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# POSITIVE UNBOUNDED SOLUTIONS OF SINGULAR BOUNDARY VALUE PROBLEMS ON THE HALF-LINE

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**Abstract.** In this paper, by constructing a special cone and using fixed point theorem on cone, the author established the existence of at least one positive unbounded solutions for infinite boundary value problems.

## 1. Introduction

In this paper, we are concerned with the existence of positive solutions for the following nonlinear singular boundary value problems (BVP) on the half-line:

$$\begin{cases} (p(t)x'(t))' + \phi(t)f(t,x(t)) = 0, \quad 0 < t < +\infty, \\ ax(0) - b\lim\limits_{t\to 0^+} p(t)x'(t) = y_0 \geq 0, \\ \lim\limits_{t\to +\infty} p(t)x'(t) = k > 0, \end{cases} \tag{1.1}$$

where $a,\ b > 0,\ \ \phi,\ f$ are continuous functions and $\phi$ may be singular at $t = 0$.

Boundary value problems on the half-line have been studied extensively by many authors over the last two decades due to its extensive application in the study of radial solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium[1-4]. In [3], with $p(t) \equiv 1, \phi(t) \equiv 1$, $b = 0$, Ning and Wang discussed the fixed theorem for BVP(1.1) and presented the sufficient conditions for the existence of positive solutions to BVP(1.1) with the superlinearity and sublinearity conditions $f(t,x) = f_1(t,x) + f_2(t,x)$. In [5], Yan et al. studied the equation $x''(t) + \phi(t)f(t,x(t)) = 0$ and proved the positive unbounded solutions by the lower and upper solution technique. In

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this paper, unbounded solutions means the solution $x(t)$ in $[0, +\infty)$ satisfying $\lim_{t\to+\infty} x(t) = +\infty$, which not means the solution blow-up in a finite time, it depicts the solution(the system trajectory) removes far away from origins as time increases by the determinable gradual velocities.

**Lemma 1.1.** *Let $P$ be a positive cone in a real Banach space $E$, $\Omega_1$, $\Omega_2$ are bounded open sets in $E$, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : P \cap \overline{\Omega}_2 \backslash \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied:*

$\|Ax\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_1$; $\quad \|Ax\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_2$

*or*

$\|Ax\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_1$; $\quad \|Ax\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

*then $A$ has at least one fixed points in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

## 2. Preliminaries and some lemmas

Let $p \in C([0, +\infty), \mathbf{R}) \cap C^1(0, +\infty)$, $p(t) > 0$, $t \in (0, +\infty)$, and $\int_0^1 \frac{1}{p(t)} dt < +\infty$, $\int_1^\infty \frac{1}{p(t)} dt = +\infty$.

$$G(t, s) = \begin{cases} \tau_0(s), & 0 \le s \le t < +\infty, \\ \tau_0(t), & 0 \le t \le s < +\infty, \end{cases} \tag{2.1}$$

$$e(t) = k\tau_0(t) + \frac{a}{b}k + \frac{y_0}{a}, \tag{2.2}$$

here $\tau_0(t) = \int_0^t \frac{1}{p(s)} ds + \frac{a}{b}$, then there exists $[a^*, b^*] \subset (0, +\infty)$, $0 < c^* = c^*(a^*, b^*) \le 1$, such that

$$\frac{G(t, s)}{1 + \tau_0(t)} \ge c^* \frac{G(t', s)}{1 + \tau_0(t')}, \quad t \in [a^*, b^*], \quad t',\ s \in [0, +\infty). \tag{2.3}$$

$$\frac{e(t)}{1 + \tau_0(t)} \ge c^* \frac{e(s)}{1 + \tau_0(s)}, \quad t \in [a^*, b^*],\ s \in [0, +\infty). \tag{2.4}$$

This paper, we use the basic space:

$$C_\infty([0, +\infty), \mathbf{R}) = \left\{ x \in C([0, +\infty), \mathbf{R}) : \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1 + \tau_0(t)} < +\infty \right\}.$$

Obviously, for any $x \in C_\infty([0, +\infty), \mathbf{R})$, it is a Banach space with the norm $\|x\|_\infty =: \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+\tau_0(t)}$. Define

$$P = \left\{ x \in C_\infty([0, +\infty), \mathbf{R}),\ x(t) \ge 0,\ t \ge 0,\ \frac{x(t)}{1+\tau_0(t)} \ge c^* \|x\|_\infty,\ t \in [a^*, b^*] \right\}.$$

In evidence, $P$ is a cone of the Banach space $C_\infty([0, +\infty), \mathbf{R})$.

In this paper, we discuss the existence of positive solutions for the BVP(1.1) in the space $C_\infty([0, +\infty), \mathbf{R})$. If $x \in C^2([0, +\infty), \mathbf{R})$ and satisfies BVP (1.1), then $x$ is the solution of BVP(1.1). What's more, If $x$ satisfies $x(t) > 0$ when $t > 0$, then we say $x$ is the positive solution of BVP(1.1). Especially, If $b \neq 0$, the positive solution is certainly unbounded.

## 3. Main results

Let us list the following assumptions:
$(\mathbf{H}_1)$ $f(t,u) \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $\phi \in C((0,+\infty),[0,+\infty))$, there are constants $\lambda_1, \mu_1 (1 < \lambda_1 \leq \mu_1 < +\infty)$, such that

$$c^{\mu_1} f(t,u) \leq f(t,cu) \leq c^{\lambda_1} f(t,u), \quad \forall c \in [0.1],$$

$$c^{\lambda_1} f(t,u) \leq f(t,cu) \leq c^{\mu_1} f(t,u), \quad \forall c \in [1,+\infty).$$

$(\mathbf{H}_2)$ There are constants $\lambda_2, \mu_2 (0 < \lambda_2 \leq \mu_2 < 1)$, such that

$$c^{\mu_2} f(t,u) \leq f(t,cu) \leq c^{\lambda_2} f(t,u), \quad \forall c \in [0.1],$$

$$c^{\lambda_2} f(t,u) \leq f(t,cu) \leq c^{\mu_2} f(t,u), \quad \forall c \in [1,+\infty).$$

$(\mathbf{H}_3)$ $0 < \int_0^\infty (1+\tau_0(s))^{\mu_1} \phi(s) f(s,1) ds < +\infty$.

**Theorem 3.1.** *Assume $(\boldsymbol{H}_1),(\boldsymbol{H}_3)$ hold, when $k$, $y_0$ are sufficient small, BVP(1.1) has at least one positive solution for any $x \in P$. Especially when $y_0 > 0$, BVP(1.1) exists the unbounded positive solutions $x \in C^2([0,+\infty), [0,+\infty))$.*

*Proof.* By $(\mathbf{H}_1)(\mathbf{H}_3)$, let

$$(Ax)(t) = e(t) + \int_0^\infty G(t,s)\phi(s) f(s,x(s)) ds, \quad x \in P, \ t \in [0,+\infty).$$

Obviously the BVP(1.1) has a solution $x$ if and only if $x \in P$ is a fixed point of the operator $A$. In the following, we prove $A : P \to P$ is a completely continuous operator.

First, we prove that $A(P) \subset P$. Clearly, $(Ax)(t) \geq 0$, $t \in [0,+\infty)$. For any $x \in P$, there exists a constant $c > 0$, such that $c\|x\|_\infty \leq 1$, $\frac{1}{c} \geq 1$,then from $(\mathbf{H}_1)$,

$$\begin{aligned} f(t,x(t)) =& f\left(t, \frac{cx(t)}{1+\tau_0(t)} \cdot \frac{1+\tau_0(t)}{c}\right) \leq c^{\lambda_1} \left|\frac{x(t)}{1+\tau_0(t)}\right|^{\lambda_1} f\left(t, \frac{1+\tau_0(t)}{c}\right) \\ \leq& c^{\lambda_1-\mu_1} \|x\|_\infty{}^{\lambda_1} (1+\tau_0(t))^{\mu_1} f(t,1). \end{aligned}$$

Then for any $t \in [0, +\infty)$,

$$\begin{aligned} &\frac{(Ax)(t)}{1+\tau_0(t)} \\ &= \frac{e(t)}{1+\tau_0(t)} + \int_0^\infty \frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds \\ &\le k + \frac{y_0}{a+b} + c^{\lambda_1-\mu_1}\|x\|_\infty{}^{\lambda_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds \\ &< +\infty. \end{aligned} \tag{3.1}$$

Thus $\sup_{t\in[0,+\infty)}\frac{(Ax)(t)}{1+\tau_0(t)} < +\infty$, $(Ax)(t) \in C_\infty([0,+\infty), \mathbf{R})$.
For any $x \in P$, $t \in [a^*, b^*]$, from (2.3)(2.4)

$$\begin{aligned} \frac{(Ax)(t)}{1+\tau_0(t)} &= \frac{e(t)}{1+\tau_0(t)} + \int_0^\infty \frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds \\ &\ge c^*\frac{e(t')}{1+\tau_0(t')} + c^*\int_0^\infty \frac{G(t',s)}{1+\tau_0(t')}\phi(s)f(s,x(s))ds \\ &= c^*\left[\frac{e(t')}{1+\tau_0(t')} + \int_0^\infty \frac{G(t',s)}{1+\tau_0(t')}\phi(s)f(s,x(s))ds\right] \\ &= c^*\frac{(Ax)(t')}{1+\tau_0(t')}, \quad \forall t' \in [0,+\infty). \end{aligned}$$

So $\frac{(Ax)(t)}{1+\tau_0(t)} \ge c^*\|Ax\|_\infty$, $A(P) \in P$.

Next, we prove $A : P \to P$ is a bounded operator, let $M \subset P$ is a bounded set, there exists $N > 0$, for any $x \in M$, we have $\|x\|_\infty \le N$, From(3.1), for any $x \in M, t \in [0, +\infty)$,

$$\begin{aligned} \left|\frac{(Ax)(t)}{1+\tau_0(t)}\right| =& \frac{e(t)}{1+\tau_0(t)} + \int_0^\infty \frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds \\ \le& k + \frac{y_0}{a+b} + c^{\lambda_1-\mu_1}\|x\|_\infty{}^{\lambda_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds \\ \le& k + \frac{y_0}{a+b} + c^{\lambda_1-\mu_1}N^{\lambda_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds, \end{aligned}$$

i.e. $AM$ is bounded in $C_\infty([0,+\infty), \mathbf{R})$, so $A : P \to P$ is a bounded operator. At last, just as [3, 4], by lemma 2.1, (3.1), we can see $A : P \to P$ is completely continuous.

Choose the constant $c_1 > 1$, so as to $c_1c^* \ge 1$, $\frac{1+\tau_0(t)}{c_1} \le \frac{1+\tau_0(b^*)}{c_1} \le 1$, when $t \in [a^*, b^*]$, then for $x \in P$ and $\|x\|_\infty > 1$, $t \in [a^*, b^*]$, we have

$$c_1\frac{x(t)}{1+\tau_0(t)} \ge c_1c^*\|x\|_\infty \ge c_1c^* > 1.$$

From $(\mathbf{H}_1)$,

$$f(t,x(t)) \geq c_1^{\lambda_1-\mu_1}\left|\frac{x(t)}{1+\tau_0(t)}\right|^{\lambda_1}(1+\tau_0(t))^{\mu_1}f(t,1), \quad t\in[a^*,b^*],\ \|x\|_\infty>1.$$

Thus, for any $x\in P$, $\|x\|_\infty>1$, $t\in[a^*,b^*]$,

$$\begin{aligned}\frac{(Ax)(t)}{1+\tau_0(t)} &= \frac{e(t)}{1+\tau_0(t)}+\int_0^\infty \frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds\\ &\geq \int_{a^*}^{b^*}\frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds\\ &\geq c_1^{\lambda_1-\mu_1}c^{*\lambda_1}\|x\|_\infty{}^{\lambda_1}\int_{a^*}^{b^*}\frac{G(t,s)}{1+\tau_0(t)}\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds\\ &\geq c_1^{\lambda_1-\mu_1}c^{*\lambda_1}\|x\|_\infty{}^{\lambda_1}\min_{t\in[a^*,b^*]}\frac{G(t,s)}{1+\tau_0(t)}\int_{a^*}^{b^*}\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds.\end{aligned}$$

In that $\lambda_1>1$, choose

$$R =: \max\left\{2,\left[c_1^{\lambda_1-\mu_1}c^{*\lambda_1}\min_{t\in[a^*,b^*]}\frac{G(t,s)}{1+\tau_0(t)}\int_{a^*}^{b^*}\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds\right]^{\frac{1}{1-\lambda_1}}\right\}$$
$$>1.$$

Then, we obtain

$$\|Ax\|_\infty \geq \|x\|_\infty, \quad \forall x\in P,\ \|x\|_\infty = R. \tag{3.2}$$

On the other hand, get the constant $c_2 (0<c_2\leq 1)$, by $(H_1)$, when $x\in P$, $\|x\|_\infty\leq 1$, $t\in[0,+\infty)$,we can obtain

$$f(t,x(t)) = f\left(t,\frac{c_2x(t)}{1+\tau_0(t)}\cdot\frac{1+\tau_0(t)}{c_2}\right)\leq c_2^{\lambda_1-\mu_1}\|x\|_\infty{}^{\lambda_1}(1+\tau_0(t))^{\mu_1}f(t,1).$$

Thus

$$\begin{aligned}&\frac{(Ax)(t)}{1+\tau_0(t)}\\ &= \frac{e(t)}{1+\tau_0(t)}+\int_0^\infty\frac{G(t,s)}{1+\tau_0(t)}\phi(s)f(s,x(s))ds\\ &\leq k+\frac{y_0}{a+b}+\int_0^\infty p(s)\phi(s)f(s,x(s))ds\\ &\leq k+\frac{y_0}{a+b}+c_2^{\lambda_1-\mu_1}\|x\|_\infty{}^{\lambda_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds.\end{aligned} \tag{3.3}$$

Take

$$r =: \min\left\{\frac{1}{2},\ \left[c_2^{\lambda_1-\mu_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds\right]^{\frac{1}{1-\lambda_1}}\right\} < 1.$$

We can see from (3.3), when $k,\ y_0$ are sufficient small, i.e.,

$$0 \le k + \frac{y_0}{a} \le r - r^{\lambda_1}c_2^{\lambda_1-\mu_1}\int_0^\infty p(s)\phi(s)(1+\tau_0(s))^{\mu_1}f(s,1)ds,$$

we have

$$\|Ax\|_\infty \le \|x\|_\infty, \quad \forall x \in P,\ \|x\|_\infty = r. \tag{3.4}$$

From (3.2), (3.4), Lemma 1.1, we can obtain that the operator $A$ has fixed point $x \in P$ when $a,\ y_0$ are sufficient small, which satisfying $r < \|x\|_\infty < R$. It is easy to see that $x$ is a positive solution of BVP (1.1). □

Similarly, we can get the following theorem.

**Theorem 3.2.** *Assume* $(\boldsymbol{H}_2), (\boldsymbol{H}_3)$ *hold, then for any* $a \ge 0$, $y_0 \ge 0$, *BVP(1.1) has at least one positive solution for any* $x \in P$. *Especially* $y_0 > 0$, *BVP(1.1) exists the unbounded positive solutions* $x \in C^2([0,+\infty),[0,+\infty))$.

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