Nonlinear Functional Analysis and Applications
Vol. 16, No. 2 (2011), pp. 253-271

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON THE CONVERGENCE OF NEWTON–LIKE METHODS USING OUTER INVERSES BUT NOT LIPSCHITZ CONDITIONS

Ioannis K. Argyros[1], Saïd Hilout[2] and Sanjay K. Khattri[3]

[1]Department of Mathematics Sciences, Cameron University
Lawton, OK 73505, USA
e-mail: iargyros@cameron.edu

[2]Laboratoire de Mathématiques et Applications, Poitiers University
Bd. Pierre et Marie Curie, Téléport 2, B.P. 30179
86962 Futuroscope Chasseneuil Cedex, France
e-mail: said.hilout@math.univ-poitiers.fr

[3]Department of Engineering, Stord Haugesund University College
Bjørnsonsgt. 45, N–5528, Haugesund, Norway
e-mail: sanjay.khattri@hsh.no

Abstract. We provide new semilocal convergence results for Newton–like method using outer inverses but no Lipschitz conditions in a Banach space setting. The first is the Kantorovich–type approach, whereas the second uses our new concept of recurrent functions. Comparisons are given between the two techniques. Our results are compared favorably with earlier ones using the information and requiring the same computational cost. Numerical examples are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$\mathcal{Q}\ F(x) = 0, \tag{1.1}$$

where, F is a Fréchet–differentiable operator defined on an open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in Banach space $\mathcal{Y}$, and $\mathcal{Q} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ the space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$.

[0]Received August 27, 2010. Revised December 28, 2010.
[0]2000 Mathematics Subject Classification: 65H10, 65J15, 65G99, 65B05, 65N30, 47H17, 49M15.
[0]Keywords: Newton–like methods, Kantorovich hypothesis, recurrent functions, Banach space, semilocal convergence, majorizing sequences, outer inverses.

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

We shall use the Newton–like method (NLM)

$$x_{n+1} = x_n - A(x_n)^{\#}\, F(x_n) \quad (n \geq 0) \quad (x_0 \in D) \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $A(x_n) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is an approximation of the Fréchet–derivative $F'(x_n)$, and $A(x_n)^{\#}$ denotes an outer inverse of $A(x_n)$, i.e.,

$$A(x_n)^{\#}\, A(x_n)\, A(x_n)^{\#} = A(x_n)^{\#} \qquad (n \geq 0).$$

This general setting includes generalized Newton methods (GNM) for undetermined systems, the Gauss–Newton method (GNM) for nonlinear least–squares problems, a Newton–type method suitable for ill–posed equations, and Newton–type method for solving equations provided that $A(x)^{\#} = A(x)^{-1}$ $(x \in \mathcal{D})$ [1]–[23]. Outer inverses and generalized inverses have been used by several authors in connection with (NLM). A survey of such results can be found in [3] (see also [4], [7], [17]).

The Lipschitz condition

$$\parallel F'(x) - F'(y) \parallel \leq H \ \parallel x - y \parallel \quad \text{for all} \quad x, y \in \mathcal{D} \tag{1.3}$$

is the crucial hypothesis in the convergence analysis of (NLM). However, they are many examples in the litterature, where (1.3) is vilated [3] (see also Example 5.1 in this study). Here, we expand the applicability of (NLM) by considering instead of (1.3) condition

$$\| F(x) - F(y) - F'(x)\,(x-y) \| \leq H \ \| x-y \|^2 \quad \text{for all} \quad x, y \in \mathcal{D}. \tag{1.4}$$

Note that (1.3) implies (1.4) but not necessarily vice versa. Simply, let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, and consider the function

$$F(x) = \frac{H}{2}\, x^2.$$

It turns out that (1.4) can replace (1.3) in all convergence results [14], [17], involving the latter without changing the rest of the hypotheses (see Theorems 3.1, 3.3, 4.4, and Corollary 3.2).

The paper is organized as follows. Section 2 contains some Banach–type perturbations lemmas for outer inverses. A Kantorovich–type semilocal convergence analysis for (NLM) is provided in Section 3. Using our new concept of recurrent functions, we provide in Section 4 a different semilocal convergence than in Section 3. Comparisons between the two techniques are provided. Finally, in Section 5, we present some numerical examples.

2. Preliminaries

In order for us to make the study as self contained as possible, we provide some results on perturbations bounds for outer inverses that can originally be found in [17] (see also [3]). For a comprehensive theory of various inverses in Banach spaces, see [7], [16]. Let A be a linear operator. Then, $\mathcal{N}(A)$, $\mathcal{R}(A)$ denote the null space, and range of A, respectively. We need the following Lemmas. The proofs can be found in [17].

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $A^{\#}$ is a bounded outer inverse of A. Then the following direct sum decomposition hold:*

$$\mathcal{X} = \mathcal{R}(A^{\#}) \oplus \mathcal{N}(A^{\#}\,A),$$

and

$$\mathcal{Y} = \mathcal{N}(A^{\#}) \oplus \mathcal{R}(A\,A^{\#}).$$

Lemma 2.2. *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $A^{\#}$ is a bounded outer inverse of A. Let $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be such that $\| A^{\#}\,(B-A) \| < 1$. Then $B^{\#} := (I + A^{\#}\,(B-A))^{-1}\,A^{\#}$ is a bounded outer inverse of B, with $\mathcal{N}(B^{\#}) = \mathcal{N}(A^{\#})$, and $\mathcal{R}(B^{\#}) = \mathcal{R}(A^{\#})$. Moreover, the following hold:*

$$\| B^{\#} - A^{\#} \| \leq \frac{\| A^{\#}\,(B-A)\,A^{\#} \|}{1 - \| A^{\#}\,(B-A) \|} \leq \frac{\| A^{\#}\,(B-A) \| \, \| A^{\#} \|}{1 - \| A^{\#}\,(B-A) \|},$$

and

$$\| B^{\#}\,A \| \leq \frac{1}{1 - \| A^{\#}\,(B-A) \|}.$$

Lemma 2.3. *Let $A, B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $A^{\#}$, $B^{\#}$ are a bounded outer inverses of A and B, respectively. Then $B^{\#}\,(I - A\,A^{\#}) = 0$ if and only if $\mathcal{N}(A^{\#}) \subset \mathcal{N}(B^{\#})$.*

3. Semilocal analysis of (NLM)

We shall show the following semilocal convergence theorem for (NLM).

Theorem 3.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a Fréchet–differentiable operator. Assume:*

(a) *there exist an approximation $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ of $F'(x)$, an open convex subset $\mathcal{D}_0$ of $\mathcal{D}$, $x_0 \in \mathcal{D}_0$, a bounded outer inverse $A^{\#}$ of $A(x_0) := A$, and constants $\eta > 0$, $K > 0$, $M > 0$, $L > 0$, $\mu \geq 0$, $\ell \geq 0$ such that for all $x, y \in \mathcal{D}_0$, the following hold:*

$$\| A^{\#}\ F(x_0) \| \leq \eta, \tag{3.1}$$

$$\| A^{\#}\ (F(x) - F(y) - F'(y)\,(x - y)) \| \leq \frac{K}{2}\ \| x - y \|^2, \tag{3.2}$$

$$\| A^{\#}\ (F'(x) - A(x)) \| \leq M\ \ \| x - x_0 \| + \mu, \tag{3.3}$$

and

$$\| A^{\#}\ (A(x) - A(x_0)) \| \leq L\ \ \| x - x_0 \| + \ell; \tag{3.4}$$

(b)

$$b = \mu + \ell < 1, \tag{3.5}$$

$$h = \sigma\ \eta \leq \frac{(1-b)^2}{2}, \tag{3.6}$$

where,

$$\sigma := \max\ \{K, M + L\},$$

$$\overline{U}(x_0, t^\star) = \{x \in \mathcal{X} : \| x - x_0 \| \leq t^\star\} \subseteq \mathcal{D}_0,$$

and

$$t^\star = \frac{1 - b - \sqrt{(1-b)^2 - h}}{\sigma}.$$

Then,

(i) *Sequence $\{x_n\}$ $(n \geq 0)$ generated by (NLM) with*

$$A(x_n)^{\#} = [I + A^{\#}\ (A(x_n) - A(x_0))]^{-1}\ A^{\#}$$

is well defined, remains in $U(x_0, t^\star)$ for all $n \geq 0$, and converges to a solution $x^\star$ of equation $A^{\#}\ F(x) = 0$;

(ii) *The solution* $x^\star$ *is unique in* $\tilde{U}(x_0, t^\star) \cap R(A^\#, x_0)$, *where,*

$$\tilde{U}(x_0, t^\star) = \begin{cases} \overline{U}(x_0, t^\star) \cap \mathcal{D}_0 & \text{if} \quad h = \dfrac{(1-b)^2}{2} \\ U(x_0, t^{\star\star}) \cap \mathcal{D}_0 & \text{if} \quad h < \dfrac{(1-b)^2}{2}, \end{cases}$$

$t^{\star\star}$ *is the large zero of function* f *given by*

$$f(s) = \frac{\sigma}{2}\, s^2 - (1-b)\, s + \eta$$

and

$$R(A^\#, x_0) \equiv R(A^\#) + x_0 = \{x + x_0 : \; x \in R(A^\#)\}.$$

Moreover, define function q *by*

$$q(s) = 1 - L\, s - \ell \tag{3.7}$$

and

sequence $\{t_n\}$ $(n \geq 0)$ *by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{f(t_{n+1})}{q(t_{n+1})} \quad (n \geq 0). \tag{3.8}$$

Then, the following estimates hold for all $n \geq 0$

$$\| \, x_{n+1} - x_n \, \| \leq t_{n+1} - t_n, \tag{3.9}$$

and

$$\| \, x_n - x^\star \, \| \leq t^\star - t_n. \tag{3.10}$$

Proof. We shall show using induction on m, that (3.9) holds. Estimate (3.10) will then follow from (3.9) using standard majorization techniques [3], [15].

By the initial conditions, we have

$$\| \, x_1 - x_0 \, \| \leq t_1 - t_0,$$

and (3.9) holds for $m = 0$. Using (3.4), we get:

$$\| \, A^\# \; (A(x_1) - A) \, \| \leq L \; \| \, x_1 - x_0 \, \| + \ell \leq L\; t_1 + \ell \leq L\; t^\star + \ell < 1. \tag{3.11}$$

From perturbation Lemma 2.2, and (3.11), we obtain that $A(x_1)^\# := (I + A^\# \; (A(x_1) - A))^{-1} \; A^\#$ is an outer inverse of $A(x_1)$. Moreover

$$\| \, A(x_1)^\# \; A \, \| \leq (1 - L \; \| \, x_1 - x_0 \, \| - \ell)^{-1} \leq (1 - L\; t_1 - \ell)^{-1},$$

and $\mathcal{N}\,(A(x_1)^\#) = \mathcal{N}\,(A^\#)$. Assume that for $1 \leq m \leq k$:

$$\| \, x_m - x_{m-1} \, \| \leq t_m - t_{m-1},$$

and

$$\mathcal{N}\,(A(x_{m-1})^\#) = \mathcal{N}\,(A^\#).$$

Then

$$\| x_m - x_0 \| \le \sum_{i=1}^{m} (t_i - t_{i-1}) \le t_m - t_0 = t_m, \tag{3.12}$$

and

$$\mathcal{N}\,(A(x_m)^{\#}) = \mathcal{N}\,(A(x_{m-1})^{\#}) = \mathcal{N}\,(A^{\#}). \tag{3.13}$$

Hence, we have by (1.2) and Lemma 2.3:

$$A(x_m)^{\#}\ (I - A(x_{m-1})\ A(x_{m-1})^{\#}) = 0$$

and

$$\begin{aligned}
&x_{m+1} - x_m \\
&= -A(x_m)^{\#}\, F(x_m) \\
&= -A(x_m)^{\#}\, (F(x_m) - A(x_{m-1})\, (x_m - x_{m-1}) \\
&\quad -A(x_{m-1})\, A(x_{m-1})^{\#} F(x_{m-1})) \\
&= -A(x_m)^{\#}\, (F(x_m) - F(x_{m-1}) - A(x_{m-1})\, (x_m - x_{m-1})) \\
&= -A(x_m)^{\#}\, (F(x_m) - F(x_{m-1}) - F'(x_{m-1})\, (x_m - x_{m-1}) \\
&\quad +(F'(x_{m-1}) - A(x_{m-1}))\, (x_m - x_{m-1}))
\end{aligned} \tag{3.14}$$

We have by (3.13), and Lemma 2.3:

$$A(x_m)^{\#}\ (I - A\ A^{\#}) = 0.$$

In view of (3.2)–(3.4), (3.12), and (3.14)

$$\begin{aligned}
&\| x_{m+1} - x_m \| \\
&\le \| A(x_m)^{\#} A \| \Big(\| A^{\#}(F(x_m) - F(x_{m-1}) \\
&\quad -F'(x_{m-1})(x_m - x_{m-1})) \| \\
&\quad + \| A^{\#}(F'(x_{m-1}) - A(x_{m-1})) \| \| x_m - x_{m-1} \| \Big) \\
&\le \frac{1}{1 - Lt_m - \ell} \left(\frac{K}{2} \| x_m - x_{m-1} \|^2 \right. \\
&\quad \left. +(M \| x_{m-1} - x_0 \| + \mu) \| x_m - x_{m-1} \| \right) \\
&\le \frac{1}{1 - Lt_m - \ell} \left(\frac{\sigma}{2} (t_m - t_{m-1}) + M t_{m-1} + \mu \right) (t_m - t_{m-1}) \\
&= t_{m+1} - t_m,
\end{aligned} \tag{3.15}$$

which completes the induction. Hence, we have for any m:

$$\| x_{m+1} - x_m \| \le t_{m+1} - t_m,$$

$$\| A^{\#}\ (A(x_{m+1}) - A) \| \le L\ \ \| x_{m+1} - x_0 \| + \ell \le L\ t_{m+1} + \ell \le L\ t^{\star} + \ell < 1,$$

$$\| x_m - x_0 \| \le t_m \le t^{\star},$$

and $A(x_{m+1})^{\#} := (I + A^{\#}\ (A(x_{m+1}) - A))^{-1}\ A^{\#}$ is an outer inverse of $A(x)$. It follows that $x_m \in U(x_0, t^\star)$, $m \geq 0$, and $\{x_m\}$ converges to a point $x^\star$ in $\overline{U}(x_0, t^\star)$. The point $x^\star$ is a solution of $A^{\#}\ F(x) = 0$. Indeed, by definition

$$A(x_m)^{\#} = (I + A^{\#}\ (A(x_m) - A))^{-1}\ A^{\#}, \quad \text{for all} \quad m,$$

and

$$\begin{aligned} 0 &= \lim_{m \longrightarrow \infty} (I + A^{\#}\, (A(x_m) - A))\, (x_m - x_{m-1}) \\ &= \lim_{m \longrightarrow \infty} A^{\#}\, F(x_m) = A^{\#}\, F(x^\star). \end{aligned}$$

Hence, $x^\star$ solves equation $A^{\#}\ F(x^\star) = 0$.

To show that $x^\star$ is the unique solution of equation (1.1) in $\tilde{U}(x_0, t^\star) \cap R(A^{\#}, x_0)$, let $y^\star \neq x^\star$ such that $y^\star \in \tilde{U}(x_0, t^\star) \cap R(A^{\#}, x_0)$, and $A^{\#}\ F(y^\star) = 0$. Then $y^\star - x^\star \in \mathcal{R}\,(A^{\#})$, and

$$A^{\#}\, A\, (y^\star - x_k) = A^{\#}\, A\, (y^\star - x_0) + A^{\#}\, A\, (x_k - x_0) = y^\star - x_k, \quad \text{for}\, k \geq 0.$$

By Lemma 2.2, we have $\mathcal{R}\,(A(x_m)^{\#}) = \mathcal{R}\,(A^{\#})$, for all $m \geq 0$, so

$$x_{m+1} - x_m = -A(x_m)^{\#}\, F(x_m) \in \mathcal{R}\,(A(x_m)^{\#}) = \mathcal{R}\,(A^{\#}).$$

Furthemore, using Lemma 2.1, we have $\mathcal{R}\,(A^{\#}) = \mathcal{R}\,(A^{\#}\, A)$, and $x_{m+1} \in x_m + \mathcal{R}\,(A^{\#})$, for all $m \geq 0$. We also have the estimate

$$\begin{aligned} \| y^\star - x_1 \| &= \| y^\star - x_0 + A^{\#}\, F(x_0) - A^{\#}\, F(y^\star) \| \\ &\leq \| A^{\#}\, (F(y^\star) - F(x_0) - F'(x_0)\, (y^\star - x_0)) \| \\ &\quad + \| A^{\#}\, (F'(x_0) - A) \| \, \| y^\star - x_0 \| \\ &\leq \left(\frac{\sigma}{2} \| y^\star - x_0 \| + \ell + \mu \right) \| y^\star - x_0 \| \\ &= \phi(\| y^\star - x_0 \|), \end{aligned} \tag{3.16}$$

where,

$$\phi(s) = \frac{\sigma}{2}\, s^2 + (\ell + \mu)\, s.$$

Since

$$\| y^\star - x_0 \| \leq \| y^\star - x_1 \| + \| x_1 - x_0 \| \leq \phi(\| y^\star - x_0 \|) + \eta$$

then, $\phi(\| y^\star - x_0 \|) \geq 0$. Consequently, $y^\star \in \overline{U}(x_0, t^\star)$.

We prove by induction that

$$\| y^\star - x_m \| \leq t^\star - t_m, \quad \text{for} \quad m \geq 0. \tag{3.17}$$

Inequality (3.17) holds for $m = 0$ since $y^\star \in \overline{U}(x_0, t^\star)$. Suppose that (3.17) holds for m. As in (3.14) and (3.15), we have the estimation:

$$\begin{aligned}
&\| y^\star - x_{m+1} \| \\
&\leq \| A(x_m)^{\#}\, A \| \Bigg(\| A^{\#}\,(F(y^\star) - F(x_m) - F'(x_m)\,(y^\star - x_m)) \| \\
&\quad + \| A^{\#}\,(F'(x_m) - A(x_m)) \| \, \| y^\star - x_m \| \Bigg) \\
&\leq (1 - \ell - L t_m)^{-1} \Bigg(\frac{K}{2} \| y^\star - x_m \|^2 \\
&\quad + (M \| x_m - x_0 \| + \mu) \| y^\star - x_m \| \Bigg) \\
&\leq (1 - \ell - L t_m)^{-1} \Big(\frac{\sigma}{2}(t^\star - t_m) + M t_m + \mu \Big)(t^\star - t_m) \\
&\leq t^\star - t_{m+1}.
\end{aligned} \tag{3.18}$$

which complete the induction for (3.17). It follows by (3.18) that $\lim_{m \longrightarrow \infty} x_m = y^\star$. But we showed $\lim_{m \longrightarrow \infty} x_m = x^\star$. Hence, we deduce $x^\star = y^\star$. That completes the proof of Theorem 3.1. □

Corollary 3.2. *Under the hypotheses of Theorem 3.1 with $A(x) = F'(x)$ $(x \in \mathcal{D}_0)$, the method (1.2) with $A(x_n) := F'(x_n)$ $(n \geq 0)$ converges quadratically to a solution $x^\star \in \tilde{U}(x_0, t^\star) \cap R(A^{\#}, x_0)$ of equation $F'(x_0)^{\#}\, F(x) = 0$.*

Proof. Hypothese (3.1)–(3.6) of Theorem 3.1 become (for $A(x) = F'(x)$, $x \in \mathcal{D}_0$):

$$\| F'(x_0)^{\#}\, F(x_0) \| \leq \eta,$$

$$\| F'(x_0)^{\#}\,(F(x) - F(y) - F'(y)\,(x - y)) \| \leq \frac{K}{2} \; \| x - y \|^2, \tag{3.19}$$

$$\| F'(x_0)^{\#}\,(F'(x) - F'(x_0)) \| \leq K_0 \;\; \| x - x_0 \|, \tag{3.20}$$

$$K\,\eta \leq \frac{1}{2}, \tag{3.21}$$

$$U(x_0, t^\star) \subseteq \mathcal{D}_0,$$

and

$$t^\star = \frac{1 - \sqrt{1 - K\,\eta}}{K}.$$

Using the uniqueness part of the proof of Theorem 3.1, and (3.18), we have $y^\star = x^\star$, and

$$
\begin{aligned}
& \| x^\star - x_{m+1} \| \\
& \leq \| F'(x_m)^{\#} \, F'(x_0) \| \\
& \quad \times \| F'(x_0)^{\#} \, (F(x^\star) - F(x_m) - F'(x_m)\,(x^\star - x_m)) \| \\
& \leq (1 - K_0 \, t^\star)^{-1} \, \frac{K}{2} \, \| x^\star - x_m \|^2 \, .
\end{aligned} \tag{3.22}
$$

That completes the proof of Corollary 3.2. □

We now state a generalization of an affine invariant version of Mysovskii–type theorem. The proof as similar to one in Theorem 3.1 is omitted (see also [17]).

Theorem 3.3. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a Fréchet–differentiable operator. Assume there exist an approximation $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ of $F'(x)$, an open convex subset $\mathcal{D}_0$ of $\mathcal{D}$, $x_0 \in \mathcal{D}_0$, a bounded outer inverse $A^{\#}$ of $A(x_0) := A$, and constants $\eta > 0$, $K > 0$, such that for all $x, y \in \mathcal{D}_0$, the following hold:*

$$
\mathcal{N}(A(x)^{\#}) = \mathcal{N}(A^{\#}), \qquad \| A^{\#} \; F(x_0) \| \leq \eta,
$$

$$
\| A^{\#} \; (F(x) - F(y) - F'(y)\,(x-y)) \| \leq \frac{K}{2} \; \| x - y \|^2,
$$

$$
h \equiv \frac{1}{2} \, K \, \eta \leq 1,
$$

and

$$
\overline{U}(x_0, r) \subseteq \mathcal{D}_0,
$$

where,

$$
r = \frac{\eta}{1-h}.
$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (NLM) with $\mathcal{N}(A(x_k)^{\#}) = \mathcal{N}(A^{\#})$ is well defined, remains in $U(x_0, r)$ for all $n \geq 0$, and converges to a solution $x^\star$ of equation $A^{\#} \, F(x) = 0$.

Remark 3.4.

(i) *If (3.2) is replaced by stronger condition*

$$
\| A^{\#} \; (F'(x) - F'(y)) \| \leq K \; \| x - y \|, \quad \text{for all} \quad x, y \in \mathcal{D}_0, \tag{3.23}
$$

then, our Theorem 3.1 reduces to [17, Theorem 3.1 p. 141, 142]. *Otherwise it constitues an improvement, since (3.23) implies (3.2), but not vice versa.*

(ii) *Condition (3.20) is only used to show (3.22). However, in the next section, we shall show that condition (3.20) in combination with (3.2) can be used to generate more precise majorizing sequences than Theorem 3.1, and weaker sufficient convergence conditions than in Theorem 3.1 and Corollary 3.2 provided that $K_0 < K$.*

(iii) *Condition (3.2) can be replaced by*

$$\| A^{\#}\ (F(x) - F(y) - F'(y)\,(x-y)) \| \leq K_1\ \| F'(y)\,(x-y) \|\, \| x-y \|, \quad (3.24)$$

and

$$\| F(y) \| \leq K_2, \quad (3.25)$$

for all $x, y \in \mathcal{D}_0$, *(see* [14]*),*

or

$$\| A^{\#}(F(x) - F(y) - F'(y)(x-y)) \| \leq K_3 \| F(x) - F(y) \| \| x-y \|, \quad (3.26)$$

and

$$\| F(x) - F(y) \| \leq K_4, \quad (3.27)$$

for all $x, y \in \mathcal{D}_0$.

In the case of conditions (3.24), and (3.25), we can set $K = 2\,K_1\,K_2$, *wheras when (3.26), and (3.27) hold, we let* $K = 2\,K_3\,K_4$.

(iv) *If Lipschitz–type condition of Theorem 3.3 is replaced by stronger*

$$\| A^{\#}\ (F'(x + t\,(y-x)) - F'(x)) \| \leq K\ t\ \| x-y \|,$$

for all $x, y \in \mathcal{D}_0$, *and* $t \in [0,1]$, *then Theorem 3.3 reduces to* [17, Theorem 3.2, p. 247].

4. Semilocal convergence for (GNLM)

We shall consider the more general equation

$$A^{\#}\,(F(x) + G(x)) = 0, \quad (4.1)$$

where, $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a continuous operator.

The corresponding general Newton–like method (GNLM) to (4.1) is given by

$$x_{n+1} = x_n - A(x_n)^{\#}\,(F(x_n) + G(x_n)) \quad (n \geq 0) \quad (x_0 \in D). \quad (4.2)$$

We use our new concept of recurrent function to study the semilocal convergence of (GNLM). First, we need the following results on the convergnce of majorizing sequences for (GNLM).

Lemma 4.1. ([5]) *Assume there exist constants* $K > 0$, $M > 0$, $\mu \geq 0$, $L > 0$, *and* $\eta > 0$, *such that:*

$$2\ M < K; \quad (4.3)$$

Quadratic polynomial f_1 *given by*

$$f_1(s) = 2\ L\ \eta\ s^2 - \bigg(2\ (1 - L\ \eta) - K\ \eta\bigg)\ s + 2\ (M\ \eta + \mu), \quad (4.4)$$

has a root in $(0,1)$*, denoted by* $\dfrac{\delta}{2}$*,*

and

for

$$\delta_0 = \frac{K\ \eta + 2\ \mu}{1 - L\ \eta}, \tag{4.5}$$

$$\alpha = \frac{2\ (K - 2\ M)}{K + \sqrt{K^2 + 8\ L\ (K - 2\ M)}}, \tag{4.6}$$

the following holds

$$\delta_0 \le \delta \le 2\ \alpha. \tag{4.7}$$

Then, scalar sequence $\{t_n\}$ $(n \ge 0)$ *given by*

$$\begin{aligned} &t_0 = 0, \quad t_1 = \eta, \\ &t_{n+2} = t_{n+1} + \frac{K\ (t_{n+1} - t_n) + 2\ (M\ t_n + \mu)}{2\ (1 - L\ t_{n+1})}\ (t_{n+1} - t_n) \end{aligned} \tag{4.8}$$

is increasing, bounded from above by

$$t^{\star\star} = \frac{2\ \eta}{2 - \delta}, \tag{4.9}$$

and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$*. Moreover the following estimates hold for all* $n \ge 1$*:*

$$t_{n+1} - t_n \le \frac{\delta}{2}\ (t_n - t_{n-1}) \le \left(\frac{\delta}{2}\right)^n\ \eta, \tag{4.10}$$

and

$$t^\star - t_n \le \frac{2\ \eta}{2 - \delta}\ \left(\frac{\delta}{2}\right)^n. \tag{4.11}$$

Remark 4.2. *The hypotheses of Lemma 4.1 have been left as uncluttered as possible. Note that these hypotheses involve only computations only at the initial point* x_0*. Next, we shall provide some simpler but stronger hypotheses under which the hypotheses of Lemma 4.1 hold.*

Lemma 4.3. ([5]) *Let* $K > 0$*,* $M > 0$*,* $\mu > 0$*, with* $L > 0$*, and* $\eta > 0$*, be such that:*

$$\mu < \alpha, \quad 2\ M < K,$$

and

$$0 < h_A = a\ \eta \le \frac{1}{2}, \tag{4.12}$$

where,

$$a = \frac{1}{4\,(\alpha-\mu)}\ \max\{2\ L\ \alpha^2 + 2\ L\ \alpha + K\ \alpha + 2\ M,\ K + 2\ \alpha\ L\}. \tag{4.13}$$

Then, the following hold:

$$f_1 \text{ has a positive root } \frac{\delta}{2},$$

$$\max\{\delta_0,\ \delta\} \le 2\ \alpha,$$

and

the conclusions of Lemma 4.1 hold, with α *replacing* $\frac{\delta}{2}$.

We show the semilocal convergence theorem for (GNLM) using outer inverses.

Theorem 4.4. *Let* $F\ :\ \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ *be a Fréchet–differentiable operator, and* $G : \mathcal{D} \to \mathcal{Y}$ *a continuous operator. Assume there exist an approximation* $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *of* $F'(x)$*, an open convex subset* $\mathcal{D}_0$ *of* $\mathcal{D}$*,* $x_0 \in \mathcal{D}_0$*, a bounded outer inverse* $A^{\#}$ *of* $A(x_0) := A$*, and constants* $\eta > 0$*,* $K > 0$*,* $M > 0$*,* $L > 0$*,* $\mu_0, \mu_1 \ge 0$*,* $\ell \ge 0$ *such that for all* $x, y \in \mathcal{D}_0$*:*

$$\parallel A^{\#}\, F(x_0) \parallel \le \eta, \tag{4.14}$$

$$\parallel A^{\#}\ (F(x) - F(y - F'(y)\,(x-y)) \parallel \le \frac{K}{2}\ \parallel x - y \parallel^2, \tag{4.15}$$

$$\parallel A^{\#}\ (F'(x) - A(x)) \parallel \le M\ \parallel x - x_0 \parallel + \mu_0, \tag{4.16}$$

$$\parallel A^{\#}\ (G(x) - G(y)) \parallel \le \mu_1\ \parallel x - y \parallel, \tag{4.17}$$

$$\parallel A^{\#}\, (A(x) - A(x_0))] \parallel \le L\ \parallel x - x_0 \parallel + \ell, \tag{4.18}$$

and Hypotheses of Lemmas 4.1 or 4.3. Then, sequence $\{x_n\}$ $(n \ge 0)$ *generated by (GNLM) with*

$$A(x_n)^{\#} = [I + A^{\#}\ (A(x_n) - A(x_0))]^{-1}\ A^{\#} \tag{4.19}$$

is well defined, remains in $U(x_0, t^\star)$ *for all* $n \ge 0$*, and converges to a solution* $x^\star$ *of equation* $A^{\#}\, F(x) = 0$*. Moreover, the following estimates hold for all* $n \ge 0$

$$\parallel x_{n+1} - x_n \parallel \le t_{n+1} - t_n, \tag{4.20}$$

and

$$\parallel x_n - x^\star \parallel \le t^\star - t_n, \tag{4.21}$$

where, $\{t_n\}$ *is given by (4.8), with* $\mu = \mu_0 + \mu_1$*.*

Furthemore, the solution $x^\star$ *of equation (4.1) is unique in* $\overline{U}(x_0, t^\star)$ *provided that*

$$\left(\frac{K}{2} + M + L\right) t^\star + \mu + \ell < 1. \tag{4.22}$$

Proof. We shall show using induction on m, that (4.20) holds. Estimate (4.21) will then follow from (3.9) using standard majorization techniques [3], [15]. By the initial conditions, we have

$$\| x_1 - x_0 \| \leq t_1 - t_0,$$

and (4.20) holds for $m = 0$. Using (4.18), we get:

$$\| A^{\#} (A(x_1) - A) \| \leq L \ \| x_1 - x_0 \| + \ell \leq L\, t_1 + \ell \leq L\, t^\star + \ell < 1. \quad (4.23)$$

From Lemma 2.2, and (4.23), we obtain that $A(x_1)^{\#} := (I + A^{\#}\ (A(x_1) - A))^{-1}\ A^{\#}$ is an outer inverse of $A(x_1)$. Moreover

$$\| A(x_1)^{\#}\ A \| \leq (1 - L\ \ \| x_1 - x_0 \| - \ell)^{-1} \leq (1 - L\ t_1 - \ell)^{-1},$$

and $\mathcal{N}\,(A(x_1)^{\#}) = \mathcal{N}\,(A^{\#})$. Assume that for $1 \leq m \leq k$:

$$\| x_m - x_{m-1} \| \leq t_m - t_{m-1},$$

and

$$\mathcal{N}\,(A(x_{m-1})^{\#}) = \mathcal{N}\,(A^{\#}).$$

Then

$$\| x_m - x_0 \| \leq t_m - t_{m-1},$$

and

$$\mathcal{N}\,(A(x_m)^{\#}) = \mathcal{N}\,(A(x_{m-1})^{\#}) = \mathcal{N}\,(A^{\#}).$$

Hence, we have by (4.2), and Lemma 2.3:

$$A(x_m)^{\#}\ (I - A(x_{m-1})\ A(x_{m-1})^{\#}) = 0$$

and

$$\begin{aligned} & x_{m+1} - x_m \\ & = -A(x_m)^{\#}(F(x_m) + G(x_m)) \\ & = -A(x_m)^{\#}\Big((F(x_m) - F(x_{m-1}) - F'(x_{m-1})(x_m - x_{m-1})) \\ & \quad + (F'(x_{m-1}) - A(x_{m-1}))(x_m - x_{m-1}) + (G(x_m) - G(x_{m-1}))\Big), \end{aligned} \quad (4.24)$$

We have by Lemma 2.3:

$$A(x_m)^{\#}\ (I - A\ A^{\#}) = 0.$$

In view of hypotheses of Thoerem and (4.24)

$$\begin{aligned}
&\| x_{m+1} - x_m \| \\
&\leq \| A(x_m)^{\#} A \| \Big\{ (F(x_m) - F(x_{m-1}) - F'(x_{m-1})\,(x_m - x_{m-1})) \\
&\quad + \| A^{\#}\,(F'(x_{m-1}) - A(x_{m-1})) \| \| x_m - x_{m-1} \| \\
&\quad + \| A^{\#}\,(G(x_m) - G(x_{m-1})) \| \Big\} \\
&\leq \frac{1}{1 - L\ t_m - \ell} \left(\frac{K}{2}\ \| x_m - x_{m-1} \|^2 \right. \\
&\quad \left. + (M\ \| x_{m-1} - x_0 \| + \mu)\ \| x_m - x_{m-1} \| \right) \\
&\leq \frac{1}{1 - L\ t_m - \ell} \left(\frac{K}{2}\,(t_m - t_{m-1}) + M\,t_{m-1} + \mu \right) (t_m - t_{m-1}) \\
&= t_{m+1} - t_m,
\end{aligned} \tag{4.25}$$

which completes the induction. Hence, we have for any m:

$$\| x_{m+1} - x_m \| \leq t_{m+1} - t_m,$$

$$\| A^{\#}\ (A(x_{m+1}) - A) \| \leq L\ \ \| x_{m+1} - x_0 \| + \ell \leq L\ t_{m+1} + \ell \leq L\ t^{\star} + \ell < 1,$$

$$\| x_m - x_0 \| \leq \sum_{i=1}^{m} (t_i - t_{i-1}) \leq t_m - t_0 = t_m,$$

and $A(x_{m+1})^{\#} := (I + A^{\#}\ (A(x_{m+1}) - A))^{-1}\ A^{\#}$ is an outer inverse of $A(x)$. It follows that $x_m \in U(x_0, t^{\star})$, $m \geq 0$, and $\{x_m\}$ converges to a point $x^{\star}$ in $\overline{U}(x_0, t^{\star})$. The point $x^{\star}$ is a solution of $A^{\#}\ (F(x) + G(x)) = 0$. Indeed, by definition

$$A(x_m)^{\#} = (I + A^{\#}\ (A(x_m) - A))^{-1}\ A^{\#}, \quad \text{for all} \quad m,$$

and

$$\begin{aligned}
0 &= \lim_{m \longrightarrow \infty} (I + A^{\#}\ (A(x_m) - A))\ (x_m - x_{m-1}) \\
&= \lim_{m \longrightarrow \infty} A^{\#}\ (F(x_m) + G(x_m)) = A^{\#}\ (F(x^{\star}) + G(x^{\star})).
\end{aligned}$$

Hence, $x^{\star}$ solves equation $A^{\#}\ (F(x^{\star}) + G(x^{\star})) = 0$.

Finally to show that $x^{\star}$ is the unique solution of equation (1.1) in $\overline{U}(x_0, t^{\star})$, as in (3.14) and (3.15), we get in turn for $y^{\star} \in \overline{U}(x_0, t^{\star})$, with $A^{\#}\ (F(y^{\star}) +$

$G(y^\star)) = 0$, the estimation:

$$\begin{aligned}
&\| y^\star - x_{m+1} \| \\
&\leq \| A(x_m)^{\#} A \| \Big\{ (F(y^\star) - F(x_m) - F'(x_m)\,(y^\star - x_m)) \\
&\quad + \| A^{\#}(F'(x_m) - A(x_m)) \| \| y^\star - x_m \| \\
&\quad + \| A^{\#}(G(x_m) - G(y^\star)) \| \Big\} \\
&\leq (1 - \ell - L\, t_m)^{-1} \left(\frac{K}{2} \| y^\star - x_m \|^2 \right. \\
&\quad \left. + (M \, \| x_m - x_0 \| + \mu) \, \| y^\star - x_m \| \right) \\
&\leq (1 - \ell - L\, t_m)^{-1} \left(\frac{K}{2} (t^\star - t_m) + M\, t_m + \mu \right) \| y^\star - x_m \| \\
&\leq (1 - \ell - L\, t^\star)^{-1} \left(\frac{K}{2} (t^\star - t_0) + M\, t^\star + \mu \right) \| x^\star - x_m \| \\
&< \| y^\star - x_m \|,
\end{aligned} \tag{4.26}$$

by the uniqueness hypothesis (4.22). It follows by (4.26) that $\lim_{m \longrightarrow \infty} x_m = y^\star$. But we showed $\lim_{m \longrightarrow \infty} x_m = x^\star$. Hence, we deduce $x^\star = y^\star$. That completes the proof of Theorem 4.4. □

Remark 4.5. (i) *The point $t^\star$ can be replaced by $t^{\star\star}$, given in closed form by (4.9) in all hypotheses of Theorem 4.4.*

(ii) *If (4.15) is replaced by stronger condition*

$$\| A^{\#} \, (F'(x) - F'(y)) \| \leq K \, \| x - y \|, \quad \text{for all} \quad x, y \in \mathcal{D}_0,$$

then, our Theorem 4.4 reduces to [17, Theorem 2.4].

(iii) *If $G = 0$, majorizing sequence $\{t_n\}$ given by (4.8) is finer than $\{t_n\}$ given in Theorem 3.1 (see also* [5, Proposition 2.7, p. 123] *for the proof).*

(iv) *If $A(x)^{\#} = A^{\#}$ ($x_0 \in \mathcal{D}_0$), and $G = 0$ (Newton's method), then Kantorovich hypothesis (3.21) can be replaced by weaker (if $K_0 < K$)*

$$h_K = \overline{K}\, \eta \leq \frac{1}{2}, \tag{4.27}$$

where,

$$\overline{K} = \frac{1}{8} \left(K + 4\, K_0 + \sqrt{K^2 + 8\, K\, K_0} \right),$$

and the constant K_0 satisfies the center–Lipschitz condition

$$\| A^{\#} \, (F'(x) - F'(x_0)) \| \leq K \, \| x - x_0 \|, \quad \text{for all} \quad x \in \mathcal{D}_0. \tag{4.28}$$

Note also that

$$K_0 \le K \tag{4.29}$$

holds in general, and $\dfrac{K}{K_0}$ *can be arbitrarily large* [2]–[5].

5. Examples

In this section, we provide some examples. For simplicity, we set $A(x) = F'(x)$, $A^{\#}(x) = A(x)^{-1}$, and $G(x) = 0$ $(x \in \mathcal{D}_0)$.

Example 5.1. *We consider the integral equation*

$$u(s) = f(s) + \varrho \int_a^b \mathcal{G}(s,t)\, u(t)^{1+\frac{1}{n}}\, dt, \quad n \in \mathbb{N}.$$

Here, f is a given continuous function satisfying $f(s) > 0$, $s \in [a,b]$, ϱ is a real number, and the kernel $\mathcal{G}$ is continuous and positive in $[a,b] \times [a,b]$. For example, when $\mathcal{G}(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} &u'' = \varrho\, u^{1+\frac{1}{n}}, \\ &u(a) = f(a),\ u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [3].

We consider F as follows

$$F\colon \mathcal{D} \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b], \quad \mathcal{D} = \{u \in \mathcal{C}[a,b]\colon u(s) \ge 0,\ s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \varrho \int_a^b \mathcal{G}(s,t)\, u(t)^{1+\frac{1}{n}}\, dt. \tag{5.1}$$

$\mathcal{C}[a,b]$ is equipped with the max–norm. The derivative F' is given by

$$F'(u)\, v(s) = v(s) - \varrho \left(1 + \frac{1}{n}\right) \int_a^b \mathcal{G}(s,t)\, u(t)^{\frac{1}{n}} v(t)\, dt, \quad v \in \mathcal{D}. \tag{5.2}$$

First, note that F' does not satisfy a Lipschitz condition (3.23) in $\mathcal{D}$. Let us consider, for instance, $[a,b] = [0,1]$, $\mathcal{G}(s,t) = 1$ and $y(t) = 0$. Then $F'(y)\, v(s) = v(s)$, and

$$\| F'(x) - F'(y) \| = |\varrho| \left(1 + \frac{1}{n}\right) \int_0^1 x(t)^{\frac{1}{n}}\, dt. \tag{5.3}$$

We suppose that F' is a Lipschitz function, then

$$\| F'(x) - F'(y) \| \le K \ \| x - y \| .$$

Consequently, we obtain

$$\int_0^1 x(t)^{\frac{1}{n}}\, dt \leq \frac{K}{|\varrho|\left(1+\frac{1}{n}\right)} \max_{x\in[0,1]} x(s), \tag{5.4}$$

would hold for all $x \in \mathcal{D}$. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (5.4), we obtain

$$\frac{1}{j^{1/n}\left(1+\frac{1}{n}\right)} \leq \frac{K}{j\,|\varrho|\left(1+\frac{1}{n}\right)} \Longleftrightarrow j^{1-\frac{1}{n}} \leq \frac{K}{|\varrho|}, \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$. Therefore, condition (3.23) fails in this case. However, condition (4.28) holds. To show this, let $x_0(t) = f(t)$ and $\Xi = \min_{s\in[a,b]} f(s) > 0$. Then, for $v \in \mathcal{D}$,

$$\begin{aligned}
&\| \,(F'(x) - F'(x_0))\, v \,\| \\
&= |\varrho| \left(1+\frac{1}{n}\right) \max_{s\in[a,b]} \left| \int_a^b \mathcal{G}(s,t)\big(x(t)^{\frac{1}{n}} - f(t)^{\frac{1}{n}}\big) v(t) dt \right| \\
&\leq |\varrho| \left(1+\frac{1}{n}\right) \Upsilon \;\| v \|,
\end{aligned}$$

where,

$$\Upsilon = \max_{s\in[a,b]} \int_a^b \frac{\mathcal{G}(s,t)\,|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}}\, dt.$$

Hence,

$$\begin{aligned}
\| \,F'(x) - F'(x_0)\, \| \;&\leq\; \frac{|\varrho|\left(1+\frac{1}{n}\right)}{\Xi^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b \mathcal{G}(s,t)\, dt \;\| x - x_0 \| \\
&=\; K_0 \;\| x - x_0 \|,
\end{aligned}$$

where

$$K_0 = \frac{|\varrho|\,(1+\frac{1}{n})}{\Xi^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b \mathcal{G}(s,t)\, dt.$$

and (4.28) is satisfied.

We finally provide a numerical example to show how the parameters in Theorem 4.4 can be computed.

Example 5.2. *Let* $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$*,* $x_0 = (.495, .495)^T$*, and* $\mathcal{D} = \overline{U}(x_0, \frac{7}{8})$*. Define function* F *on* $\mathcal{D}$ *by*

$$F(x) = (\xi_1^3 - \frac{1}{8}, \xi_2^3 - \frac{1}{8})^T, \qquad x = (\xi_1, \xi_2)^T. \tag{5.5}$$

Using hypotheses of Theorem 4.4, and Remark 4.5 (iv), we get

$$L = 5.595, \quad \eta = .0050506751, \quad \ell = \mu_0 = \mu_1 = 0,$$
$$K = 90.0912, \quad M = 0, \quad \delta = .90861451,$$
$$h = .22751069028 < .5.$$

That is, all hypotheses of Theorem 4.4 are satisfied. Hence, (NLM) starting at x_0 *converges quadratically to* $x^\star = (.5, .5)^T$*.*

Conclusion

We exploited our new concept of recurrent functions, and new condition (3.2), instead of Lipschitz condition (3.23) used in [17], in order to study a semilocal convergence analysis for (NLM) and (GNLM) using outer or generalized inverses in Banach spaces. This analysis has the following advantages over the work in [17]: weaker sufficient convergence conditions (in particular in some interesting cases (e.g., when $G = 0$)), and finer majorizing sequences. Numerical examples further validating the results are also provided in this study.

References

[1] I. K. Argyros, *Convergence domain for some iterative processes in Banach spaces using outer and generalized inverses*, J. Comput. Anal. Appl., **1** (1999), 87–104.

[2] I. K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397.

[3] I. K. Argyros, *Computational theory of iterative methods, Series: Studies in Computational Mathematics*, **15**, Editors: C. K. Chui and L. Wuytack, Elsevier Publ. Co., New York, USA, 2007.

[4] I. K. Argyros and S. Hilout, *Efficient methods for solving equations and variational inequalities*, Polimetrica Publisher, Milano, Italy, 2009.

[5] I. K. Argyros and S. Hilout, *A convergence analysis of Newton–like method for singular equations using recurrent functions*, Numer. Funct. Anal. Optimiz., **31** (2010), 112–130.

[6] I. K. Argyros and F. Szidarovszky, *The Theory and Applications of Iteration Methods*, C.R.C. Press, Boca Raton, Florida, 1993.

[7] A. Ben–Israel and N. E. Greville, *Generalized inverses: theory and applications, Pure and Applied Mathematics*, Wiley–Interscience, John Wiley and Sons, New York, London, Sydney, 1974.

[8] X. Chen and T. Yamamoto, *Convergence domains of certain iterative methods for solving nonlinear equations*, Numer. Funct. Anal. Optimiz., **10** (1989), 37–48.

[9] J. E. Dennis, *On Newton–like methods*, Numer. Math., **11** (1968), 324–330.

[10] P. Deuflhard and G. Heindl, *Convergence theorems for Newton's method and extensions to related methods*, SIAM J. Numer. Anal., **16** (1979), 1–10.

[11] J. M. Gutiérrez, *A new semilocal convergence theorem for Newton's method*, J. Comp. Appl. Math., **79** (1997), 131–145.

[12] J. M. Gutiérrez, M. A. Hernández and M. A. Salanova, *Accessibility of solutions by Newton's method*, Internat. J. Comput. Math., **57** (1995), 239–247.

[13] W. H. Häubler, *A Kantorovich–type convergence analysis for the Gauss–Newton methods*, Numer. Math., **48** (1986), 119–125.

[14] Q. Jin, *A convergence analysis of the iteratively regularized Gauss–Newton method under the Lipschitz condition*, Inverse Problems, **24** (2008), 1–24.

[15] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1964.

[16] M. Z. Nashed, *Generalized inverse mapping theorems and related applications of generalized inverses in nonlinear analysis.* Nonlinear equations in abstract spaces (Proc. Internat. Sympos., Univ. Texas, Arlington, Tex., 1977), 217–252, Academic Press, New York, 1978.

[17] M. Z. Nashed and X. Chen, *Convergence of Newton–like methods for singular operator equations using outer inverses*, Numer. Math., **66** (1993), 235–257.

[18] F. A. Potra and V. Ptàk, *Sharp error bounds for Newton's process*, Numer. Math., **34** (1980), 67–72.

[19] W. C. Rheinboldt, *A unified convergence theory for a class of iterative processes*, SIAM J. Numer. Anal., **5** (1968), 42–63.

[20] W. C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, Polish Academy of Sciences, Banach Ctr. Publ., **3** (1977), 129–142.

[21] T. Yamamoto, *A method for finding sharp error bounds for Newton's method under the Kantorovich assumptions*, Numer. Math., **49** (1987), 203–230.

[22] T. Yamamoto, *A convergence theorem for Newton–like methods in Banach spaces*, Numer. Math., **51** (1987), 545–557.

[23] T. Yamamoto, *Uniqueness of the solution in a Kantorovich–type theorem of Häubler for the Gauss–Newton method*, Japan J. Appl. Math., **6** (1989), 77–81.