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RANDOM COINCIDENCE POINTS AND INVARIANT APPROXIMATION THEOREMS

Hemant Kumar Nashine

Department of Mathematics, Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud
Raipur-492101(Chhattisgarh), India
e-mail: drhknashine@gmail.com, nashine_09@rediffmail.com

Abstract. Some random coincidence point theorems satisfying a generalized contractive condition have been established. As application, random best approximation results have also been derived.

1. Introduction

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [15] and references mentioned therein. These results mainly rely on geometric properties of Banach spaces. These results were the starting point for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. The fixed point theory of multi-valued nonexpansive mappings is much more complicated than the corresponding theory of single-valued nonexpansive mappings.

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the $1950s$. The interest in this subject enhanced after publication of the survey paper by Bharucha Reid [7]. Random fixed

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point theorems for contraction mappings were first studies by Spacek [30] and Hans [11, 12]. Itoh [16, 17, 18] gave several random fixed point theorems for various single and multivalued random operators. Random fixed point theory has received much attention in recent years(see [2, 23, 24, 32]).

Random coincidence point theorems and random approximations are stochastic generalization of classical coincidence point and approximation theorems, and have application in probability theory and nonlinear analysis. The random fixed point theory for self-maps and non-self maps has been developed during the last decade by various author, (see e.g. [2, 13]). Recently, this theory has been further extended for 1-set contractive, nonexpansive, semi-contractive and completely continuous random maps, etc.

Random fixed point theorems have been applied in many instances in the field of random best approximation theory and several interesting and meaningful results have been studied. The theory of approximation has become so vast that it intersects with every other branch of analysis and plays an important role in the applied sciences and engineering. Approximation theory is concerned with the approximation of functions of a certain kind by other functions. In this point of view, in the year 1963, Meinardus [21] was first to observe the general principle and to use a Schauder fixed point theorem for finding deterministic version fixed point theorem as best approximation. Afterwards, number of results were developed in this direction under different conditions following the line made by Meinardus (see [8, 26, 27]).

On the other hand, In the year 2000, Shahzad and Latif [28, Theorem 3.2] proved the random coincidence point which is further extended by Shahzad and Nawab [29, Theorem 3.1]. Shahzad and Nawab [29] has also given the invariant approximation result (Theorem 3.8) for single-valued mappings and extends and complements the results of Beg and Shahzad [4, 6]. The result of Shahzad and Latif [28, Theorem 3.2] and Xu [31, Theorem 1] is also generalized and improved by Khan et al. [19, Theorem 3.13], in the sense that the maps $\mathcal{S}$ and $\mathcal{T}$ need not be commuting for the existence of random coincidence, $\mathcal{T}(\omega, .)$ is not necessarily $\mathcal{S}(\omega, .)$- nonexpansive, and $\mathcal{S}$ is not affine. As application, random invariant approximation results has also been obtained for single-valued mappings.

The purpose of this paper is to generalize Khan et al. [19] for more generalized nonexpansive mappings. In this way, related results of Beg and Shahzad [3, 4, 5], Nashine [22], Shahzad and Latif [28], Shahzad and Latif [29] and Xu [31] are improve and generalized for multivalued random operators. Incidently, these results also give multivalued random version generalization of Dotson [9], Sahab et al. [25] and Singh [26, 27] and many more related results.

2. Preliminaries

In the material to be produced here, the following definitions have been used:

Definition 2.1. [6]. *Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let $\mathcal{M}$ be a subset of metric space $(\mathcal{X}, d)$. We denote by 2^X the family of all subsets of $\mathcal{X}$, by $CB(\mathcal{X})$ the family of all nonempty closed and bounded subsets of $\mathcal{X}$, and by $\mathcal{H}$ the Hausdorff metric on $CB(\mathcal{X})$, induced by the metric d. For any $x \in \mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}$, by $d(x, \mathcal{A})$ we denote the distance between x and $\mathcal{A}$, that is, $d(x, \mathcal{A}) = \inf\{d(x, a) : a \in \mathcal{A}\}$.*

A mapping $\mathcal{T} : \Omega \to 2^{\mathcal{X}}$ is called measurable (respectively, weakly measurable) if, for any closed (respectively, open) subset $\mathcal{B}$ of $\mathcal{X}$, $\mathcal{T}^{-1}(\mathcal{B}) = \{\omega \in \Omega : \mathcal{T}(\omega) \cap \mathcal{B} \neq \phi\} \in \Sigma$. Note that, if $\mathcal{T}(\omega) \in \mathcal{C}(\mathcal{X})$ for every $\omega \in \Omega$, then $\mathcal{T}$ is weakly measurable if and only if measurable.

A mapping $\xi : \Omega \to \mathcal{X}$ is said to be measurable selector of a measurable mapping $\mathcal{T} : \Omega \to 2^{\mathcal{X}}$, if ξ is measurable and, for any $\omega \in \Omega, \xi(\omega) \in \mathcal{T}(\omega)$. A mapping $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ is called a random operator if, for any $x \in \mathcal{X}$, $\mathcal{T}(., x)$ is measurable. A mapping $\mathcal{T} : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ is called a multivalued random operator if for every $x \in X$, $\mathcal{T}(., x)$ is measurable. A measurable mapping $\xi : \Omega \to \mathcal{X}$ is called a random fixed point of a random operator $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$, if for every $\omega \in \Omega$, $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$. A measurable mapping $\xi : \Omega \to \mathcal{X}$ is called a random coincidence of $\mathcal{T} : \Omega \times CB(\mathcal{X}) \to \mathcal{X}$ and $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ if $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$ for all $\omega \in \Omega$. We denote by $\mathcal{F}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$ and by $\mathcal{C}(\mathcal{S} \cap \mathcal{T})$ the set of coincidence points of $\mathcal{S}$ and $\mathcal{T}$.

A map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be $\mathcal{S}$-nonexpansive, if there exists a self-map $\mathcal{S}$ on $\mathcal{X}$ such that $d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{S}x, \mathcal{S}y)$ for all $x, y \in \mathcal{X}$.

Let $\mathcal{T} : \mathcal{M} \to CB(\mathcal{M})$. The mapping $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is said to be $\mathcal{T}$-weakly commuting if for all $x \in \mathcal{M}$, $\mathcal{S}\mathcal{S}x \in \mathcal{T}\mathcal{S}x$.

A set $\mathcal{M}$ is said to have property (N) [14], if

(1) $\mathcal{T} : \mathcal{M} \to CB(\mathcal{M})$,

(2) $(1-k_n)p + k_n\mathcal{T}x \subseteq \mathcal{M}$, for some $p \in \mathcal{M}$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in \mathcal{M}$. Each p-starshaped set has the property $(\mathcal{N})$ with respect to any map $\mathcal{T} : \mathcal{M} \to CB(\mathcal{M})$ but the converse does not hold, in general.

A random operator $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ is continuous (respectively, nonexpansive, $\mathcal{S}$-nonexpansive) if, for each $\omega \in \Omega$, $\mathcal{T}(\omega, .)$ is continuous(respectively, nonexpansive, $\mathcal{S}$-nonexpansive). Let $\mathcal{T}:\Omega\times\mathcal{X}\to CB(\mathcal{X})$ be a random operator. Then, random operators $\mathcal{S}:\Omega\times\mathcal{X}\to\mathcal{X}$ is $\mathcal{T}$-weakly commuting, if $\mathcal{S}(\omega,.)$ is $\mathcal{T}$-weakly commuting, for each $\omega\in\Omega$.

Definition 2.2. [6]. *Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$. For $x_0\in\mathcal{X}$, let us define*

$$dist(x_0,\mathcal{M})=\inf_{y\in\mathcal{M}}\|x_0-y\|$$

and

$$\mathcal{P}_{\mathcal{M}}(x_0)=\{y\in\mathcal{M}:\|x_0-y\|=dist(x_0,\mathcal{M})\}.$$

An element $y\in\mathcal{P}_{\mathcal{M}}(x_0)$ is called a best approximant of x_0 out of $\mathcal{M}$. The set $\mathcal{P}_{\mathcal{M}}(x_0)$ is the set of all best approximation of x_0 out of $\mathcal{M}$.

The following result is also needed in the sequel:

Theorem 2.3. [10]. *Let $(\mathcal{X},d)$ be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $\mathcal{T}:\Omega\times\mathcal{X}\to CB(\mathcal{X})$ and $\mathcal{S}:\Omega\times\mathcal{X}\to\mathcal{X}$ be mappings such that*

(i) $\mathcal{T}(\omega,.)$, $\mathcal{S}(\omega,.)$ are continuous for all $\omega\in\Omega$,

(ii) $\mathcal{T}(\omega,.)$, $\mathcal{S}(\omega,.)$ are measurable for all $x\in\mathcal{X}$,

(iii) they satisfy for each $\omega\in\Omega$

$$\begin{aligned}&\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\&\le\alpha(\omega)\max\{d(\mathcal{S}(\omega,x),\mathcal{S}(\omega,y)),d(\mathcal{S}(\omega,x),\mathcal{T}(\omega,x)),d(\mathcal{S}(\omega,y),\mathcal{T}(\omega,y)),\\&\qquad\qquad\frac{1}{2}[d(\mathcal{S}(\omega,x),\mathcal{T}(\omega,y))+d(\mathcal{S}(\omega,y),\mathcal{T}(\omega,x))]\}\\&\quad+\beta(\omega)\max\{d(\mathcal{S}(\omega,x),\mathcal{T}(\omega,x)),d(\mathcal{S}(\omega,y),\mathcal{T}(\omega,y))\}\\&\quad+\gamma(\omega)[d(\mathcal{S}(\omega,x),\mathcal{T}(\omega,y))+d(\mathcal{S}(\omega,y),\mathcal{T}(\omega,x))]\end{aligned}$$

for every $x,y\in\mathcal{X}$, where $\alpha,\beta,\gamma:\Omega\to[0,1)$ are measurable mappings such that for all $\omega\in\Omega$, $\beta(\omega)>0,\gamma(\omega)>0,\alpha(\omega)+\beta(\omega)+2\gamma(\omega)=1$. If $\mathcal{S}(\Omega\times\mathcal{X})=\mathcal{X}$ for each $\omega\in\Omega$, then there is a measurable mapping $\xi:\Omega\to\mathcal{X}$ such that $\mathcal{S}(\omega,\xi(\omega))\in\mathcal{T}(\omega,\xi(\omega))$ for all $\omega\in\Omega$ (i.e., $\mathcal{T}$ and $\mathcal{S}$ have a random coincidence point).

3. Main Results

Following is a common random fixed point theorem for multivalued random operator:

Theorem 3.1. *Let (Ω, Σ) be a measurable space, let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$, and $\mathcal{S} : \Omega \times \mathcal{M} \to \mathcal{M}$ a random operators such that $\mathcal{S}(\omega, \mathcal{M}) = \mathcal{M}$ for each $\omega \in \Omega$. Assume that $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ a continuous random operators and satisfies, for each $\omega \in \Omega$, $x, y \in \mathcal{M}$ and $\lambda \in [0,1]$*

$$\begin{aligned}
&\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\
&\le \max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|, dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)),\\
&\qquad\qquad \frac{1}{2}[dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))]\}\\
&\quad +[\frac{1-k(\omega)}{2k(\omega)}]\max\{dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y))\}\\
&\quad +[\frac{1-k(\omega)}{4k(\omega)}][dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))] \qquad (3.1)
\end{aligned}$$

where $k : \Omega \to (0,1)$ are measurable mappings such that for all $\omega \in \Omega$. Suppose that $\mathcal{M}$ has the property (N), then $\mathcal{S}$ and $\mathcal{T}$ have a random coincidence point, if one of the following conditions is satisfied:

(1) *$\mathcal{M}$ is separable compact and $\mathcal{S}$ is continuous;*

(2) *$\mathcal{X}$ is Banach space, $\mathcal{M}$ is separable weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S} - \mathcal{T})(\omega, .)$ is demiclosed at 0;*

(3) *$\mathcal{X}$ is Banach space, $\mathcal{M}$ is separable weakly compact, $\mathcal{T}$ is completely continuous, and $\mathcal{S}$ is continuous;*

(4) *$\mathcal{M}$ is separable complete, $\mathcal{T}(\mathcal{M})$ is bounded and $(\mathcal{S} - \mathcal{T})(\mathcal{M})$ is closed.*

Moreover, if for each $\omega \in \Omega$ and any $x \in \mathcal{M}$, $\mathcal{S}(\omega,x) \in \mathcal{T}(\omega,x)$ implies $\mathcal{S}(\omega, \mathcal{S}(\omega,x)) = \mathcal{S}(\omega,x)$, and $\mathcal{S}$ is $\mathcal{T}$-weakly commuting random operator, then $\mathcal{T}$ and $\mathcal{S}$ have a common random fixed point.

Proof. Choose a fixed sequence of measurable mappings $k_n : \Omega \to (0,1)$ such that $k_n(\omega) \to 1$ as $n \to \infty$. For $n \ge 1$, define a sequence of random operators $\mathcal{T}_n : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ as

$$\mathcal{T}_n(\omega,x) = k_n(\omega)\mathcal{T}(\omega,x) + (1-k_n(\omega))p \qquad (3.2)$$

for all $x \in \mathcal{M}$. Then, each $\mathcal{T}_n$ is a well-defined map from $\mathcal{M}$ into $CB(\mathcal{M})$ and $\omega \in \Omega$ as $\mathcal{M}$ has property (N). It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\mathcal{H}(\mathcal{T}_n(\omega,x),\mathcal{T}_n(\omega,y)) = k_n(\omega)\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\
&\le k_n(\omega)\max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|,\\
&\qquad\qquad\qquad \frac{1}{2}[\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,y)\|+\|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,x)\|]\}\\
&\quad +[\frac{1-k_n(\omega)}{2}]\max\{\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,x)\|,\|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,y)\|\}\\
&\quad +[\frac{1-k_n(\omega)}{4}][\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,y)\|+\|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,x)\|]
\end{aligned}$$

for all $x,y\in\mathcal{M}$ and $\omega\in\Omega$. Note that for all $\omega\in\Omega$, $(\frac{1-k_n(\omega)}{2})>0$, $(\frac{1-k_n(\omega)}{4})>0$ and $[k_n(\omega)+(\frac{1-k_n(\omega)}{2})+2(\frac{1-k_n(\omega)}{4})]=1$ for each n.

(1) Since $\mathcal{M}$ is compact, thus, all the condition of the Theorem 2.3 are satisfied on $\mathcal{M}$ and so, there exists a coincidence random fixed point ξ_n of $\mathcal{T}_n$ and $\mathcal{S}$ such that $\mathcal{S}_n(\omega,\xi_n(\omega))\in\mathcal{T}(\omega,\xi_n(\omega))$.

For each n, define $\mathcal{G}_n:\Omega\to\mathcal{C}(\mathcal{M})$ by $\mathcal{G}_n = cl\{\xi_i(\omega): i\ge n\}$ where $\mathcal{C}(\mathcal{M})$ is the set of all nonempty compact subset of $\mathcal{M}$. Let $\mathcal{G}:\Omega\to\mathcal{C}(\mathcal{M})$ be a mapping defined as $\mathcal{G}(\omega)=\cap_{n=1}^{\infty}\mathcal{G}_n(\omega)$. By Himmelberg [13, Theorem 4.1] implies that $\mathcal{G}$ is measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [20] further implies that $\mathcal{G}$ has a measurable selector $\xi:\Omega\to\mathcal{M}$. We show that ξ is the random fixed point of $\mathcal{T}$ and $\mathcal{S}$. Fix $\omega\in\Omega$. Since $\xi(\omega)\in\mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges to $\xi(\omega)$; that is $\xi_m(\omega)\to\xi(\omega)$. Also, for every $\omega\in\Omega$, Since $\xi_m(\omega)\in\mathcal{T}_m(\omega,\xi_m(\omega))$, we have

$$\mathcal{T}_m(\omega,\xi_m(\omega)) = k_m(\omega)\mathcal{T}(\omega,\xi_m(\omega))+(1-k_m(\omega))p\to\mathcal{T}(\omega,\xi(\omega))$$

as $k_m(\omega)\to 1$ and $\mathcal{H}(\mathcal{T}(\omega,\xi_m(\omega)),\mathcal{T}(\omega,\xi(\omega)))\to 0$, for every $\omega\in\Omega$. Now,

$$\begin{aligned}
\|\xi(\omega)-\mathcal{T}(\omega,\xi(\omega))\| &\le \|\xi(\omega)-\xi_m(\omega)\|+\|\xi_m(\omega)-\mathcal{T}(\omega,\xi(\omega))\|\\
&\le \|\xi(\omega)-\xi_m(\omega)\|+\mathcal{H}(\mathcal{T}_m(\omega,\xi_m(\omega)),\mathcal{T}(\omega,\xi(\omega)))\to 0
\end{aligned}$$

as $m\to\infty$, for every $\omega\in\Omega$. Since $\mathcal{T}(\omega,\xi(\omega))$ is closed for each $\omega\in\Omega$. Hence $\xi(\omega)\in\mathcal{T}(\omega,\xi(\omega))$. Also from the continuity of $\mathcal{S}$, we have

$$\mathcal{S}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\lim_{m\to\infty}\xi_m(\omega)) = \lim_{m\to\infty}\mathcal{S}(\omega,\xi_m(\omega)) = \lim_{m\to\infty}\xi_m(\omega)=\xi(\omega).$$

If $\mathcal{S}$ is $\mathcal{T}$-weakly commuting at $v(\omega)\in C(\mathcal{S},\mathcal{T})$, then

$$\mathcal{S}(\omega,\mathcal{S}(\omega,v(\omega))) = \mathcal{T}(\omega,\mathcal{S}(\omega,v(\omega)))$$

and hence

$$\mathcal{S}(\omega,v(\omega)) = \mathcal{S}(\omega,\mathcal{S}(\omega,v(\omega)))\in\mathcal{T}(\omega,\mathcal{S}(\omega,v(\omega))).$$

Thus $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \phi$.

(2) Since weak topology is Hausdorff and $\mathcal{M}$ is weakly compact, it follows that $\mathcal{M}$ is strongly closed and is a completely metric space. Thus by weakly continuity of $\mathcal{S}$ and Theorem 2.3, there exists a random fixed point ξ of $\mathcal{T}_n$ such that $\mathcal{S}(\omega, \xi_n(\omega)) \in \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. By the definition of $\mathcal{T}(\omega, \xi_n(\omega))$, there is a $\zeta_n(\omega) \in \mathcal{T}(\omega, \xi_n(\omega)$.

For each n, define $\mathcal{G}_n : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}_n = w - cl\{\xi_i(\omega) : i \geq n\}$, where $\mathcal{WC}(\mathcal{M})$ is the set of all nonempty weakly compact subset of $\mathcal{M}$ and $w - cl$ denotes the weak closure. Defined a mapping $\mathcal{G} : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then by Himmelberg [13, Theorem 4.1] implies that $\mathcal{G}$ is $w-$measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [20] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We show that ξ is the random fixed point of $\mathcal{T}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$; that is $\xi_m(\omega) \to \xi(\omega)$. Now, from weakly continuity of $\mathcal{S}$, we have

$$\mathcal{S}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \lim_{m\to\infty} \xi_m(\omega)) = \lim_{m\to\infty} \mathcal{S}(\omega, \xi_m(\omega)) = \lim_{m\to\infty} \xi_m(\omega) = \xi(\omega).$$

Now,

$$\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega) = k_n(\omega)\zeta_m(\omega) + (1 - k_n(\omega))p - \zeta_m(\omega)$$

$$= (1 - k_m(\omega))(p - \zeta_m(\omega)).$$

Since $\mathcal{M}$ is bounded and $k_m(\omega) \to 1$, it follows that

$$\|\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)\| \to 0.$$

Now, $y_m = \mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)) = (\mathcal{S} - \mathcal{T})(\omega, \xi_m(\omega))$ and $y_m \to 0$. Since $(\mathcal{S} - \mathcal{T})(\omega, .)$ is demiclosed at 0, so $0 \in (\mathcal{S} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$. As in the proof of (i), thus $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \phi$.

(3) As in (ii) there exists a random fixed point ξ_n of $\mathcal{T}_n$ such that $\xi_n = \mathcal{S}(\omega, \xi_n(\omega)) = \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. For each n, define $\mathcal{G}_n : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}_n = w - cl\{\xi_i(\omega) : i \geq n\}$, where $WC(\mathcal{M})$ is the set of all nonempty weakly compact subset of $\mathcal{M}$ and $w - cl$ denotes the weak closure. Defined a mapping $\mathcal{G} : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then by Himmelberg([13], Theorem 4.1) implies that $\mathcal{G}$ is $w-$measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [20] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We show that ξ is the random fixed

point of $\mathcal{T}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$; that is $\xi_m(\omega) \to^w \xi(\omega)$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}(\omega, \xi_m(\omega)) \to \mathcal{T}(\omega, \xi(\omega))$ as $m \to \infty$. Since $k_m \to 1$, we get

$$\xi_m(\omega) = (1 - k_m)q + k_m\mathcal{T}(\omega, \xi_m(\omega)) = \mathcal{T}(\omega, \xi(\omega)).$$

Thus $\mathcal{T}(\omega, \xi_m(\omega)) \to \mathcal{T}^2(\omega, \xi(\omega))$ as $m \to \infty$ and consequently $\mathcal{T}^2(\omega, \xi(\omega)) = \mathcal{T}(\omega, \xi(\omega))$ implies that $\mathcal{T}(\omega, \zeta(\omega)) = \zeta(\omega)$, where $\zeta(\omega) = \mathcal{T}(\omega, \xi(\omega))$. But, since $\mathcal{S}(\omega, \xi_m(\omega)) = \xi_m(\omega) \to \mathcal{T}(\omega, \xi(\omega)) = \zeta(\omega)$, using the continuity of $\mathcal{S}$ and the uniqueness of the limit, we have $\mathcal{S}(\omega, \zeta(\omega)) = \zeta(\omega)$. Hence $\mathcal{S}(\omega, \zeta(\omega)) = \mathcal{T}(\omega, \zeta(\omega)) = \zeta(\omega)$.

(4) By Theorem 2.3, for each $n \geq 1$, there exists $\xi_n(\omega) \in \mathcal{M}$ such that $\mathcal{S}(\omega, \xi_n(\omega)) \in \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. This implies that there is a $\zeta_n(\omega) \in \mathcal{T}(\omega, \xi_n(\omega)$ such that

$$\begin{aligned}\mathcal{S}(\omega, \xi_n(\omega)) - \zeta_n(\omega) &= k_n(\omega)\zeta_n(\omega) + (1 - k_n(\omega))p - \zeta_n(\omega) \\ &= (1 - k_n(\omega))(p - \zeta_n(\omega)).\end{aligned}$$

Since $\mathcal{T}(\mathcal{M})$ is bounded and $k_n(\omega) \to 1$, it follows that $\|\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)\| \to 0.$ as $n \to \infty$. As $(\mathcal{S} - \mathcal{T})(\omega, .)$ is closed, so $0 \in (\mathcal{S} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$. As in the proof of (i), thus $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \phi$. $\square$

Remark 3.2. *If in Theorem 3.1, $\mathcal{S}(\omega, x) = x$ for all $(\omega, x) \in \Omega \times \mathcal{M}$, then we get the following random fixed point theorem.*

Corollary 3.3. *Let (Ω, Σ) be a measurable space, let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$, and $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ a continuous random operators and satisfies, for each $\omega \in \Omega$, $x, y \in \mathcal{M}$ and $\lambda \in [0, 1]$*

$$\begin{aligned}&\mathcal{H}(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \\ &\leq \max\{\|x - y\|, dist(x, \mathcal{T}_\lambda(\omega, x)), dist(y, \mathcal{T}_\lambda(\omega, y)), \\ &\qquad \frac{1}{2}[dist(x, \mathcal{T}_\lambda(\omega, y)) + dist(y, \mathcal{T}_\lambda(\omega, x))]\} \\ &\quad + [\frac{1 - k(\omega)}{2k(\omega)}] \max\{dist(x, \mathcal{T}_\lambda(\omega, x)), dist(y, \mathcal{T}_\lambda(\omega, y))\} \\ &\quad + [\frac{1 - k(\omega)}{4k(\omega)}][dist(x, \mathcal{T}_\lambda(\omega, y)) + dist(y, \mathcal{T}_\lambda(\omega, x))] \qquad (3.3)\end{aligned}$$

where $k : \Omega \to (0, 1)$ are measurable mappings such that for all $\omega \in \Omega$. Suppose that $\mathcal{M}$ has the property (N), then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) \in \mathcal{T}(\omega, \xi(\omega))$, if one of the following conditions is satisfied:

(1) $\mathcal{M}$ *is separable compact;*
(2) $\mathcal{X}$ *is Banach space,* $\mathcal{M}$ *is separable weakly compact, and* $(\mathcal{I}-\mathcal{T})(\omega,.)$ *is demiclosed at* 0*, where* $\mathcal{I}$ *is identity operator;*
(3) $\mathcal{X}$ *is Banach space,* $\mathcal{M}$ *is separable weakly compact,* $\mathcal{T}$ *is completely continuous;*
(4) $\mathcal{M}$ *is separable complete,* $\mathcal{T}(\mathcal{M})$ *is bounded and* $(\mathcal{I}-\mathcal{T})(\mathcal{M})$ *is closed, where* $\mathcal{I}$ *is identity operator;*

Corollary 3.4. *Let* (Ω,Σ) *be a measurable space, let* $\mathcal{M}$ *be a nonempty subset of a normed space* $\mathcal{X}$*, and* $\mathcal{S}:\Omega\times\mathcal{M}\to\mathcal{M}$ *a random operators such that* $\mathcal{S}(\omega,\mathcal{M})=\mathcal{M}$ *for each* $\omega\in\Omega$*. Assume that* $\mathcal{T}:\Omega\times\mathcal{M}\to CB(\mathcal{M})$ *a continuous random operators and satisfies, for each* $\omega\in\Omega$*,* $x,y\in\mathcal{M}$ *and* $\lambda\in[0,1]$

$$\begin{aligned}&\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\&\le\max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|,dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)),dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)),\\&\qquad\frac{1}{2}[dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))]\}.\end{aligned}\tag{3.4}$$

Suppose that $\mathcal{M}$ *has the property* (N)*, then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a random coincidence point under each of the condition of Theorem 3.1. Moreover, if for each* $\omega\in\Omega$ *and any* $x\in\mathcal{M}$*,* $\mathcal{S}(\omega,x)\in\mathcal{T}(\omega,x)$ *implies* $\mathcal{S}(\omega,\mathcal{S}(\omega,x))=\mathcal{S}(\omega,x)$*, and* $\mathcal{S}$ *is* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a common random fixed point.*

Remark 3.5. *If in Corollary 3.4,* $\mathcal{S}(\omega,x)=x$ *for all* $(\omega,x)\in\Omega\times\mathcal{M}$*, then we obtain the following random fixed point theorem.*

Corollary 3.6. *Let* (Ω,Σ) *be a measurable space, let* $\mathcal{M}$ *be a nonempty subset of a normed space* $\mathcal{X}$*, and* $\mathcal{T}:\Omega\times\mathcal{M}\to CB(\mathcal{M})$ *a continuous random operators and satisfies, for each* $\omega\in\Omega$*,* $x,y\in\mathcal{M}$ *and* $\lambda\in[0,1]$

$$\begin{aligned}\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\le\max\{&\|x-y\|,dist(x,\mathcal{T}_\lambda(\omega,x)),dist(y,\mathcal{T}_\lambda(\omega,y)),\\&\frac{1}{2}[dist(x,\mathcal{T}_\lambda(\omega,y))+dist(y,\mathcal{T}_\lambda(\omega,x))]\}\end{aligned}\tag{3.5}$$

then there exists a measurable map $\xi:\Omega\to\mathcal{M}$ *such that* $\xi(\omega)\in\mathcal{T}(\omega,\xi(\omega))$ *under each of the condition of Theorem 3.3.*

As application of Theorem 3.1, following are common random fixed point theorems for random invariant approximation:

Theorem 3.7. *Let* $\mathcal{M}$ *be a subset of normed space of* $\mathcal{X}$*,* $\mathcal{S}:\Omega\times\mathcal{M}\to\mathcal{M}$ *and* $\mathcal{T}:\Omega\times\mathcal{M}\to CB(\mathcal{M})$ *is continuous. Suppose that*

(a) $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy for all* $x,y\in\mathcal{P}_{\mathcal{M}}(x_0)$ *and* $\lambda\in[0,1]$

$$\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))$$
$$\leq \begin{cases} \max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,x_0)\|, \\ \ dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)), \\ \ \frac{1}{2}[dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))]\} \\ \ +[\frac{1-k(\omega)}{2k(\omega)}]\max\{dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y))\} \\ \ +[\frac{1-k(\omega)}{4k(\omega)}][dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))] \end{cases} \tag{3.6}$$

where $k:\Omega\to(0,1)$ are measurable mappings such that for all $\omega\in\Omega$;
(b) $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N);
(c) $\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}-$invariant and $\mathcal{S}-$invariant.

Then $\mathcal{P}_{\mathcal{M}}(x_0)\cap C(\mathcal{S},\mathcal{T})\neq\phi$, if one of the following conditions is satisfied:

(1) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is separable compact and $\mathcal{S}$ is continuous;*
(2) *$\mathcal{X}$ is Banach space, $\mathcal{P}_{\mathcal{M}}(x_0)$ is separable weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S}-\mathcal{T})(\omega,.)$ is demiclosed at 0;*
(3) *$\mathcal{X}$ is Banach space, $\mathcal{P}_{\mathcal{M}}(x_0)$ is separable weakly compact, $\mathcal{T}$ is completely continuous, and $\mathcal{S}$ is continuous;*
(4) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is separable complete, $\mathcal{T}(\mathcal{M})$ is bounded and $(\mathcal{S}-\mathcal{T})(\mathcal{M})$ is closed.*

Moreover, if for each $\omega\in\Omega$ and any $x\in\mathcal{M}$, $\mathcal{S}(\omega,x)\in\mathcal{T}(\omega,x)$ implies $\mathcal{S}(\omega,\mathcal{S}(\omega,x))=\mathcal{S}(\omega,x)$, and $\mathcal{S}$ is $\mathcal{T}$-weakly commuting random operator, then $\mathcal{P}_{\mathcal{M}}(x_0)\cap\mathcal{F}(\mathcal{S})\cap\mathcal{F}(\mathcal{T})\neq\phi$.

Proof. Since $\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}-$invariant and $\mathcal{S}-$invariant, it follows that $\mathcal{S}:\Omega\times\mathcal{P}_{\mathcal{M}}(x_0)\to\mathcal{P}_{\mathcal{M}}(x_0)$, $\mathcal{T}:\Omega\times\mathcal{P}_{\mathcal{M}}(x_0)\to CB(\mathcal{P}_{\mathcal{M}}(x_0))$. The results now follow from Theorem 3.1. $\square$

Theorem 3.8. *Let $\mathcal{M}$ be a subset of normed space of $\mathcal{X}$, $\mathcal{S}:\Omega\times\mathcal{M}\to\mathcal{M}$, $\mathcal{T}:\Omega\times\mathcal{M}\to CB(\mathcal{M})$ such that $\mathcal{T}(\omega,.):\partial\mathcal{M}\cap\mathcal{M}\to\mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0\in\mathcal{X}$ and $\mathcal{S}(\omega,x_0)\in\mathcal{T}(\omega,x_0)=\{x_0\}$ $\omega\in\Omega$. Suppose that*
(a) $\mathcal{T}$ and $\mathcal{S}$ satisfy for all $x\in\mathcal{P}_{\mathcal{M}}(x_0)\cup\{x_0\}$ and $\lambda\in[0,1]$

$$\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))$$

$$\leq \begin{cases} \|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,x_0)\|, \quad \textit{if} \quad y=x_0, \\ \max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|, \\ \quad dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)), \\ \quad \frac{1}{2}[dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))]\} \\ \quad +[\frac{1-k(\omega)}{2k(\omega)}]\max\{dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y))\} \\ \quad +[\frac{1-k(\omega)}{4k(\omega)}][dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+dist(\mathcal{S}(\omega,y),x\mathcal{T}_\lambda(\omega,x))], \\ \qquad\qquad \textit{if } y\in\mathcal{P}_\mathcal{M}(x_0), \end{cases} \tag{3.7}$$

where $k:\Omega\to(0,1)$ *are measurable mappings such that for all* $\omega\in\Omega$*;*

(b) $\mathcal{P}_\mathcal{M}(x_0)$ *is nonempty and has the property* (N)*;*

(c) $\mathcal{S}(\omega,\mathcal{P}_\mathcal{M}(x_0))=\mathcal{P}_\mathcal{M}(x_0)$*, i.e.,* $\mathcal{P}_\mathcal{M}(x_0)$ *is* $\mathcal{S}-$*invariant.*

Then $\mathcal{P}_\mathcal{M}(x_0)\cap C(\mathcal{S},\mathcal{T})\neq\phi$*, under each of the conditions of Theorem 3.7.*

Moreover, if for each $\omega\in\Omega$ *and any* $x\in\mathcal{M}$*,* $\mathcal{S}(\omega,x)\in\mathcal{T}(\omega,x)$ *implies* $\mathcal{S}(\omega,\mathcal{S}(\omega,x))=\mathcal{S}(\omega,x)$*, and* $\mathcal{S}$ *is* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{P}_\mathcal{M}(x_0)\cap\mathcal{F}(\mathcal{S})\cap\mathcal{F}(\mathcal{T})\neq\phi$*.*

Proof. Let $y\in\mathcal{P}_\mathcal{M}(x_0)$. Then $\|y-x_0\|=dist(x,\mathcal{M})$. Note that for any $t(\omega)\in(0,1)$,

$$\|t(\omega)x_0+(1-t(\omega))y-x_0\|=(1-t(\omega))\|y-x_0\|<dist(x_0,\mathcal{M}).$$

It follows that the line segment $\{t(\omega)x_0+(1-t(\omega))y:0<t(\omega)<1\}$ and the set $\mathcal{M}$ are disjoint. Thus y is not in the interior of $\mathcal{M}$ and so $y\in\partial\mathcal{M}\cap\mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M}\cap\mathcal{M})\subset\mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Let $z\in\mathcal{T}(\omega,y)$.

$$\|z-x_0\|\leq\mathcal{H}(\mathcal{T}(\omega,y),\mathcal{T}(\omega,x_0)$$

$$\leq\|\mathcal{S}(\omega,y)-\mathcal{S}(\omega,x_0)\|=\|\mathcal{S}(\omega,y)-x_0\|=dist(x_0,\mathcal{M}).$$

Now $z\in\mathcal{M}$ and $\mathcal{S}(\omega,y)\in\mathcal{P}_\mathcal{M}(x_0)$, imply that $z\in\mathcal{P}_\mathcal{M}(x_0)$. Thus $\mathcal{T}(\omega,\mathcal{P}_\mathcal{M}(x_0))\subseteq\mathcal{P}_\mathcal{M}(x_0)$. Hence $\mathcal{T}$ maps $\mathcal{P}_\mathcal{M}(x_0))$ into $CB(\mathcal{P}_\mathcal{M}(x_0))$. Thus, the result follows from Theorem 3.1. □

Define $\mathcal{C}^\mathcal{S}_\mathcal{M}(x_0)=\{x\in\mathcal{M}:\mathcal{S}x\in\mathcal{P}_\mathcal{M}(x_0)\}$ and $\mathcal{D}^\mathcal{S}_\mathcal{M}(x_0)=\mathcal{P}_\mathcal{M}(x_0)\cap\mathcal{C}^\mathcal{S}_\mathcal{M}(x_0)$ [1].

Theorem 3.9. *Let $\mathcal{M}$ be a subset of normed space of $\mathcal{X}$, $\mathcal{S} : \Omega \times \mathcal{M} \to \mathcal{M}$, $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $\mathcal{S}(\omega, x_0) \in \mathcal{T}(\omega, x_0) = \{x_0\}$ $\omega \in \Omega$. Suppose that*
(a) $\mathcal{T}$ and $\mathcal{S}$ satisfy for all $x \in \mathcal{D}^{\mathcal{S}}_{\mathcal{M}}(x_0)(= \mathcal{D}) \cup \{x_0\}$ and $\lambda \in [0, 1]$

$$\mathcal{H}(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y))$$
$$\leq \begin{cases} \|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, x_0)\|, \quad if \quad y = x_0, \\ \max\{\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, y)\|, \\ \quad dist(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), dist(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y)), \\ \quad \frac{1}{2}[dist(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + dist(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))]\} \\ \quad +[\frac{1-k(\omega)}{2k(\omega)}] \max\{dist(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), dist(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y))\} \\ \quad +[\frac{1-k(\omega)}{4k(\omega)}][dist(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + dist(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))], \\ \qquad\qquad if\ y \in \mathcal{D}, \end{cases} \tag{3.8}$$

where $k : \Omega \to (0, 1)$ are measurable mappings such that for all $\omega \in \Omega$;
(b) $\mathcal{D}$ is nonempty and has the property (N);
(c) $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$, i.e., $\mathcal{D}$ is $\mathcal{S}-$invariant;
(d) $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$.
Then $\mathcal{P}_{\mathcal{M}}(x_0) \cap C(\mathcal{S}, \mathcal{T}) \neq \phi$, under each of the conditions of Theorem 3.7. Moreover, if for each $\omega \in \Omega$ and any $x \in \mathcal{M}$, $\mathcal{S}(\omega, x) \in \mathcal{T}(\omega, x)$ implies $\mathcal{S}(\omega, \mathcal{S}(\omega, x)) = \mathcal{S}(\omega, x)$, and $\mathcal{S}$ is $\mathcal{T}$-weakly commuting random operator, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \phi$.

Proof. Let $y \in \mathcal{D}$, then $\mathcal{S}(\omega, y) \in \mathcal{D}$, since $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$ for each $\omega \in \Omega$. Also, if $y \in \partial\mathcal{M}$ and so $\mathcal{T}(\omega, y) \in \mathcal{M}$, since $\mathcal{T}(\omega, \partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Let $z \in \mathcal{T}(\omega, y)$.

$$\|z - x_0\| \leq \mathcal{H}(\mathcal{T}(\omega, y), \mathcal{T}(\omega, x_0) \leq \|\mathcal{S}(\omega, y) - \mathcal{S}(\omega, x_0)\|$$
$$= \|\mathcal{S}(\omega, y) - x_0\| = dist(x_0, \mathcal{M}).$$

Now $z \in \mathcal{M}$ and $\mathcal{S}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$, imply that $z \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega, y)$ is also closest to x_0, so, $\mathcal{T}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$; consequently $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}(\omega, .)$-invariant, that is, $\mathcal{T}(\omega, .) \subseteq \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup$

$\{x_0\}$, so for each $\omega \in \Omega$, we have

$$\begin{aligned}\|\mathcal{S}\mathcal{T}(\omega, y) - x_0\| &= \|\mathcal{S}\mathcal{T}(\omega, y) - \mathcal{S}(\omega, x_0)\| \leq \|\mathcal{T}(\omega, y) - x_0\| \\ &= \|\mathcal{T}(\omega, y) - \mathcal{T}(\omega, x_0)\| \leq \|\mathcal{S}(\omega, y) - \mathcal{S}(\omega, x_0)\| \\ &= \|\mathcal{S}(\omega, y) - x_0\|.\end{aligned}$$

Thus, $\mathcal{S}\mathcal{T}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega, y) \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ and hence $\mathcal{T}(\omega, y) \in \mathcal{D}$. So, $\mathcal{T}$ maps $\mathcal{P}_{\mathcal{M}}(x_0)$ into $CB(\mathcal{P}_{\mathcal{M}}(x_0))$ and $\mathcal{S}(\omega, .)$ is self-map on $\mathcal{D}$. Hence, all the condition of the Theorem 3.7 are satisfied. Thus, there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega))$ for each $\omega \in \Omega$. $\square$

Remark 3.10. *If in Theorem 3.7, $\mathcal{S}(\omega, x) = x$ for all $(\omega, x) \in \Omega \times \mathcal{P}_{\mathcal{M}}(x_0)$, then we get the following random best approximation theorem:*

Corollary 3.11. *Let $\mathcal{M}$ be a subset of normed space of $\mathcal{X}$, $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ and satisfies for all $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$ and $\lambda \in [0, 1]$*

$$\mathcal{H}(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq \begin{cases} \max\{\|x - y\|, dist(x, \mathcal{T}_\lambda(\omega, x)), dist(y, \mathcal{T}_\lambda(\omega, y)), \\ \frac{1}{2}[dist(x, \mathcal{T}_\lambda(\omega, y)) + dist(y, \mathcal{T}_\lambda(\omega, x))]\} \\ +[\frac{1-k(\omega)}{2k(\omega)}] \max\{dist(x, \mathcal{T}_\lambda(\omega, x)), dist(y, \mathcal{T}_\lambda(\omega, y))\} \\ +[\frac{1-k(\omega)}{4k(\omega)}][dist(x, \mathcal{T}_\lambda(\omega, y)) + dist(y, \mathcal{T}_\lambda(\omega, x))] \end{cases} \tag{3.9}$$

where $k : \Omega \to (0, 1)$ are measurable mappings such that for all $\omega \in \Omega$. If $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N) and $\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}-$invariant, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \phi$, under each of the conditions of Theorem 3.7.

Corollary 3.12. *Let $\mathcal{M}$ be a subset of normed space of $\mathcal{X}$, $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $\mathcal{T}(\omega, x_0) = \{x_0\}$, for all $\omega \in \Omega$. Suppose that $\mathcal{T}$ satisfy for all $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$ and $\lambda \in [0, 1]$*

$$\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))$$

$$\leq \begin{cases} \|x-x_0\|, \quad if \quad y = x_0, \\ \max\{\|x-y\|, dist(x,\mathcal{T}_\lambda(\omega,x)), dist(y,\mathcal{T}_\lambda(\omega,y)), \\ \quad \frac{1}{2}[dist(x,\mathcal{T}_\lambda(\omega,y)) + dist(y,\mathcal{T}_\lambda(\omega,x))]\} \\ \quad +[\frac{1-k(\omega)}{2k(\omega)}]\max\{dist(x,\mathcal{T}_\lambda(\omega,x)), dist(y,\mathcal{T}_\lambda(\omega,y))\} \\ \quad +[\frac{1-k(\omega)}{4k(\omega)}][dist(x,\mathcal{T}_\lambda(\omega,y)) + dist(y,\mathcal{T}_\lambda(\omega,x))], \\ \qquad\qquad\qquad if\ y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases} \tag{3.10}$$

where $k : \Omega \to (0,1)$ *are measurable mappings such that for all* $\omega \in \Omega$. *If* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has the property* (N). *Then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \phi$, *under each of the conditions of Theorem 3.11.*

Corollary 3.13. *Let* $\mathcal{M}$ *be a subset of normed space of* $\mathcal{X}$, $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$ *such that* $\mathcal{T}(\omega,.) : \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathcal{X}$ *and* $\mathcal{T}(\omega,x_0) = \{x_0\}$, *for all* $\omega \in \Omega$. *Suppose that* $\mathcal{T}$ *satisfies for all* $x \in \mathcal{D} \cup \{x_0\}$ *and* $\lambda \in [0,1]$

$$\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))$$

$$\leq \begin{cases} \|x-x_0\|, \quad if \quad y = x_0, \\ \max\{\|x-y\|, dist(x,\mathcal{T}_\lambda(\omega,x)), dist(y,\mathcal{T}_\lambda(\omega,y)), \\ \frac{1}{2}[dist(x,\mathcal{T}_\lambda(\omega,y)) + dist(y,\mathcal{T}_\lambda(\omega,x))]\} \\ +[\frac{1-k(\omega)}{2k(\omega)}]\max\{dist(x,\mathcal{T}_\lambda(\omega,x)), dist(y,\mathcal{T}_\lambda(\omega,y))\} \\ +[\frac{1-k(\omega)}{4k(\omega)}][dist(x,\mathcal{T}_\lambda(\omega,y)) + dist(y,\mathcal{T}_\lambda(\omega,x))], \quad if\ y \in \mathcal{D}, \end{cases} \tag{3.11}$$

where $k : \Omega \to (0,1)$ *are measurable mappings such that for all* $\omega \in \Omega$. *If* $\mathcal{D}$ *nonempty and has the property* (N). *Then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \phi$, *under each of the conditions of Theorem 3.7.*

Corollary 3.14. *Let* $\mathcal{M}$ *be a subset of normed space of* $\mathcal{X}$, $\mathcal{S} : \Omega \times \mathcal{M} \to \mathcal{M}$, $\mathcal{T} : \Omega \times \mathcal{M} \to CB(\mathcal{M})$. *Suppose that* $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy for all* $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$ *and* $\lambda \in [0,1]$

$$
\begin{aligned}
&\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y)) \\
&\leq \max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,x_0)\|, dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), \\
&\qquad dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)), \tfrac{1}{2}[dist(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) \\
&\quad + dist(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))]\}
\end{aligned}
\tag{3.12}
$$

where $k:\Omega\to(0,1)$ are measurable mappings such that for all $\omega\in\Omega$. If $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N) and $\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}-$invariant and $\mathcal{S}-$invariant. Then $\mathcal{P}_{\mathcal{M}}(x_0)\cap C(\mathcal{S},\mathcal{T})\neq\phi$, under each of the conditions of Theorem 3.7. Moreover, if for each $\omega\in\Omega$ and any $x\in\mathcal{M}$, $\mathcal{S}(\omega,x)\in\mathcal{T}(\omega,x)$ implies $\mathcal{S}(\omega,\mathcal{S}(\omega,x))=\mathcal{S}(\omega,x)$, and $\mathcal{S}$ is $\mathcal{T}$-weakly commuting random operator, then $\mathcal{P}_{\mathcal{M}}(x_0)\cap\mathcal{F}(\mathcal{S})\cap\mathcal{F}(\mathcal{T})\neq\phi$.

Corollary 3.15. *Let $\mathcal{M}$ be a subset of normed space of $\mathcal{X}$, $\mathcal{T}:\Omega\times\mathcal{M}\to CB(\mathcal{M})$ and satisfies for all $x,y\in\mathcal{P}_{\mathcal{M}}(x_0)$ and $\lambda\in[0,1]$*

$$
\begin{aligned}
&\mathcal{H}(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y)) \\
&\leq \begin{cases} \max\{\|x-x_0\|, dist(x,\mathcal{T}_\lambda(\omega,x)), dist(y,\mathcal{T}_\lambda(\omega,y)), \\ \tfrac{1}{2}[dist(x,\mathcal{T}_\lambda(\omega,y))+dist(y,\mathcal{T}_\lambda(\omega,x))]\} \end{cases}
\end{aligned}
\tag{3.13}
$$

where $k:\Omega\to(0,1)$ are measurable mappings such that for all $\omega\in\Omega$. $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N) and $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}-$invariant. Then $\mathcal{P}_{\mathcal{M}}(x_0)\cap\mathcal{F}(\mathcal{T})\neq\phi$, under each of the conditions of Theorem 3.11.

References

[1] M. A. Al-Thagafi, *Common fixed points and best approximation,* J. Approx. Theory, **85** (1996), 318–320.

[2] I. Beg and N. Shahzad, *Random fixed points of random multivalued operators on polish spaces,* Nonlinear Anal., **20** (1993), 835–847.

[3] I. Beg and N. Shahzad, *Applications of the proximity map to Random fixed points theorems in Hilbert space,* J. Math. Anal. Appl., **19** (1995), 606–613.

[4] I. Beg and N. Shahzad, *On invariant random approximations,* Approx. Theory Appl., **12** (1996), 68–72.

[5] I. Beg and N. Shahzad, *Common random fixed points of noncommuting random operrators,* Random Oper. Stoch. Equ., **7** (1999), 367–372.

[6] I. Beg and N. Shahzad, *An application of a random fixed point theorem to random best approximation,* Arch. Math., **74** (2000), 298–301.

[7] A. T. Bharucha-Reid, *Fixed point theorem in probabilistic analysis,* Bull. Amer. Math. Soc., **82** (1976), 641–645.

[8] B. Brosowski, *Fixpunktsätze in der Approximationstheorie,* Mathematica (Cluj), **11** (1969), 165–220.

[9] W. G. Dotson, *Fixed point theorems for nonexpasive mappings on starshaped subsets of Banach space,* J. London Math. Soc., **4**(2) (1972), 408–410.
[10] L. B. Ćirić, J. S. Ume and S. N. Ješić, *On random coincidence and fixed points for a pair of multivalued and single-valued mappings,* J. Inequal. Appl., **2006** (2006), 1–12
[11] O. Hans, *Reduzierende Zufallige Transformationen,* Czeh. Math. J., **7** (1957), 154–158.
[12] O. Hans, *Random operator equations,* Proc. 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part I, pp. 185–202, University of California Press, Berkeley, 1961.
[13] C. J. Himmerberg, *Measurable relations.* Fund. Math., **87** (1975), 53–72.
[14] N. Hussain, D. O'Regan and R. P. Agarwal, *Common fixed point and invariant approximation results on non-starshaped domain,* Georgian Math. J., **12**(4) (2005), 559–569.
[15] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, Cambridge, 1990.
[16] S. Itoh, *A random fixed point theorem for a multivalued contraction mapping,* Pacific J. Math., **68** (1977), 85–90.
[17] S. Itoh, *Random fixed point theorems with an application to random differential equations in Banach spaces,* J. Math. Anal. Appl., **67** (1979) 261–273.
[18] S. Itoh, *Measurable or condensing multivalued mappings and random fixed point theorems,* Kodai Math. J., **2**(3) (1979), 293–299.
[19] A. R. Khan, F. Akbar, N. Sultana and N. Hussain, *Coincidence and invariant approximation theorems for generalized f-nonexpansive multivalued mappings,* Inter. J. Math. Math. Sci., **2006** (2006), 1–18.
[20] K. Kuratoski and C. Ryll-Nardzewski, *A general theorem on selectors,* Bull Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys., **13** (1965), 397–403.
[21] G. Meinardus, *Invarianze bei Linearen Approximationen,* Arch. Rational Mech. Anal., **14** (1963), 301–303.
[22] H. K. Nashine, *Existence of random best approximation for noncommutative maps,* African Diaspora J. Math., **4**(1) (2007), 47–53.
[23] N. S. Papagergiou, *Random fixed point theorems for measurable multifunctions in Banach spaces,* Proc. Amer. Math., Soc. **97** (1986), 507–514.
[24] N. S. Papagergiou, *Random fixed points and random differential inclusions,* Internat. J. Math. Math. Sci., **11** (1988), 551–560.
[25] S. A. Sahab, M. S. Khan and S. Sessa, *A result in best approximation theory.,* J. Approx. Theory, **55** (1988), 349–351.
[26] S. P. Singh, *An application of a fixed point theorem to approximation theory,* J. Approx. Theory, **25** (1979), 89–90.
[27] S. P. Singh, *Application of fixed point theorems to approximation theory* in: V. Lakshmikantam(Ed.), Applied Nonlinear Analysis, Academic Press, New York, 1979.
[28] N. Shahzad and A. Latif, *A random coincidence point theorem,* J. Math. Anal. Appl., **245**(2) (2000), 633-638.
[29] N. Shahzad and A. Latif, *Deterministic and random coincidence point results for f-nonexpansive maps,* J. Math. Anal. Appl., 323 (2006), 1038–1046
[30] A. Spacek, *Zufallige Gleichungen,* Czech. Math. J., **5** (1955),462–466.
[31] H. K. Xu, *Some random fixed point theorems for condensing and nonexpansive operators,* Proc. Amer. Math. Soc., **110**(2) (1990), 395-400.
[32] K. K. Tan and X. Z. Yuan, *Random fixed point theorems and approximation in cones,* J. Math. Anal. Appl., **185** (1994), 378–390.