$ denote the space of all continuous linear mappings from X into Y. Let $P : K \to 2^Y$ be such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$ and let $P_- = \bigcap_{x \in K} P(x)$.

We recall the following concepts and results which are needed in the sequel.

Definition 2.1 ([15], Definition 2.3)**.** *Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let T :*

$K \to L(X,Y)$ and $\eta : K \times K \to X$ be two mappings. T is said to be η-pseudomonotone, if for any $x, y \in K$

$$\langle T(x), \eta(y,x)\rangle \geq_{P_-} 0 \Longrightarrow \langle T(y), \eta(x,y)\rangle \leq_{P_-} 0, \text{ where } P_- = \bigcap_{x\in K} P(x).$$

Remark that, if $\eta(y,x) = y - x$, $\forall x, y \in K$, then η-pseudo monotonicity of T reduces to pseudo monotonicity of T.

Definition 2.2. *A mapping $f : K \to Y$ is said to be*

(i) *P_--convex, if $f(tx + (1-t)y) \leq_{P_-} tf(x) + (1-t)f(y)$, $\forall x, y \in K$, $t \in [0,1]$;*

(ii) *P_--concave, if $-f$ is P_--convex.*

Definition 2.3. *A mapping $\eta : K \times K \to X$ is said to be affine in first argument, if for any $x_i \in K$ and $\lambda_i \geq 0$, $(1 \leq i \leq n)$, with $\sum_{i=1}^{n} \lambda_i = 1$ and any $y \in K$,*

$$\eta(\sum_{i=1}^{n} \lambda_i x_i, y) = \sum_{i=1}^{n} \lambda_i \eta(x_i, y).$$

Definition 2.4. *Let $T : X \to 2^Y$ be a set-valued mapping. Then T is said to be lower semicontinuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_0 \in T(x_0)$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y_0$. T is called lower semicontinuous on X if it is lower semicontinuous at each point of X.*

Definition 2.5. *A mapping $T : K \to L(X,Y)$ is said to be η-hemicontinuous, if for any $x, y \in K$, the mapping $t \to \langle Tx + t(y-x), \eta(y,x)\rangle$ is continuous at 0^+.*

The next two lemmas will be needed in the proof of the main results of this paper.

Lemma 2.6. *Let $K \subset X$ be a nonempty, closed, and convex subset of X. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Suppose following conditions hold:*

(i) *$\eta : K \times K \to X$ is an affine mapping in first argument, with the condition $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in K$;*

(ii) *The set-valued mapping* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus \{-int\, P(x)\}$, $\forall x \in K$, *such that graph of* W *is weakly closed in* $X \times Y$;

(iii) $T : K \to L(X, Y)$ *is* η*-hemicontinuous and* η*-pseudomonotone mapping.*

Then following two problems are equivalent:

(A) $x \in K,\ \langle Tx, \eta(y, x)\rangle \not\leq_{\text{int}\, P(x)} 0,\ \forall y \in K,$

(B) $x \in K,\ \langle Ty, \eta(y, x)\rangle \not\leq_{\text{int}\, P(x)} 0,\ \forall y \in K.$

Lemma 2.7. *Let* X *be real reflexive Banach space and* Y *be a Banach space. Let* $K \subset X$ *be a nonempty, bounded, closed and convex subset of* X*. Let* $P : K \to 2^Y$ *be such that for each* $x \in K, P(x)$ *is a proper, closed, convex cone with int* $P(x) \neq \emptyset$*. Suppose following conditions hold:*

(i) $\eta : K \times K \to X$ *is affine mapping in first argument with the condition* $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in K$ *and lower semicontinuous mapping in second argument;*

(ii) *The set-valued mapping* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus \{-int\, P(x)\}$, $\forall x \in K$, *such that graph of* W *is weakly closed in* $X \times Y$;

(iii) $T : K \to L(X, Y)$ *is* η*-hemicontinuous and* η*-pseudomonotone mapping.*

Then there exist $x \in K$*, such that*

$$\langle Tx, \eta(y, x)\rangle \not\leq_{int\, P(x)} 0,\ \forall y \in K.$$

3. Main Results

We now give the concepts of pseudo semi-monotone and η-pseudo semi-monotone mappings.

Definition 3.1. *Let* $K \subset X$ *be a nonempty, closed and convex subset of* X*. Let* $P : K \to 2^Y$ *be such that for each* $x \in K, P(x)$ *is a proper, closed, convex cone with int* $P(x) \neq \emptyset$*. Let* $\eta : K \times K \to X$ *be a mapping. A mapping* $A : K \times K \to L(X, Y)$ *is said to be* η*-pseudo semi-monotone mapping, if following conditions hold:*

(i) *for every* $u \in K$, $A(u, .)$ *is* η*-pseudomonotone mapping; i.e.*

$$\langle A(u, x), \eta(y, x)\rangle \geq_{P_-} 0 \implies \langle A(u, y), \eta(x, y)\rangle \leq_{P_-} 0,\ \forall x, y \in K;$$

(ii) *for every* $y \in K$, $A(., y)$ *is completely continuous, i.e., when* $u_n \to^w u$, $A(u_n, y) \to A(u, y)$ *(by the norm of operators), where* $\to^w$ *denotes the weak convergence.*

Definition 3.2. *If we take* $\eta(y,x) = y - x$, $\forall x, y \in K$, *then* $A : K \times K \to L(X,Y)$ *is said to be pseudo semi-monotone mapping.*

Example 3.3. *Let* $X=\mathbb{R}$, $K=\mathbb{R}_+$, $Y = \mathbb{R}^2$, $P_- = \mathbb{R}^2_+$. *Let* $\eta : K \times K \to X$ *is defined by* $\eta(y,x) = y - (\sqrt{x}+1)^2$, *for all* $x, y \in K$ *and let* $A : K \times K \to L(X,Y)$ *be defined by*

$$A(x,y) = \begin{pmatrix} 3 + \sin\, x + \cos\, y \\ 3 + \cos\, x + \sin\, y \end{pmatrix}, \ \forall x, y \in K.$$

Also, the norm of A *is defined as* $\|A\| = |3 + \sin\, x + \cos\, y| + |3 + \cos\, x + \sin\, y|$, $\forall x, y \in K$.

First, we show that A *is* η*-pseudomonotone mapping. Indeed, for each* $u, v \in K$

$$\langle A(y,u), \eta(v,u)\rangle = \begin{pmatrix} 3 + \sin\, y + \cos\, u \\ 3 + \cos\, y + \sin\, u \end{pmatrix} \begin{pmatrix} v - (\sqrt{u}+1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (3 + \sin\, y + \cos\, u)(v - (\sqrt{u}+1)^2) \\ (3 + \cos\, y + \sin\, u)(v - (\sqrt{u}+1)^2) \end{pmatrix} \geq_{P_-} 0.$$

The inequality implies that $v \geq (\sqrt{u}+1)^2 > u$. *It follows that*

$$\langle A(y,v), \eta(u,v)\rangle = \begin{pmatrix} 3 + \sin\, y + \cos\, v \\ 3 + \cos\, y + \sin\, v \end{pmatrix} \begin{pmatrix} u - (\sqrt{v}+1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (3 + \sin\, y + \cos\, v)(u - (\sqrt{v}+1)^2) \\ (3 + \cos\, y + \sin\, v)(u - (\sqrt{v}+1)^2) \end{pmatrix} \leq_{P_-} 0.$$

This shows that $A(y,.)$ *is* η*-pseudomonotone.*
Now, for fixed $v \in K$, *if* $u_n \in K$, $u \in K$, $u_n \to^w u$, *it is easy to prove that*

$$\|A(u_n, v) - A(u,v)\| \to 0.$$

Hence, for every $v \in K$, $A(.,v)$ *is completely continuous. Therefore,* A *is* η*-pseudo semi-monotone mapping.*

Now we will pose the main problem of our study. In this paper, we investigate the following *vector variational-like inequality problem* (for short, VVLIP) is to find a vector $u \in K$ satisfying

$$\langle A(u,u), \eta(v,u)\rangle \not\leq_{\mathrm{int}\, P(u)} 0, \ \forall v \in K.$$

where $A : K \times K \to L(X,Y)$ be a nonlinear mapping and $\eta : K \times K \to X$ is a vector-valued bi-mapping.

We recall the following fixed point theorem, by *Kakutani-Fan-Glicksberg* [14], which will play an important role in establishing our existing results for VVLIP.

Theorem 3.4 ([14])**.** *Suppose that X is a Hausdorff locally convex space and K is a nonempty convex compact subset of X. If $F : K \to K$ is an upper semi-continuous mapping with nonempty convex closed values, then F has a fixed point in K, i.e., there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

Now, by using *Kakutani-Fan-Glicksberg* fixed point theorem[14] we prove the existence result for VVLIP .

Theorem 3.5. *Let X be real reflexive Banach space and Y be a Banach space. Let $K \subset X$ be a nonempty, bounded, closed and convex subset of X. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Suppose following conditions hold:*

(i) *$\eta : K \times K \to X$ is an affine mapping with the condition $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in K$ and lower semi-continuous mapping in second argument;*
(ii) *The set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus \{-int\, P(x)\}$, $\forall x \in K$, such that graph of W is weakly closed in $X \times Y$ and concave;*
(iii) *$A : K \times K \to L(X, Y)$ is η-pseudo semi-monotone mapping;*
(iv) *For each fixed $v \in K$, $A(v, .) : K \times K \to L(X, Y)$ is continuous on each finite dimensional subspace of X.*

Then VVLIP has a solution in K.

Proof. Let F be a finite dimensional subspace of X and $K_F = K \bigcap F \neq \emptyset$. For each $v \in K_F$, we consider the following vector variational-like inequality problem: Find $u_0 \in K_F$ such that

$$\langle A(v, u_0), \eta(u, u_0)\rangle \nleq_{\text{int}\, P(u_0)} 0,\ \forall u \in K_F.$$

Since $K_F \subset F$ is bounded, closed and convex, $A(v, .)$ is continuous on K_F and η-pseudomonotone for each fixed $v \in K$. From Lemma 2.7, we know our problem has solution $u_0 \in K_F$.

Define a set-valued mapping $T : K_F \to 2^{K_F}$ as follows:

$$F(v) = \{w \in K_F : \langle A(v, w), \eta(u, w)\rangle \nleq_{\text{int}\, P(w)} 0,\ \forall u \in K_F\}$$

It follows from Lemma 2.6, that for each fixed $v \in K_F$

$$\{w \in K_F : \langle A(v, w), \eta(u, w)\rangle \nleq_{\text{int}\, P(w)} 0,\ \forall u \in K_F\}$$
$$= \{w \in K_F : \langle A(v, u), \eta(u, w)\rangle \nleq_{\text{int}\, P(w)} 0,\ \forall u \in K_F\}$$

Now we shall use the fixed point theorem to verify the existence of solution of problem in a finite dimensional. Since F is of finite dimensional, hence K_F is compact. First, we claim that

$$F(v) = \{w \in K_F : \langle A(v,u), \eta(u,w)\rangle \ \in Y\backslash\{-\text{int } P(w)\} = W(w)\}$$

is convex. Indeed, let $w_1, w_2 \in F(v)$ and $m, n \geq 0$, such that $m + n = 1$

$$m[\langle A(v,u), \eta(u,w_1)\rangle] = mW(w_1)$$

$$n[\langle A(v,u), \eta(u,w_2)\rangle] = mW(w_2)$$

Since η is affine mapping, then from preceeding two inclusions we have

$$\langle A(v,u), \eta(u, mw_1 + nw_2)\rangle \in mW(w_1) + nW(w_2)$$

Since W is concave, we have $mw_1 + nw_2 \in F(v)$, i.e., $F(v)$ is convex and our claim is then verified. Now, we claim that $F(v)$ is closed. Let $w_j \in F(v)$ such that $w_j \to w$, then

$$\langle A(v,u), \eta(u,w_j)\rangle \in Y\backslash\{-\text{int } P(w_j)\} \in W(w_j)$$

Since $A(v,u) \in L(X,Y)$ and W is weakly closed, therefore

$$\langle A(v,u), \eta(u,w_j)\rangle \to \langle A(v,u), \eta(u,w)\rangle \in W(w)$$

This implies $w \in F(v)$, hence $F(v)$ is closed. Next, we claim that F is upper semi-continuous. Let $v_j \to v$ and $w_j \to F(v_j),\ w_j \to w$, then we have

$$\langle A(v_j,u), \eta(u,w_j)\rangle \in W(w_j)$$

From the complete continuity of $A(.,u)$ and lower semicontinuity of $\eta(u,.)$, we have

$$\langle A(v_j,u), \eta(u,w_j)\rangle \to \langle A(v,u), \eta(u,w)\rangle$$

Also W is weakly closed, which implies that

$$\langle A(v,u), \eta(u,w)\rangle \in W(w)$$

i.e., $w \in F(v)$, thus our claim is then verified. Hence F is upper semicontinuos. By Fan-Glicksberg fixed point theorem, there exists a $v_0 \in F(v_0)$ i.e., there exists a $v_0 \in K_F$ such that

$$\langle A(v_0,v_0), \eta(u,v_0)\rangle \not\leq_{\text{int } P(v_0)} \ 0,\ \forall u \in K_F.$$

Now we generalize this result to whole space.

Let $\Omega \equiv \{F \subset X :\ F$ is finite dimensional, $F \bigcap K \neq \emptyset\}$ and let

$$\Gamma_F \equiv \big\{w \in K : \langle A(w,u), \eta(u,w)\rangle \not\leq_{\text{int } P(w)} \ 0,\ \forall u \in K_F\},\ F \in \Omega.$$

From above we know that $\forall F \in \Omega,\ \Gamma_F \neq \emptyset$. Let $\overline{\Gamma_F}^w$ denotes the weak closure of Γ_F. For any $F_i \in \Omega,\ i = 1, ..., n$, we know that $\Gamma_{\bigcup_{i=1}^n F_i} \subseteq \bigcap_{i=1}^n \Gamma_{F_i} \subseteq$

$\bigcap_{i=1}^{n} \overline{\Gamma_{F_i}}^{w}$. Therefore, $\bigcap_{i=1}^{n} \overline{\Gamma_{F_i}}^{w} \neq \emptyset$. Since K is weakly compact, from the finite intersection property, we have $\bigcap_{F\in\Omega} \overline{\Gamma_F}^{w} \neq \emptyset$. Let $w_0 \in \bigcap_{F\in\Omega} \overline{\Gamma_F}^{w}$, we see that

$$\langle A(w_0,w_0),\eta(u,w_0)\rangle \not\leq_{\operatorname{int} P(w_0)} 0.$$

Indeed, for each $u \in K$, let $F \in \Omega$, such that $u \in K_F$, $w_0 \in K_F$. From $w_0 \in \overline{\Gamma_F}^{w}$, there exists $w_j \in \Gamma_F$ i.e.,

$$\langle A(w_j,u),\eta(u,w_j)\rangle \in Y\backslash\{-\mathrm{int}P(w_j)\} \in W(w_j),$$

such that $w_j \rightarrow^{w} w_0$, from the complete continuity of $A(.,v)$, lower semicontinuity of $\eta(u,.)$ and weakly closedness of W, we have

$$\langle A(w_0,u),\eta(u,w_0)\rangle \in W(w_0)$$

From Lemma 2.6, we have

$$\langle A(w_0,w_0),\eta(u,w_0)\rangle \not\leq_{\operatorname{int} P(w_0)} 0,\ \forall u \in K.$$

This completes the proof. □

If the boundedness of K is dropped off, then we have the following theorem under certain coercivity condition:

Theorem 3.6. *Let X be real reflexive Banach space and Y be a Banach space. Let $K \subset X$ be a nonempty, unbounded, closed and convex subset of X and $0 \in K$. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Suppose following conditions hold:*

- (i) *$\eta : K \times K \to X$ is an affine mapping with the condition $\eta(x,y) + \eta(y,x) = 0$, $\forall x,y \in K$ and lower semi-continuous mapping in second argument;*
- (ii) *The set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y\backslash\{-int\, P(x)\}$, $\forall x \in K$, such that graph of W is weakly closed in $X \times Y$ and concave;*
- (iii) *$A : K \times K \to L(X,Y)$ is η-pseudo semi-monotone mapping;*
- (iv) *For each fixed $v \in K$, $A(v,.) : K \times K \to L(X,Y)$ is continuous on each finite dimensional subspace of X.*
- (v) $\lim_{\|u\|\to\infty} \langle A(u,u),\eta(u,0)\rangle \geq_{int P_-} 0$, *where* $P_- = \bigcap_{u\in K} P(u)$.

Then VVLIP has a solution in K.

Proof. For each $r > 0$, let $B[0,r]$ denote the closed ball in the Banach space X with center 0 and radius r. By Theorem 3.5, for each $r \in \mathbb{N}$, there exists $u_r \in B[0,r] \cap K$ such that

$$\langle A(u_r,u_r),\eta(v,u_r)\rangle \not\leq_{\operatorname{int} P(u_r)} 0,\ \forall v \in B[0,r] \cap K.$$

Since $0 \in K$, we have

$$\langle A(u_r, u_r), \eta(0, u_r)\rangle \not\leq_{\text{int}\, P(u_r)} 0.$$

From condition (i), above inclusion implies that,

$$\langle A(u_r, u_r), \eta(u_r, 0)\rangle \not\leq_{-\text{int}\, P(u_r)} 0, \ \forall v \in K \cap B_r.$$

We say that $\{u_r\}_{r\in\mathbb{N}}$ is bounded. If not, without loss of generality, we assume that $\|u_r\| \to \infty$, when $r \to \infty$. Now, from condition (v), we have

$$\lim_{n\to\infty}\langle A(u_r, u_r), \eta(u_r, 0)\rangle \geq_{\text{int}\, P_-} 0,$$

From this, we know that when r is sufficiently large,

$$\langle A(u_r, u_r), \eta(u_r, 0)\rangle \geq_{\text{int}\, P_-} 0,$$

This is a contradiction. Since X is reflexive, we may suppose that u_r converges to u as $r \to \infty$. Since $A(., v)$, $\eta(v, .)$ are completely continuous and lower semi-continuous, respectively, also by the weak closedness of W, it follows that

$$\langle A(u, v), \eta(v, u)\rangle \not\leq_{\text{int}\, P(u)} 0.$$

Again from Lemma 2.6, we get

$$\langle A(u, u), \eta(v, u)\rangle \not\leq_{\text{int}\, P(u)} 0, \ \forall v \in K.$$

This completes the proof. □

Remark 3.7. *Theorem 3.5 and Theorem 3.6 improve and generalize Theorem 3.1 and Theorem 3.2 of Zheng* [16] *and Theorem 2.1 to Theorem 2.6 of Chen* [2].

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