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CONTROLLABILITY OF DAMPED SECOND ORDER NONLINEAR IMPULSIVE SYSTEMS

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Abstract. In this paper sufficient conditions for the controllability of second order damped impulsive systems are established. The results are obtained by using the Leray-Schauder alternative and the theory of strongly continuous cosine families.

1. Introduction

In recent years, the study of impulsive control systems has received increasing interest, since dynamical system with impulsive effects have great importance in applied sciences [15, 25]. The controllability problems of first order equation have been extensively studied [2, 6, 16]. In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert first order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine families. The basic results concerning strongly continuous cosine families have been established in [22, 23]. Motivation for second order systems and damped second order differential equations can be found in [3, 4, 8, 9, 12, 17, 21, 24]. The existence and controllability results for second order nonlinear systems with and without impulses has been studied by several authors [1, 5, 7, 11, 13, 19]. Li et al. [16] discussed the controllability of first order impulsive functional differential systems in Banach spaces. Park and Han [18] investigated the

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controllability of second order differential equations, whereas in [20] the controllability problem is discussed for second order nonlinear impulsive systems. The purpose of this work is to study the controllability of damped second order system with impulses. Sufficient conditions are formulated and the results are established by using a fixed point approach and the cosine function theory.

2. Preliminaries

Consider the damped second order nonlinear abstract control system of the form

$$\begin{aligned} x''(t) &= Ax(t) + Gx'(t) + Bu(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \\ &\quad t \in I = [0, T], t \neq t_i, \; i = 1, 2, \dots, n, && (2.1) \\ x(0) &= y_0, \quad x'(0) = y_1, && (2.2) \\ \triangle x(t_i) &= I_i(x(t_i), x'(t_i^-)), \; i = 1, 2, \dots, n, && (2.3) \\ \triangle x'(t_i) &= J_i(x(t_i), x'(t_i^-)), \; i = 1, 2, \dots, n, && (2.4) \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ of bounded linear operators in a Banach space X, G is a bounded linear operator on X. The control function $u(\cdot)$ is given in $L^2(I, U)$, and a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ is a bounded linear operator. $y_0, y_1 \in X, 0 < t_1 < \ldots < t_n < T$ are prefixed numbers, $a(\cdot), b(\cdot), f(\cdot), I_i(\cdot)$ and $J_i(\cdot)$ are appropriate functions and the symbol $\triangle\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

In this section we introduce some basic concepts, definitions, lemmas and hypothesis which are needed to establish our main result. Throughout this paper, X will be a Banach space endowed with a norm $\|\cdot\|$. In what follows we put $t_0 = 0, t_{n+1} = T$ and we denote by $\mathcal{PC}$ the space formed by the functions $x : I \to X$ such that $x(\cdot)$ is continuous at $t \neq t_i, x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i = 1, 2, \dots, n$. It is clear that $\mathcal{PC}$ endowed with the norm $\|x\|_{\mathcal{PC}} = \sup_{t \in I} \|x(t)\|$ is a Banach space. Similarly, $\mathcal{PC}^1$ will be the space of the functions $x(\cdot) \in \mathcal{PC}$ such that $x(\cdot)$ is continuously differentiable on $I \backslash \{t_i : i = 1, \dots, n\}$ and the lateral derivatives $x'_R(t) = \lim_{s \to 0^+} \frac{x(t+s) - x(t^+)}{s}, x'_L(t) = \lim_{s \to 0^-} \frac{x(t+s) - x(t^-)}{s}$ are continuous functions on $[t_i, t_{i+1})$ and $(t_i, t_{i+1}]$ respectively. Next, for $x \in \mathcal{PC}^1$ we represent by $x'(t)$ the left derivative at $t \in (0, T]$ and by $x'(0)$ the right derivative at zero. It is easy to see that $\mathcal{PC}^1$ provided with the norm $\|x\|_{\mathcal{PC}^1} = \|x\|_{\mathcal{PC}} + \|x'\|_{\mathcal{PC}}$ is a Banach space.

For $x \in \mathcal{PC}$, we denote by $\tilde{x_i}, i = 0, 1, \ldots, n$, the unique continuous function $\tilde{x_i} \in C([t_i, t_{i+1}]; X)$ such that

$$\tilde{x_i}\,(t) = \begin{cases} x(t), & \text{for } t \in (t_i, t_{i+1}], \\ x(t_i^+), & \text{for } t = t_i. \end{cases}$$

In this paper we say that the family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators in a Banach space X is called a strongly continuous cosine family iff
i) $C(0) = I$,
ii) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathbb{R}$,
iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.
We denote by $\{S(t), t \in \mathbb{R}\}$ the sine function associated to $\{C(t), t \in \mathbb{R}\}$ which is defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X$ and $t \in \mathbb{R}$.

Assume the following conditions on A.
(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ of bounded linear operators from X into itself and the adjoint operator A^* is densely defined i.e. $\overline{D(A^*)} = X^*$.

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x \bigg|_{t=0}, \; x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|, x \in D(A)$. Moreover, the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1 on R. We know from Kisynski [14], that $E = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$ endowed with the graph norm $\|x\|_E = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|, x \in E$ is a Banach space.

To establish our main theorem, we need the following lemmas and definitions.

Lemma 2.1 (See [23]). *Let $(H1)$ hold. Then*
i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Me^{\omega|t|} \text{ and } |S(t) - S(t^*)| \leq M \left| \int_t^{t^*} e^{\omega|s|} ds \right|, \quad \text{for } t, t^* \in \mathbb{R};$$

ii) $S(t)X \subset E$ and $S(t)E \subset D(A)$, for $t \in \mathbb{R}$;
iii) $\frac{d}{dt} C(t)x = AS(t)x$, for $x \in E$ and $t \in \mathbb{R}$;
iv) $\frac{d^2}{dt^2} C(t)x = AC(t)x$, for $x \in D(A)$ and $t \in \mathbb{R}$.

Lemma 2.2 (See [23]). *Let $(H1)$ hold, let $v : \mathbb{R} \to X$ be such that v is continuously differentiable, and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then q is twice* *continuously differentiable and for* $t \in \mathbb{R}$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t-s)v(s)ds \quad \text{and} \quad q''(t) = Aq(t) + v(t).$$

Lemma 2.3 (Leray-Schauder Alternative [10])**.** *Let S be a convex subset of a Banach space Y and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let*

$$\zeta(F) = \{x \in S : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Definition 2.4. *A function $x \in \mathcal{PC}^1$ is said to be a mild solution of the impulsive abstract cauchy problem* (2.1) − (2.4)*, if the conditions* (2.2) − (2.4) *are satisfied and*

$$\begin{aligned} x(t) &= C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)[Gx'(s) + Bu(s) \\ &\quad + f(s, x(s), x(a(s)), x'(s), x'(b(s)))]ds + \sum_{t_i<t} C(t-t_i)I_i(x(t_i), x'(t_i^-)) \\ &\quad + \sum_{t_i<t} S(t-t_i)J_i(x(t_i), x'(t_i^-)),\ t \in I. \end{aligned}$$

or equivalently

$$\begin{aligned} x(t) &= (C(t) - S(t)G)y_0 + S(t)y_1 + \int_0^t C(t-s)Gx(s)ds \\ &\quad + \int_0^t S(t-s)[Bu(s) + f(s, x(s), x(a(s)), x'(s), x'(b(s)))]ds \\ &\quad + \sum_{t_i<t} C(t-t_i)I_i(x(t_i), x'(t_i^-)) + \sum_{t_i<t} S(t-t_i)J_i(x(t_i), x'(t_i^-)). \end{aligned}$$

Definition 2.5. *The system* (2.1) − (2.4) *is said to be controllable on the interval I, if for every $y_0 \in D(A), y_1 \in E$ and $x_1 \in X$, there exists a control $u \in L^2(I, U)$ such that the mild solution $x(t)$ of* (2.1)−(2.4) *satisfies $x(T) = x_1$.*

Further, we assume the following hypotheses :

(H2) $C(t), t > 0$ is compact.

(H3) The linear operator $W : L^2(I, U) \to X$, defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds,$$

has an induced invertible operator W^{-1}, which takes values in $L^2(I, U)/kerW$ and there exists a positive constant M_1 such that $\|BW^{-1}\| \le M_1$.

(H4) The functions $a(\cdot), b(\cdot) : I \to I$ are continuous and $\max\{a(t), b(t)\} \leq t$ for every $t \in I$.

(H5) The function $f : I \times X^4 \to X$ satisfies the following conditions :
(i) The function $f(t, \cdot) : X^4 \to X$ is continuous a.e. $t \in I$;
(ii) The function $f(\cdot, x) : I \to X$ is strongly measurable for every $x \in X^4$;
(iii) There exist a continuous function $m : I \to [0, \infty)$ and a continuous non-decreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x_1, x_2, x_3, x_4)\| \leq m(t)W(\sum_{i=1}^{4} \|x_i\|), \ t \in I, x_i \in X.$$

(H6) For every positive constant k, there exists $\alpha_k \in L^1(I)$ such that

$$\sup_{\|x_i\| \leq k} \|f(t, x_i)\| \leq \alpha_k(t), \text{ for a.a. } \ t \in I \text{ and } i = 1, 2, 3, 4.$$

(H7) (a) The functions $I_i, J_i : X^2 \to X, i = 1, ..., n$, are continuous.
(b) There exists positive constants $c_i^j, d_i^j, j = 1, 2, i = 1, ..., n$ such that $\|I_i(x, x')\| \leq c_i^1(\|x\| + \|x'\|) + c_i^2, \|J_i(x, x')\| \leq d_i^1(\|x\| + \|x'\|) + d_i^2$, for every $x, x' \in X$.

(H8) The number $\mu = \sum_{i=1}^{n}[(M + N_1)c_i^1 + (N + M)d_i^1] < 1$ and

$$\int_0^T \psi(s)ds < \int_c^\infty \frac{ds}{s + W(2s)},$$

where $\psi(t) = \max\{l_1, l_2, l_3\}$,

$$\begin{aligned}
l_1 &= \frac{1}{1-\mu}(M + N)m(t), \ \ l_2 = \frac{1}{1-\mu}(M + N_1)\|G\|, \\
l_3 &= \frac{1}{1-\mu}\|G\|, \ \ M = \sup\{\|C(t)\| : t \in I\}, \\
N &= \sup\{\|S(t)\| : t \in I\}, \ \ N_1 = \sup\{\|AS(t)\| : t \in I\}, \\
c &= \frac{1}{1-\mu}\Big[(M + N_1 + (M + N)\|G\|)\|y_0\| + (M + N)\|y_1\| \\
&\quad +(N + M)M_2T + \sum_{i=1}^{n}[(M + N_1)c_i^2 + (N + M)d_i^2]\Big],
\end{aligned}$$

$$
\begin{aligned}
M_2 \;=\;& M_1\Big[\|x_1\| + (M + N\|G\|)\|y_0\| + N\|y_1\| + M\|G\|\int_0^T \|x(\tau)\|d\tau \\
& +N\int_0^T m(\tau)W(\|x(\tau)\| + \|x(a(\tau))\| + \|x'(\tau)\| + \|x'(b(\tau))\|)d\tau \\
& +\sum_{i=1}^{n}(Mc_i^1 + Nd_i^1)(\|x(t_i)\| + \|x'(t_i^-)\|) + \sum_{i=1}^{n}(Mc_i^2 + Nd_i^2)\Big].
\end{aligned}
$$

3. Controllability Result

Theorem 3.1. *If the conditions $(H1)-(H8)$ are satisfied, then the system $(2.1)-(2.4)$ is controllable on I.*

Proof. Consider the space $Z = \mathcal{PC}^1([0,T], X)$ with norm

$$
\|x\|^* = \max\left\{\|x\|, \|x'\|\right\}.
$$

Using the assumption $(H3)$, for an arbitrary function $x(\cdot)$, define the control

$$
\begin{aligned}
u(t) \;=\;& W^{-1}\Bigg[x_1 - (C(T) - S(T)G)y_0 - S(T)y_1 - \int_0^T C(T-\tau)Gx(\tau)d\tau \\
& -\int_0^T S(T-\tau)f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))d\tau \\
& -\sum_{i=1}^{n} C(T-t_i)I_i(x(t_i), x'(t_i^-)) - \sum_{i=1}^{n} S(T-t_i)J_i(x(t_i), x'(t_i^-))\Bigg](t)
\end{aligned}
$$

Using this control we shall now show that the operator $P : Z \to Z$ defined by

$$
\begin{aligned}
(Px)(t) \;=\;& (C(t) - S(t)G)y_0 + S(t)y_1 + \int_0^t C(t-s)Gx(s)ds \\
& +\int_0^t S(t-s)f(s, x(s), x(a(s)), x'(s), x'(b(s)))ds \\
& +\sum_{t_i<t} C(t-t_i)I_i(x(t_i), x'(t_i^-)) + \sum_{t_i<t} S(t-t_i)J_i(x(t_i), x'(t_i^-)) \\
& +\int_0^t S(t-s)BW^{-1}\Big[x_1 - (C(T) - S(T)G)y_0 - S(T)y_1 \\
& -\int_0^T S(T-\tau)f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))d\tau \\
& -\int_0^T C(T-\tau)Gx(\tau)d\tau - \sum_{i=1}^{n} C(T-t_i)I_i(x(t_i), x'(t_i^-))
\end{aligned}
$$

$$-\sum_{i=1}^{n} S(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds, \quad t\in I,$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (2.1) – (2.4).

Clearly, $(Px)(T)=x_1$, which means that the control u steers the system from the initial state y_0 to x_1 in time T, provided we can obtain a fixed point of the operator P which implies that the system is controllable.

In order to study the controllability problem for the system (2.1) – (2.4), we have to apply Lemma (2.3) to the following operator equation :

$$z=\lambda P(z), \quad \lambda\in(0,1).$$

Let $\alpha(t)=\sup\limits_{s\in[0,t]}\|x(s)\|$ and $\beta(t)=\sup\limits_{s\in[0,t]}\|x'(s)\|$. Then we have

$$\begin{aligned}
\|x(t)\| \;\le\;& (M+N\|G\|)\|y_0\|+N\|y_1\|+M\|G\|\int_0^t\|x(s)\|ds+N\int_0^t\|BW^{-1}\|\\
&\times\Bigg[\|x_1\|+(M+N\|G\|)\|y_0\|+N\|y_1\|+M\|G\|\int_0^T\|x(\tau)\|d\tau\\
&+N\int_0^T m(\tau)W(\|x(\tau)\|+\|x(a(\tau))\|+\|x'(\tau)\|+\|x'(b(\tau))\|)d\tau\\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|)+\sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg](s)ds\\
&+N\int_0^t m(s)W(\|x(s)\|+\|x(a(s))\|+\|x'(s)\|+\|x'(b(s))\|)ds\\
&+\sum_{t_i<t}(Mc_i^1+Nd_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|)+\sum_{t_i<t}(Mc_i^2+Nd_i^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha(t) \;\le\;& (M+N\|G\|)\|y_0\|+N\|y_1\|+M\|G\|\int_0^t\|x(s)\|ds+NM_2T\\
&+N\int_0^t m(s)W(\|x(s)\|+\|x(a(s))\|+\|x'(s)\|+\|x'(b(s))\|)ds\\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|)+\sum_{i=1}^{n}(Mc_i^2+Nd_i^2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x'(t)\| \leq & (N_1+M\|G\|)\|y_0\| + M\|y_1\| + \|G\|\|x(t)\| + N_1\|G\|\int_0^t \|x(s)\|ds \\
& +M\int_0^t m(s)W(\|x(s)\|+\|x(a(s))\|+\|x'(s)\|+\|x'(b(s))\|)ds \\
& +M\int_0^t \|BW^{-1}\|\Bigg[\|x_1\| + (M+N\|G\|)\|y_0\| + N\|y_1\| \\
& +N\int_0^T m(\tau)W(\|x(\tau)\|+\|x(a(\tau))\|+\|x'(\tau)\|+\|x'(b(\tau))\|)d\tau \\
& +M\|G\|\int_0^T \|x(\tau)\|d\tau + \sum_{i=1}^n (Mc_i^1+Nd_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|) \\
& +\sum_{i=1}^n (Mc_i^2+Nd_i^2)\Bigg](s)ds + \sum_{t_i<t}(N_1c_i^2+Md_i^2) \\
& +\sum_{t_i<t}(N_1c_i^1+Md_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|),
\end{aligned}$$

and hence

$$\begin{aligned}
\beta(t) \leq & (N_1+M\|G\|)\|y_0\| + M\|y_1\| + \|G\|\|x(t)\| + N_1\|G\|\int_0^t \|x(s)\|ds \\
& +M\int_0^t m(s)W(\|x(s)\|+\|x(a(s))\|+\|x'(s)\|+\|x'(b(s))\|)ds \\
& +\sum_{i=1}^n (N_1c_i^1+Md_i^1)(\|x(t_i)\|+\|x'(t_i^-)\|)+\sum_{i=1}^n (N_1c_i^2+Md_i^2) + MM_2T.
\end{aligned}$$

From the assumption on μ and the previous estimates, it follows that

$$\begin{aligned}
\alpha(t)+\beta(t) \leq & c + \frac{1}{1-\mu}\Bigg[\|G\|\|x(t)\| + (M+N_1)\|G\|\int_0^t \|x(s)\|ds + (M+N) \\
& \int_0^t m(s)W(\|x(s)\|+\|x(a(s))\|+\|x'(s)\|+\|x'(b(s))\|)ds\Bigg],
\end{aligned}$$

Let $\gamma(t)=\alpha(t)+\beta(t),\ t\in I$. Then $\gamma(0)=c$ and

$$\begin{aligned}
\gamma'(t) \leq & \frac{1}{1-\mu}\Big[\|G\|\beta(t)+(M+N_1)\|G\|\alpha(t)+(M+N)m(t)W(2\alpha(t)+2\beta(t))\Big] \\
= & \psi(t)\big[\gamma(t)+W(2\gamma(t))\big], \quad t\in I.
\end{aligned}$$

This implies

$$\int_{\gamma(0)}^{\gamma(t)} \frac{ds}{s+W(2s)} \le \int_0^T \psi(s)ds < \int_c^\infty \frac{ds}{s+W(2s)}, \quad t \in I.$$

This inequality implies that there is a constant K such that

$$\alpha(t)+\beta(t)=\gamma(t)\le K, \quad t \in I$$

and hence

$$\|x\|^* = \max\left\{\|x\|, \|x'\|\right\} \le K,$$

where K depends only on T and on the functions m and W.

In the second step we prove that the operator $P : Z \to Z$ is a completely continuous operator. Let $B_k = \{x \in Z : \|x\|^* \le k\}$ for some $k \ge 1$. We first show that P maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in I$, Then if $0 < t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\|(Px)(t_1) - (Px)(t_2)\| \\
&\le \|[C(t_1)-C(t_2)]y_0\| + \|[S(t_1)-S(t_2)]Gy_0\| + \|[S(t_1)-S(t_2)]y_1\| \\
&\quad + \|\int_0^{t_1}[C(t_1-s)-C(t_2-s)]Gx(s)ds\| + \|\int_{t_1}^{t_2} C(t_2-s)Gx(s)ds\| \\
&\quad + \|\int_0^{t_1}[S(t_1-s)-S(t_2-s)]f(s,x(s),x(a(s)),x'(s),x'(b(s)))ds\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2-s)f(s,x(s),x(a(s)),x'(s),x'(b(s)))ds\| \\
&\quad + \|\sum_{0<t_i<t_1}[C(t_1-t_i)-C(t_2-t_i)]I_i(x(t_i),x'(t_i^-))\| \\
&\quad + \|\sum_{t_1\le t_i<t_2} C(t_2-t_i)I_i(x(t_i),x'(t_i^-))\| \\
&\quad + \|\sum_{0<t_i<t_1}[S(t_1-t_i)-S(t_2-t_i)]J_i(x(t_i),x'(t_i^-))\| \\
&\quad + \|\sum_{t_1\le t_i<t_2} S(t_2-t_i)J_i(x(t_i),x'(t_i^-))\| + \|\int_0^{t_1}[S(t_1-s)-S(t_2-s)] \\
&\quad \times BW^{-1}\Bigg[x_1 - (C(T)-S(T)G)y_0 - S(T)y_1 - \int_0^T C(T-\tau)Gx(\tau)d\tau \\
&\quad - \int_0^T S(T-\tau)f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))d\tau
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n} C(T-t_i)I_i(x(t_i),x'(t_i^-)) - \sum_{i=1}^{n} S(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds\|\\
&+\|\int_{t_1}^{t_2} S(t_2-s)BW^{-1}\Bigg[x_1-(C(T)-S(T)G)y_0-S(T)y_1\\
&-\int_0^T S(T-\tau)f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))d\tau\\
&-\int_0^T C(T-\tau)Gx(\tau)d\tau - \sum_{i=1}^{n} C(T-t_i)I_i(x(t_i),x'(t_i^-))\\
&-\sum_{i=1}^{n} S(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds\|\\
\le\ & \|C(t_1)-C(t_2)\|\|y_0\| + \|S(t_1)-S(t_2)\|\|Gy_0\| + \|S(t_1)-S(t_2)\|\|y_1\|\\
&+k\int_0^{t_1} \|C(t_1-s)-C(t_2-s)\|\|G\|ds + k\int_{t_1}^{t_2} \|C(t_2-s)\|\|G\|ds\\
&+\int_0^{t_1} \|S(t_1-s)-S(t_2-s)\|\alpha_k(s)ds + \int_{t_1}^{t_2} \|S(t_2-s)\|\alpha_k(s)ds\\
&+\sum_{0<t_i<t_1} \|C(t_1-t_i)-C(t_2-t_i)\|[c_i^1(\alpha(t)+\beta(t))+c_i^2]\\
&+\sum_{t_1\le t_i<t_2} \|C(t_2-t_i)\|[c_i^1(\alpha(t)+\beta(t))+c_i^2]\\
&+\sum_{0<t_i<t_1} \|S(t_1-t_i)-S(t_2-t_i)\|[d_i^1(\alpha(t)+\beta(t))+d_i^2]\\
&+\sum_{t_1\le t_i<t_2} \|S(t_2-t_i)\|[d_i^1(\alpha(t)+\beta(t))+d_i^2]\\
&+M_1\int_0^{t_1} \|S(t_1-s)-S(t_2-s)\|\Bigg[\|x_1\|+(M+N\|G\|)\|y_0\|\\
&+N\|y_1\|+M\|G\|Tk+N\int_0^T \alpha_k(\tau)d\tau\\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\alpha(t)+\beta(t)) + \sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg]ds\\
&+M_1\int_{t_1}^{t_2} \|S(t_2-s)\|\Bigg[\|x_1\|+(M+N\|G\|)\|y_0\|
\end{aligned}
$$

$$+N\|y_1\| + M\|G\|Tk + N\int_0^T \alpha_k(\tau)d\tau$$
$$+\sum_{i=1}^{n}(Mc_i^1 + Nd_i^1)(\alpha(t)+\beta(t)) + \sum_{i=1}^{n}(Mc_i^2 + Nd_i^2)\Bigg] ds.$$

and similarly

$$\|(Px)'(t_1) - (Px)'(t_2)\|$$
$$\leq \|[AS(t_1) - AS(t_2)]y_0\| + \|[C(t_1) - C(t_2)]Gy_0\| + \|[C(t_1) - C(t_2)]y_1\|$$
$$+\|\int_0^{t_1}[AS(t_1-s) - AS(t_2-s)]Gx(s)ds\| + \|\int_{t_1}^{t_2} AS(t_2-s)Gx(s)ds\|$$
$$+\|\int_0^{t_1}[C(t_1-s) - C(t_2-s)]f(s,x(s),x(a(s)),x'(s),x'(b(s)))ds\|$$
$$+\|\int_{t_1}^{t_2} C(t_2-s)f(s,x(s),x(a(s)),x'(s),x'(b(s)))ds\|$$
$$+\|\sum_{0<t_i<t_1}[AS(t_1-t_i) - AS(t_2-t_i)]I_i(x(t_i),x'(t_i^-))\|$$
$$+\|\sum_{t_1\leq t_i<t_2} AS(t_2-t_i)I_i(x(t_i),x'(t_i^-))\|$$
$$+\|\sum_{0<t_i<t_1}[C(t_1-t_i) - C(t_2-t_i)]J_i(x(t_i),x'(t_i^-))\|$$
$$+\|\sum_{t_1\leq t_i<t_2} C(t_2-t_i)J_i(x(t_i),x'(t_i^-))\| + \|\int_0^{t_1}[C(t_1-s) - C(t_2-s)]$$
$$\times BW^{-1}\Bigg[x_1 - (C(T) - S(T)G)y_0 - \int_0^T C(T-\tau)Gx(\tau)d\tau$$
$$-S(T)y_1 - \int_0^T S(T-\tau)f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))d\tau$$
$$-\sum_{i=1}^{n} C(T-t_i)I_i(x(t_i),x'(t_i^-)) - \sum_{i=1}^{n} C(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds\|$$
$$+\|\int_{t_1}^{t_2} C(t_2-s)BW^{-1}\Bigg[x_1 - (C(T) - S(T)G)y_0 - S(T)y_1$$
$$-\int_0^T S(T-\tau)f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))d\tau$$

$$
\begin{aligned}
&-\int_0^T C(T-\tau)Gx(\tau)d\tau - \sum_{i=1}^{n} C(T-t_i)I_i(x(t_i),x'(t_i^-)) \\
&-\sum_{i=1}^{n} S(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds\| \\
\le\ & \|AS(t_1)-AS(t_2)\|\|y_0\| + \|C(t_1)-C(t_2)\|\|Gy_0\| + \|C(t_1)-C(t_2)\|\|y_1\| \\
&+k\int_0^{t_1} \|AS(t_1-s)-AS(t_2-s)\|\|G\|ds + k\int_{t_1}^{t_2} \|AS(t_2-s)\|\|G\|ds \\
&+\int_0^{t_1} \|C(t_1-s)-C(t_2-s)\|\alpha_k(s)ds + \int_{t_1}^{t_2} \|C(t_2-s)\|\alpha_k(s)ds \\
&+\sum_{0<t_i<t_1} \|AS(t_1-t_i)-AS(t_2-t_i)\|[c_i^1(\alpha(t)+\beta(t))+c_i^2] \\
&+\sum_{t_1\le t_i<t_2} \|AS(t_2-t_i)\|[c_i^1(\alpha(t)+\beta(t))+c_i^2] \\
&+\sum_{0<t_i<t_1} \|C(t_1-t_i)-C(t_2-t_i)\|[d_i^1(\alpha(t)+\beta(t))+d_i^2] \\
&+\sum_{t_1\le t_i<t_2} \|C(t_2-t_i)\|[d_i^1(\alpha(t)+\beta(t))+d_i^2] \\
&+M_1\int_0^{t_1} \|C(t_1-s)-S(t_2-s)\|\Bigg[\|x_1\| + (M+N\|G\|)\|y_0\| \\
&+N\|y_1\| + M\|G\|Tk + N\int_0^T \alpha_k(\tau)d\tau \\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\alpha(t)+\beta(t)) + \sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg]ds \\
&+M_1\int_{t_1}^{t_2} \|C(t_2-s)\|\Bigg[\|x_1\| + (M+N\|G\|)\|y_0\| \\
&+N\|y_1\| + M\|G\|Tk + N\int_0^T \alpha_k(\tau)d\tau \\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\alpha(t)+\beta(t)) + \sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg]ds.
\end{aligned}
$$

The right-hand sides are independent of $x \in B_k$ and tends to zero as $t_1 \to t_2$, since $C(t),\ S(t)$ are uniformly continuous for $t \in I$ and the compactness of

$C(t),\ S(t)$ for $t > 0$ imply the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$. Thus, P maps B_k into an equicontinuous family of functions. It is easy to see that the family PB_k is uniformly bounded.

Next we show $\overline{PB_k}$ is compact. Since we have shown PB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that P maps B_k into a precompact set in X.

Let $0 < t \le T$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}(P_\epsilon x)(t) = {} & (C(t) - S(t)G)y_0 + S(t)y_1 + \int_0^{t-\epsilon} C(t-s)Gx(s)ds \\ & + \int_0^{t-\epsilon} S(t-s)f(s, x(s), x(a(s)), x'(s), x'(b(s)))ds \\ & + \sum_{t_i<t} C(t-t_i)I_i(x(t_i), x'(t_i^-)) + \sum_{t_i<t} S(t-t_i)J_i(x(t_i), x'(t_i^-)) \\ & + \int_0^{t-\epsilon} S(t-s)BW^{-1}\Bigg[x_1 - (C(T) - S(T)G)y_0 - S(T)y_1 \\ & - \int_0^T S(T-\tau)f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))d\tau \\ & - \int_0^T C(T-\tau)Gx(\tau)d\tau - \sum_{i=1}^n C(T-t_i)I_i(x(t_i), x'(t_i^-)) \\ & - \sum_{i=1}^n S(T-t_i)J_i(x(t_i), x'(t_i^-))\Bigg](s)ds, \quad t \in I.\end{aligned}$$

Since $C(t),\ S(t)$ are compact operators, the set $Y_\epsilon(t) = \{(P_\epsilon x)(t) : x \in B_k\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$\begin{aligned}& \|(Px)(t) - (P_\epsilon x)(t)\| \\ & \le \int_{t-\epsilon}^t \|C(t-s)Gx(s)\|ds + \int_{t-\epsilon}^t \|S(t-s)BW^{-1}\Bigg[x_1 \\ & \quad -(C(T) - S(T)G)y_0 - S(T)y_1 - \int_0^T C(T-\tau)Gx(\tau)d\tau \\ & \quad - \int_0^T S(T-\tau)f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))d\tau \\ & \quad - \sum_{i=1}^n C(T-t_i)I_i(x(t_i), x'(t_i^-)) - \sum_{i=1}^n C(T-t_i)J_i(x(t_i), x'(t_i^-))\Bigg](s)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-\epsilon}^{t}\|S(t-s)f(s,x(s),x(a(s)),x'(s),x'(b(s)))\|ds\\
\leq\ & k\int_{t-\epsilon}^{t}\|C(t-s)\|\|G\|ds+\int_{t-\epsilon}^{t}\|S(t-s)\|\alpha_k(s)ds\\
&+M_1\int_{t-\epsilon}^{t}\|S(t-s)\|\Bigg[\|x_1\|+(M+N\|G\|)\|y_0\|\\
&+N\|y_1\|+M\|G\|Tk+N\int_0^T\alpha_k(\tau)d\tau\\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\alpha(t)+\beta(t))+\sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg]ds
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\|(Px)'(t)-(P_\epsilon x)'(t)\|\\
&\leq\int_{t-\epsilon}^{t}\|AS(t-s)Gx(s)\|ds+\int_{t-\epsilon}^{t}\|C(t-s)BW^{-1}\Bigg[x_1\\
&-(C(T)-S(T)G)y_0-S(T)y_1-\int_0^T C(T-\tau)Gx(\tau)d\tau\\
&-\int_0^T S(T-\tau)f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))d\tau\\
&-\sum_{i=1}^{n}C(T-t_i)I_i(x(t_i),x'(t_i^-))-\sum_{i=1}^{n}C(T-t_i)J_i(x(t_i),x'(t_i^-))\Bigg](s)ds\\
&+\int_{t-\epsilon}^{t}\|C(t-s)f(s,x(s),x(a(s)),x'(s),x'(b(s)))\|ds\\
&\leq k\int_{t-\epsilon}^{t}\|AS(t-s)\|\|G\|ds+\int_{t-\epsilon}^{t}\|C(t-s)\|\alpha_k(s)ds\\
&+M_1\int_{t-\epsilon}^{t}\|C(t-s)\|\Bigg[\|x_1\|+(M+N\|G\|)\|y_0\|\\
&+N\|y_1\|+M\|G\|Tk+N\int_0^T\alpha_k(\tau)d\tau\\
&+\sum_{i=1}^{n}(Mc_i^1+Nd_i^1)(\alpha(t)+\beta(t))+\sum_{i=1}^{n}(Mc_i^2+Nd_i^2)\Bigg]ds.
\end{aligned}
$$

Therefore,

$\|(Px)(t)-(P_\epsilon x)(t)\|\to 0$ and $\|(Px)'(t)-(P_\epsilon x)'(t)\|\to 0, \quad \text{as} \quad \epsilon\to 0^+,$

and there are precompact sets arbitrarily close to the set $\{(Px)(t) : x \in B_k\}$ Thus, the set $Y_\epsilon(t) = \{(P_\epsilon x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $P : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer q such that $\|x_n(t)\| \leq q, \|x_n'(t)\| \leq q$ for all n and $t \in I$, so $\|x(t)\| \leq q, \|x'(t)\| \leq q$ and $x, x' \in Z$. By $H(6)$ and $H(7)$, we have

(i) I_i, and J_i, $i = 1, 2, \ldots, n$ are continuous.

(ii) $f(t, x_n(t), x_n(a(t)), x_n'(t), x_n'(b(t))) \to f(t, x(t), x(a(t)), x'(t), x'(b(t)))$ for each $t \in I$ and since

$$\Big\| f(t, x_n(t), x_n(a(t)), x_n'(t), x_n'(b(t))) - f(t, x(t), x(a(t)), x'(t), x'(b(t))) \Big\| \leq 2\alpha_q(t)$$

We have by the dominated convergence theorem

$$\begin{aligned}
&\|Px_n - Px\| \\
&= \sup_{t\in I} \Bigg\| \int_0^t C(t-s)G\big[x_n(s) - x(s)\big]ds \\
&\quad + \int_0^t S(t-s)\big[f(s, x_n(s), x_n(a(s)), x_n'(s), x_n'(b(s))) \\
&\quad - f(s, x(s), x(a(s)), x'(s), x'(b(s)))\big]ds \\
&\quad + \sum_{0<t_i<t} C(t-t_i)\big[I_i(x_n(t_i), x_n'(t_i^-)) - I_i(x(t_i), x'(t_i^-))\big] \\
&\quad + \sum_{0<t_i<t} S(t-t_i)\big[J_i(x_n(t_i), x_n'(t_i^-)) - J_i(x(t_i), x'(t_i^-))\big] \\
&\quad + \int_0^t S(t-s)BW^{-1}\Bigg[\int_0^T C(T-\tau)G\big[x_n(\tau) - x(\tau)\big]d\tau \\
&\quad + \int_0^T S(T-\tau)\big[f(\tau, x_n(\tau), x_n(a(\tau)), x_n'(\tau), x_n'(b(\tau))) \\
&\quad - f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))\big]d\tau \\
&\quad + \sum_{i=1}^n C(T-t_i)\big[I_i(x_n(t_i), x_n'(t_i^-)) - I_i(x(t_i), x'(t_i^-))\big] \\
&\quad + \sum_{i=1}^n S(T-t_i)\big[J_i(x_n(t_i), x_n'(t_i^-)) - J_i(x(t_i), x'(t_i^-))\big]\Bigg](s)ds\Bigg\|
\end{aligned}$$

$$\begin{aligned}
&\leq M\|G\|\int_0^t \|x_n(s)-x(s)\|ds + N\int_0^t \|f(s,x_n(s),x_n(a(s)),x_n'(s),x_n'(b(s)))\\
&\quad -f(s,x(s),x(a(s)),x'(s),x'(b(s)))\|ds\\
&\quad +M\sum_{0<t_i<t}\|I_i(x_n(t_i),x_n'(t_i^-))-I_i(x(t_i),x'(t_i^-))\|\\
&\quad +N\sum_{0<t_i<t}\|J_i(x_n(t_i),x_n'(t_i^-))-J_i(x(t_i),x'(t_i^-))\|\\
&\quad +NM_1\int_0^t\bigg[M\|G\|\int_0^T \|x_n(\tau)-x(\tau)\|d\tau\\
&\quad +N\int_0^T \|f(\tau,x_n(\tau),x_n(a(\tau)),x_n'(\tau),x_n'(b(\tau)))\\
&\quad -f(\tau,x(\tau),x(a(\tau)),x'(\tau),x'(b(\tau)))\|d\tau\\
&\quad +M\sum_{i=1}^{n}\|I_i(x_n(t_i),x_n'(t_i^-))-I_i(x(t_i),x'(t_i^-))\|\\
&\quad +N\sum_{i=1}^{n}\|J_i(x_n(t_i),x_n'(t_i^-))-J_i(x(t_i),x'(t_i^-))\|\bigg]ds\\
&\to 0 \ \text{ as n} \to \infty
\end{aligned}$$

and similarly

$$\begin{aligned}
&\|(Px_n)'-(Px)'\|\\
&=\sup_{t\in I}\bigg\|\int_0^t AS(t-s)G\big[x_n(s)-x(s)\big]ds\\
&\quad +\int_0^t C(t-s)\big[f(s,x_n(s),x_n(a(s)),x_n'(s),x_n'(b(s)))\\
&\quad -f(s,x(s),x(a(s)),x'(s),x'(b(s)))\big]ds\\
&\quad +\sum_{0<t_i<t}AS(t-t_i)\big[I_i(x_n(t_i),x_n'(t_i^-))-I_i(x(t_i),x'(t_i^-))\big]\\
&\quad +\sum_{0<t_i<t}C(t-t_i)\big[J_i(x_n(t_i),x_n'(t_i^-))-J_i(x(t_i),x'(t_i^-))\big]\\
&\quad +\int_0^t C(t-s)BW^{-1}\bigg[\int_0^T C(T-\tau)G\big[x_n(\tau)-x(\tau)\big]d\tau\\
&\quad +\int_0^T S(T-\tau)\big[f(\tau,x_n(\tau),x_n(a(\tau)),x_n'(\tau),x_n'(b(\tau)))
\end{aligned}$$

$$
\begin{aligned}
&-f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))\big]d\tau \\
&+\sum_{i=1}^{n} C(T-t_i)\big[I_i(x_n(t_i), x'_n(t_i^-)) - I_i(x(t_i), x'(t_i^-))\big] \\
&+\sum_{i=1}^{n} S(T-t_i)\big[J_i(x_n(t_i), x'_n(t_i^-)) - J_i(x(t_i), x'(t_i^-))\big]\Bigg](s)ds\Bigg\| \\
\le\ & N_1\|G\|\int_0^t \|x_n(s) - x(s)\|ds \\
&+M\int_0^t \|f(s, x_n(s), x_n(a(s)), x'_n(s), x'_n(b(s))) \\
&-f(s, x(s), x(a(s)), x'(s), x'(b(s)))\|ds \\
&+N_1 \sum_{0<t_i<t} \|I_i(x_n(t_i), x'_n(t_i^-)) - I_i(x(t_i), x'(t_i^-))\| \\
&+M \sum_{0<t_i<t} \|J_i(x_n(t_i), x'_n(t_i^-)) - J_i(x(t_i), x'(t_i^-))\| \\
&+MM_1\int_0^t \Bigg[M\|G\|\int_0^T \|x_n(\tau) - x(\tau)\|d\tau \\
&+N\int_0^T \|f(\tau, x_n(\tau), x_n(a(\tau)), x'_n(\tau), x'_n(b(\tau))) \\
&-f(\tau, x(\tau), x(a(\tau)), x'(\tau), x'(b(\tau)))\|d\tau \\
&+M\sum_{i=1}^{n} \|I_i(x_n(t_i), x'_n(t_i^-)) - I_i(x(t_i), x'(t_i^-))\| \\
&+N\sum_{i=1}^{n} \|J_i(x_n(t_i), x'_n(t_i^-)) - J_i(x(t_i), x'(t_i^-))\|\Bigg]ds \\
\to\ & 0 \ \text{ as n} \to \infty.
\end{aligned}
$$

Thus, P is continuous. This completes the proof that P is completely continuous.

Finally the set $\zeta(P) = \{x \in Z : x = \lambda Px, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently by Leray-Schauder alternative, the operator P has a fixed point in Z. This means that any fixed point of P is a mild solution of (2.1) – (2.4) on I satisfying $(Px)(t) = x(t)$. Thus the system (2.1) – (2.4) is controllable on I. □

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