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FAN'S MINIMAX INEQUALITY IN MG-CONVEX SPACES

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Abstract. At the present paper, by using Fan-KKM principle in minimal generalized convex spaces, a minimax inequality of Ky Fan is proved. Also, Fan's well known two-functions minimax inequality in this new setting is given.

1. Introduction

It is known that the famous Fan-KKM principle and Fan's minimax inequality [15], have played very important roles in the study of the modern nonlinear analysis. The Fan minimax inequality has become one of the most applicable tools in mathematical economics which, its relation with many concepts in nonlinear analysis is well known. Moreover, a great deal of effort has gone into the theory and applications of the Fan-KKM theorem and Fan's minimax inequality [8, 10, 14, 34]. Some general minimax theorems and several extensions of these inequalities have been obtained in convex spaces for functions with KKM property under weaker various conditions [18, 19, 21, 31, 33, 35, 36]. Recently, these inequalities are considered in generalized convex (G-convex) spaces (generalization of many famous spaces with convex structure) which

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improved and generalized a large number of Ky Fan's type minimax inequalities [9, 13, 20, 22, 32]. In this work, we use the concept of minimal generalized convex spaces and some new result about compactness and product spaces in minimal spaces to establish generalization of the above mentioned results. We establish two minimax inequalities in minimal generalized convex spaces. The first one originally goes back to Fan [15]. The second is famous two-function minimax inequality also due to Fan in [16], which improved the Von Neumann-Sion minimax principle [30] in minimal generalized convex spaces.

The concept of minimal structures and minimal spaces, as a generalization of topology and topological spaces were introduced in [25]. For easy understanding of the material incorporated in this paper we recall some basic definitions and results. For details on the following notions we refer to [2, 5, 6, 7, 11, 24, 25] and [29] and references therein.

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a *minimal space*. For some examples in this setting see [24]. In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. We set $m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\text{-Cl}(A) = \bigcap\{F : A \subseteq F, F^c \in \mathcal{M}\}$. We say, the minimal space $(X, \mathcal{M})$ enjoys the *property* I, if any finite intersection of m-open sets is m-open. For any $x \in X$, $N(x)$ is said to be a *minimal neighborhood* of x, if for any $z \in N(x)$ there is an m-open subset $G_z \subseteq N(x)$ such that $z \in G_z$.

Proposition 1.1. ([24]) *For any two sets A and B,*

(a) $m\text{-Int}(A) \subseteq A$ *and* $m\text{-Int}(A) = A$ *if A is an m-open set.*

(b) $A \subseteq m\text{-Cl}(A)$ *and* $A = m\text{-Cl}(A)$ *if A is an m-closed set.*

(c) $m\text{-Int}(A) \subseteq m\text{-Int}(B)$ *and* $m\text{- Cl}(A) \subseteq m\text{-Cl}(B)$ *if* $A \subseteq B$.

(d) $m\text{-Int}(A \cap B) \subseteq (m\text{-Int}(A)) \cap (m\text{-Int}(B))$ *and* $(m\text{-Int}(A)) \cup (m\text{-Int}(B)) \subseteq m\text{-Int}(A \cup B)$.

(e) $m\text{-Cl}(A \cup B) \supseteq (m\text{-Cl}(A)) \cup (m\text{-Cl}(B))$ *and* $m\text{-Cl}(A \cap B) \subseteq (m\text{-Cl}(A)) \cap (m\text{-Cl}(B))$.

(f) $m\text{-Int}(m\text{-Int}(A)) = m\text{-Int}(A)$ *and* $m\text{-Cl}(m\text{-Cl}(B)) = m\text{-Cl}(B)$.

(g) $(m\text{-Cl}(A))^c = m\text{-Int}(A^c)$ *and* $(m\text{-Int}(A))^c = m\text{-Cl}(A^c)$.

Definition 1.2. ([29]) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called *minimal continuous* (briefly m-continuous) if $f^{-1}(U) \in \mathcal{M}$ for any $U \in \mathcal{N}$.

Definition 1.3. ([2]) Suppose (X, τ) is a topological space and also suppose $(Y, \mathcal{N})$ is a minimal space. A function $f : (X, \tau) \to (Y, \mathcal{N})$ is called *(τ, m)-continuous* if $f^{-1}(U) \in \tau$ for any $U \in \mathcal{N}$.

Similar to topological spaces some basic concepts can be obtained in the minimal spaces. In the following it is shown that the minimal product of any m-compact minimal spaces is m-compact; that is the Thychonoff's compactness Theorem holds for the product of minimal spaces.

Definition 1.4. ([29]) For a minimal space $(X, \mathcal{M})$,

(a) a family $\mathcal{A} = \{A_j : j \in J\}$ of m-open sets in X is called an *m-open cover* of K if $K \subseteq \bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also an m-open cover of K is called a *subcover* of $\mathcal{A}$ for K;

(b) a subset K of X is *m-compact* whenever any given m-open cover of K, has a finite subcover.

Theorem 1.5. ([29]) *Suppose that X and Y are two minimal spaces and $f : X \to Y$ is an m-continuous function. For any m-compact subset $K \subseteq X$, $f(K)$ is m-compact in Y.*

In [7] authors achieved product minimal structure for an arbitrary family $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in I\}$ of minimal spaces. Product minimal structure on $X = \prod_{\alpha\in I} X_\alpha$ is the weakest minimal structure on X (denoted by $\mathcal{M} = \prod_{\alpha\in I} \mathcal{M}_\alpha$), such that for each $\beta \in I$ the canonical projection $\pi_\beta : \prod_{\alpha\in I} X_\alpha \longrightarrow X_\beta$ is m-continuous. In fact, $\prod_{\alpha\in I} \mathcal{M}_\alpha = \{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{M}_\alpha\}$.

Theorem 1.6. *The product minimal space $(\prod_{\alpha\in I} X_\alpha, \prod_{\alpha\in I} \mathcal{M}_\alpha)$ is m-compact if and only if $(X_\alpha, \mathcal{M}_\alpha)$ is an m-compact minimal space, for any $\alpha \in I$.*

Proof. One direction is an immediate consequence of Theorem 1.5. For the converse, on the contrary suppose that $\mathcal{A} \subseteq \prod_{\alpha\in I} \mathcal{M}_\alpha$, is an m open cover of $\prod_{\alpha\in I} X_\alpha$ without any finite subcover include $\prod_{\alpha\in I} X_\alpha$. For any $\alpha \in I$, set $\mathcal{U}_\alpha = \{V \in \mathcal{M}_\alpha : \pi_\alpha^{-1}(V) \in \mathcal{A}\}$. Since $\mathcal{A}$ has no finite subcover for $\prod_{\alpha\in I} X_\alpha$, so no finite subcover of $\mathcal{U}_\alpha$ can cover X_α, for any $\alpha \in I$. m-compactness of X_α implies that $\mathcal{U}_\alpha$ can not cover X_α. Therefore there exists $x_\alpha \in X_\alpha \setminus \bigcup\{V : V \in \mathcal{U}_\alpha\}$, for any $\alpha \in I$. Set $x = (x_\alpha)_{\alpha\in I}$. Then $x \in \prod_{\alpha\in I} X_\alpha \setminus \bigcup\{A : A \in \mathcal{A}\}$, which implies that $\mathcal{A}$ is not an m-open cover for $\prod_{\alpha\in I} X_\alpha$, a contradiction. □

2. Minimax inequality and KKM maps

Recently, authors in [1, 2] and [4] introduced and investigated the notion of minimal generalized convex space as an extended version of generalized convex space. Suppose X and Y are two minimal spaces. A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the values $F(x) \subseteq Y$ for all $x \in X$. Given $A \subseteq X$, set $F(A) = \bigcup_{x \in A} F(x)$. Define $F^{-}(y) = \{x \in X : y \in F(x)\}$, for all $y \in Y$. The multimap F is said to be *minimal transfer open* [3] if for any $x \in X$ and $y \in F(x)$ there exists an $x_0 \in X$ such that $y \in m\text{-Int}(F(x_0))$. For a such multimap F, a point $x_0 \in X$ is called *fixed point* if, $x_0 \in F(x_0)$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D and let Δ_n be the n-simplex with vertices $e_0, e_1, \cdots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A \rangle$, where $A \in \langle D \rangle$; for example, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subseteq A$, then $\Delta_J = co\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$, where $co(A)$ denotes the convex hull of A. A *minimal generalized convex space* (briefly MG-convex space) (X, D, Γ) consists of a minimal space $(X, \mathcal{M})$, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ in which for $A \in \langle D \rangle$ with $n+1$ elements, there exists a (τ, m)-continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ for which $J \in \langle A \rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$. In case to emphasize $X \supseteq D$, (X, D, Γ) will be denoted by $(X \supseteq D, \Gamma)$; and if $X = D$, then $(X \supseteq X; \Gamma)$ by (X, Γ). For any MG-convex space (X, D, Γ), the multimap $F : D \multimap X$ is called a *KKM multimap* if $\Gamma_A \subseteq F(A)$ for any $A \in \langle D \rangle$. Clearly, any G-convex space is an MG-convex space. On the other hand, suppose $(X, \mathcal{M})$ is a minimal vector space which is not a topological vector space (for example see [2]). Consider the multimap $\Gamma : \langle X \rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \cdots, a_n\}) = \{\sum_{i=0}^{n} \lambda_i a_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1\}$. One can deduce that (X, Γ) is a minimal generalized convex space, but it is not a generalized convex space [2].

Theorem 2.1. ([4]) *Suppose (X, D, Γ) is an m-compact MG-convex space, Y is a minimal space, $S : X \multimap D$, $F : X \multimap Y$ and $T : X \multimap X$ are multimaps such that:*

(a) *$x \in X$ and $M \in \langle S(x) \rangle$ imply that $\Gamma_M \subseteq T(x)$,*

(b) *$F^{-} : Y \multimap X$ is minimal transfer open and $F(x)$ is nonempty,*

(c) *for any $y \in Y$ there exists $z \in D$ such that $F^{-}(y) \subseteq S^{-}(z)$.*

Then T has a fixed point.

Definition 2.2. Suppose that X is a nonempty set, Y is a minimal space and $f : X \times Y \longrightarrow \mathbb{R}$. Then f is said to be

(a) *minimal upper semicontinuous in the second variable*, if for each $\gamma \in \mathbb{R}$, $x \in X$ and $y \in Y$ with $f(x, y) < \gamma$, imply that there exist $x_0 \in X$ and a minimal neighborhood $N(y)$ containing y in Y such that $f(x_0, u) < \gamma$ for any $u \in N(y)$.

(b) *minimal lower semicontinuous in the second variable*, if for each $\gamma \in \mathbb{R}$, $x \in X$ and $y \in Y$ with $f(x,y) > \gamma$, imply that there exist $x_0 \in X$ and a minimal neighborhood $N(y)$ containing y in Y such that $f(x_0,u) > \gamma$ for any $u \in N(y)$.

Lemma 2.3. *Suppose that X is a nonempty set, Y is a minimal space and $f : X \times Y \to \mathbb{R}$ is a function. Then f is minimal upper semicontinuous in the first variable if and only if the multimap $F : Y \multimap X$ defined by $F(y) = \{x \in X : f(x,y) < \gamma\}$ is minimal transfer open.*

Proof. Let f be minimal upper semicontinuous in the first variable, $(y,x) \in Y \times X$ and $x \in F(y)$ (i.e., $f(x,y) < \gamma$). Then there is y_0 and a minimal neighborhood $N(x) \subseteq X$ containing x such that $f(y_0,u) < \gamma$ for all $u \in N(x)$. So $x \in N(x) \subseteq F(y_0)$ and hence $x \in m\text{-Int}(N(x)) \subseteq m\text{-Int}(F(y_0))$. Therefore F is minimal transfer open.
Conversely, suppose that $(y,x) \in Y \times X$ and $f(x,y) < \gamma$. So $x \in F(y)$ and by the hypothesis, there exists $\bar{y} \in Y$ such that $x \in m\text{-Int}(F(\bar{y})) \subseteq F(\bar{y})$. Set $m\text{-Int}(F(\bar{y})) = N(x)$, which is a minimal neighborhood containing x. Then $f(\bar{y},u) > \gamma$ for all $u \in N(x)$. □

Definition 2.4. Let (X,D,Γ) be an MG-convex space, Y be a nonempty set and $f : X \times Y \to \mathbb{R}$, $g : D \times Y \to \mathbb{R}$. f is said to be

(a) *minimal g-quasiconvex in the first variable* if for any $\{z_1, z_2, \ldots, z_n\} \in \langle D \rangle$ and each $y \in Y$,

$$f(u,y) \leq \max_{1 \leq i \leq n} g(z_i,y) \quad for\, all \quad u \in \Gamma(\{z_1, z_2, \ldots, z_n\}).$$

(b) *minimal g-quasiconcave in the first variable* if for any $\{z_1, z_2, \ldots, z_n\} \in \langle D \rangle$ and each $y \in Y$,

$$\min_{1 \leq i \leq n} g(z_i,y) \leq f(u,y) \quad for\, all \quad u \in \Gamma(\{z_1, z_2, \ldots, z_n\}).$$

Note that f is minimal g-quasiconvex if and only if $-f$ is minimal $-g$-quasiconcave. These concepts originally go back to the notion of g-concavity (convexity), introduced by Chang and Yen [12].

Theorem 2.5. *Suppose (X,D,Γ) is an m-compact MG-convex space and Y is a minimal space. Let $f : X \times Y \to \mathbb{R}$, $g : X \times D \to \mathbb{R}$ and $h : X \times X \to \mathbb{R}$ satisfying*

(a) *h is minimal g-quasiconvex in the second variable,*

(b) *f is minimal upper semicontinuous in the first variable,*

(c) *for any $y \in Y$ there exists $z \in D$ such that $g(x,z) \leq f(x,y)$.*

Then

$$\inf_{x \in X} h(x,x) \leq \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

Proof. Suppose that $\sup_{x\in X}\inf_{y\in Y} f(x,y) < +\infty$. Let $\gamma > \sup_{x\in X}\inf_{y\in Y} f(x,y)$ be a fixed real number. Define the multimaps $T : X \multimap X$, $S : X \multimap D$ and $F : X \multimap Y$ by

$$T(x) = \{u \in X : h(x,u) < \gamma\}$$
$$S(x) = \{z \in D : g(x,z) < \gamma\}$$
$$F(x) = \{y \in Y : f(x,y) < \gamma\}.$$

Since $\gamma > \sup_{x\in X}\inf_{y\in Y} f(x,y)$, F has nonempty values. Lemma 2.3 implies that F^- is minimal transfer open. According to (c), for any $y \in Y$ there exists $z \in D$ such that $F^-(y) \subseteq S^-(z)$. Let $x \in X$ and $\{y_1, y_2, \ldots, y_n\} \in \langle S(x)\rangle$ and $u \in \Gamma(\{y_1, y_2, \ldots, y_n\})$. According to (a) and $y_i \in S(x)$ we have

$$h(x,u) \le \max_{1\le i\le n} g(x,y_i) < \gamma.$$

Therefore, $u \in T(x)$ and by Theorem 2.1, T has a fixed point $\bar{x}$. Then $\inf_{x\in X} h(x,x) \le h(\bar{x},\bar{x}) < \gamma$ and hence we have

$$\inf_{x\in X} h(x,x) \le \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

□

Remark 2.6. Another forms of Theorem 2.5 can be found in [9, 26, 27] and [28].

3. Two-function Minimax Inequality

Theorem 3.1. ([7]) *Suppose $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in I\}$ is a family of minimal spaces and also suppose that (Y,N) is a minimal space. Then $f : (Y,N) \longrightarrow (\prod_{\alpha\in I} X_\alpha, \prod_{\alpha\in I} \mathcal{M}_\alpha)$ is m-continuous if and only if $\pi_\alpha o f$ is m-continuous for all $\alpha \in I$.*

Consider a family of minimal generalized convex spaces $\{(X_\alpha, D_\alpha, \Gamma_\alpha) : \alpha \in I\}$. Set $X = \prod_{\alpha\in I} X_\alpha$ and $D = \prod_{\alpha\in I} D_\alpha$. Choose $A = \{a_0, \ldots, a_n\} \in \langle D\rangle$ and for each $\alpha \in I$ let $A_\alpha = \{\pi_\alpha(a_{i_0}), \pi_\alpha(a_{i_1}), \ldots, \pi_\alpha(a_{i_{n_\alpha}})\}$, where $\pi_\alpha(a_{i_0}), \pi_\alpha(a_{i_1}), \ldots, \pi_\alpha(a_{i_{n_\alpha}})$ are distinct elements of $\pi_\alpha(A)$ and $0 \le i_0 < i_1 < \ldots < i_{n_\alpha} \le n$. Now, define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma(A) = \prod_{\alpha\in I} \Gamma_\alpha(A_\alpha)$.

Lemma 3.2. *Suppose $\{(X_\alpha, D_\alpha, \Gamma_\alpha) : \alpha \in I\}$ is a family of MG-convex spaces. Also suppose that X, D, A_α's and Γ are defined as the above paragraph. Then (X, D, Γ) is an MG-convex space.*

Proof. Since $(X_\alpha, D_\alpha, \Gamma_\alpha)$'s are MG-convex spaces and since $A_\alpha \in \langle D_\alpha \rangle$ for each $\alpha \in I$, so there are (τ, m)-continuous functions $\varphi_{A_\alpha} : \Delta_{n_\alpha} \longrightarrow \Gamma_\alpha(A_\alpha)$ in which $\varphi_{A_\alpha}(\Delta_{J_\alpha}) \subseteq \Gamma_\alpha(J_\alpha)$. Define $\psi_{A_\alpha} : \Delta_n \longrightarrow \Delta_{n_\alpha}$ by

$$\psi_{A_\alpha}\Big(\sum_{j=0}^{n} \lambda_j e_j\Big) = \sum_{t=0}^{n_\alpha} \left(\sum_{\pi_\alpha(a_j)=\pi_\alpha(a_{i_t})} \lambda_j \right) e_t.$$

Now, consider the map $\varphi_A : \Delta_n \longrightarrow \Gamma(A)$ defined by $\varphi_A(z) = (\varphi_{A_\alpha} o \psi_{A_\alpha}(z))_{\alpha \in I}$ for all $z \in \Delta_n$. It is easy to see that $\psi_{A\alpha}$'s are continuous and hence $\varphi_{A_\alpha} o \psi_{A_\alpha}$ is (τ, m)-continuous for all $\alpha \in I$. It follows from Theorem 3.1 that φ_A is (τ, m)-continuous. For any $J \subseteq A$, put $J_\alpha = \pi_\alpha(J)$. By the definition of φ_A and the fact that $\varphi_{A_\alpha}(\Delta_{J_\alpha}) \subseteq \Gamma_\alpha(J_\alpha)$, one can deduce that $\varphi_A(\Delta_J) \subseteq \Gamma(J)$. This completes our proof. □

Theorem 3.3. *Suppose that (X_1, D_1, Γ_1) and (X_2, D_2, Γ_2) are two m-compact MG-convex spaces with property I, Y_1 and Y_2 are two nonempty sets. Let $f_1 : Y_1 \times X_2 \to \mathbb{R}$, $f_2 : X_1 \times Y_2 \to \mathbb{R}$, $g_1 : D_1 \times X_2 \to \mathbb{R}$, $g_2 : X_1 \times D_2 \to \mathbb{R}$ and $h_1, h_2 : X_1 \times X_2 \to \mathbb{R}$ be functions satisfying*

(a) *$h_1(z_1, z_2) \le h_2(z_1, z_2)$ for each $(z_1, z_2) \in X_1 \times X_2$.*

(b) *h_1 is minimal g_1-quasiconcave in the first variable.*

(c) *h_2 is minimal g_2-quasiconvex in the second variable.*

(d) *f_1 is minimal lower semi continuous in the second variable.*

(e) *f_2 is minimal upper semi continuous in the first variable.*

(f) *for any $y_1 \in Y_1$ there is $d_1 \in D_1$ such that $g_1(d_1, x_2) \ge f_1(y_1, x_2)$ for all $x_2 \in X_2$.*

(g) *for any $y_2 \in Y_2$ there is $d_2 \in D_2$ such that $g_2(x_1, d_2) \le f_2(x_1, y_2)$ for all $x_1 \in X_1$.*

Then

$$\inf_{x_2 \in X_2} \sup_{y_1 \in Y_1} f_1(y_1, x_2) \le \sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} f_2(x_1, y_2).$$

Proof. Consider any $A = \{(x_0^1, x_0^2), (x_1^1, x_1^2), \ldots, (x_n^1, x_n^2)\} \in \langle D_1 \times D_2 \rangle$. For $j = 1, 2$, let $A_j = \{z_0^j, z_1^j, \ldots, z_{n_j}^j\}$, where $z_0^j, z_1^j, \ldots, z_{n_j}^j$ are distinct elements of $\pi_j(A)$. Theorem 1.6 and Lemma 3.2 imply that $(X_1 \times X_2, D_1 \times D_2, \Gamma)$ is m-compact MG-convex space, where, $\Gamma(\{(x_0^1, x_0^2), (x_1^1, x_1^2), \ldots, (x_n^1, x_n^2)\}) = \Gamma_1(A_1) \times \Gamma_2(A_2)$. On the contrary, suppose that the inequality is incorrect. So there exists a fixed $\gamma \in \mathbb{R}$, such that

$$\sup_{x_1 \in X_1} \inf_{y_2 \in Y_2} f_2(x_1, y_2) < \gamma < \inf_{x_2 \in X_2} \sup_{y_1 \in Y_1} f_1(y_1, x_2) \tag{3.1}$$

Consider the multimaps $T : X_1 \times X_2 \multimap X_1 \times X_2$, $S : X_1 \times X_2 \multimap D_1 \times D_2$ and $F : X_1 \times X_2 \multimap Y_1 \times Y_2$ defined by

$$T(x_1, x_2) = \{t_1 \in X_1 : h_1(t_1, x_2) > \gamma\} \times \{t_2 \in X_2 : h_2(x_1, t_2) < \gamma\}$$

$$S(x_1,x_2)=\{d_1\in D_1: g_1(d_1,x_2)>\gamma\}\times\{d_2\in D_2: g_2(x_1,d_2)<\gamma\}$$
$$F(x_1,x_2)=\{y_1\in Y_1: f_1(y_1,x_2)>\gamma\}\times\{y_2\in Y_2: f_2(x_1,y_2)<\gamma\}.$$

(3.1) implies that, F has nonempty values. By (f) and (g), for any $(y_1,y_2)\in Y_1\times Y_2$ there is $(d_1,d_2)\in D_1\times D_2$ such that $F^-(y_1,y_2)\subseteq S^-(d_1,d_2)$. F^- is minimal transfer open. Since for any $(y_1,y_2)\in Y_1\times Y_2$ and $(x_1,x_2)\in F^-(y_1,y_2)$, we have $f_1(y_1,x_2)>\gamma$ and $f_2(x_1,y_2)<\gamma$. It follows from (d) and (e), there are $y_1'\in Y_1$, $y_2'\in Y_2$ and minimal neighborhoods $N_1(x_1)$ and $N_2(x_2)$ containing x_1 and x_2 respectively, such that $f_1(y_1',u_2)>\gamma$ and $f_2(u_1,y_2')<\gamma$, for any $(u_1,u_2)\in N_1(x_1)\times N_2(x_2)$. Since, X_1 and X_2 have property I, then one can deduce that $N(x_1,x_2)=N_1(x_1)\times N_2(x_2)\subseteq F^-(y_1',y_2')$ is a minimal neighborhood of (x_1,x_2) and so

$$(x_1,x_2)\in m\text{-Int}(N(x_1,x_2))\subseteq m\text{-Int}(F^-(y_1',y_2'))$$

which it gives the result. Choose $(z_1,z_2)\in X_1\times X_2$, $\{(x_0^1,x_0^2),(x_1^1,x_1^2),\ldots,(x_n^1,x_n^2)\}\in\langle S(z_1,z_2)\rangle$ and

$$\begin{aligned}(t_1,t_2)&\in\Gamma(\{(x_0^1,x_0^2),(x_1^1,x_1^2),\ldots,(x_n^1,x_n^2)\})\\&=\Gamma_1(\{z_0^1,z_1^1,\ldots,z_{n_1}^1\})\times\Gamma_2(\{z_0^2,z_1^2,\ldots,z_{n_2}^2\}),\end{aligned}$$

where $z_0^1,z_1^1,\ldots,z_{n_1}^1$ and $z_0^2,z_1^2,\ldots,z_{n_2}^2$ are distinct elements of $\{x_0^1,x_1^1,\ldots,x_n^1\}$ and $\{x_0^1,x_1^1,\ldots,x_n^1\}$ respectively. Since $(x_i^1,x_i^2)\in S(z_1,z_2)$ for $i=0,1,\ldots,n$, so $g_2(z_1,x_i^2)<\gamma<g_1(x_i^1,z_2)$ for each $i=0,1,\ldots,n$. By (b) and (c) we have

$$h_1(t_1,z_2)\geq\min_{0\leq i\leq n_1} g_1(z_i^1,z_2)>\gamma$$

and

$$h_2(z_1,t_2)\leq\max_{0\leq i\leq n_2} g_2(z_1,z_i^2)<\gamma.$$

Therefore, $(t_1,t_2)\in T(z_1,z_2)$, hence $\Gamma_M\subseteq T(z_1,z_2)$. Apply Theorem 2.1 to obtain a point $(\bar{z_1},\bar{z_2})\in X_1\times X_2$ for which $(\bar{z_1},\bar{z_2})\in T(\bar{z_1},\bar{z_2})$. Finally (a) implies that $\gamma<h_1(\bar{z_1},\bar{z_2})\leq h_2(\bar{z_1},\bar{z_2})<\gamma$, which it is a contradiction. □

Remark 3.4. Theorem 3.3 is an extension of corresponding theorems in [9, 17, 18, 23] for minimal generalized convex spaces.

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