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AN ITERATION METHOD FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS OF BROWDER–PETRYSHYN TYPE

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Abstract. In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of strictly pseudocontractive mappings of Browder–Petryshyn type in Hilbert spaces. Some strong and weak convergence theorems are obtained. In particular, the necessary and sufficient conditions for strong convergence of our iterative scheme are obtained. The results presented in this paper improve and extend the recent corresponding results.

1. Introduction

Let H be a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$. Let K be a nonempty closed convex subset of H and $G : K \times K \to R$ be a bifunction, where R is the set of real number. The equilibrium problem (for

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short, EP) is to find $x^* \in K$ such that

$$G(x^*, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

The set of solutions of $EP(1.1)$ is denoted by $EP(G)$. Given a mapping $T : K \to K$, Let $G(x, y) =< Tx, y - x >$ for all $x, y \in K$. Then $x^* \in EP(G)$ if and only if $x^* \in K$ is a solution of the variational inequality $< Tx, y - x >\geq 0$ for all $y \in K$, i.e., x^* is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of $EP(1.1)$. It has been shown that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases [1, 12]. Equilibrium problems cover a vast range of applications. Some methods have been proposed to solve the $EP(1.1)$ (see, e.g., [3-5, 7, 17, 18] and references therein). Motivated by the work in [5, 11, 17], Takahashi and Takahashi [18] introduced a viscosity iteration scheme to find a common element of the set of solutions of the $EP(1.1)$ and the set of fixed points of a nonexpansive mapping in the setting of Hilbert spaces. they also studied the strong convergence of the sequences generated by their algorithm in Hilbert spaces. Later, Some authors used many iteration methods to approximate a solution of $EP(1.1)$. But many iteration methods that they used involved in projection operator. They had to compute many iterative elements and projection subsets(See, e.g., [6, 8, 16, 20] and reference therein) in each step iteration process. It is a very difficult work.

Let E be a real Banach space and let J denote the generalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \|f\|^2 = \|x\|^2 =< x, f >\};$$

where E^* denotes the dual space of E and $< \cdot, \cdot >$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single$-$valued. In the sequel we shall denote single$-$valued duality mapping by j.

Let E be a real Banach space, A mapping $T : E \to E$ is said to be $L-$ *Lipschitzian* if there exists constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

A mapping T is said to be nonexpansive if $L = 1$ in (1.2). The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in K : Tx = x\}$.

A mapping T with domain $D(T)$and range $R(T)$ in E is called strictly pseudocontractive in the terminology of Browder and Petryshyn [2] if $\forall x, y \in D(T)$ there exists $\lambda \geq 0$ and $j(x - y) \in J(x - y)$ such that

$$< Tx - Ty, j(x - y) >\leq \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{1.3}$$

Without loss of generality we may assume that $\lambda \in (0,1)$. If I denotes the identity mapping, then (1.3) can be written in the following form

$$< (I-T)x-(I-T)y, j(x-y) >\geq \lambda\|(I-T)x-(I-T)y\|^2. \qquad (1.4)$$

In Hilbert spaces, (1.3)(and hence (1.4)) is equivalent to the inequality

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad k=(1-2\lambda)<1. \quad (1.5)$$

It follows from (1.4) that

$$\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\| \geq \lambda\|Tx-Ty\| - \lambda\|x-y\|.$$

So

$$\|Tx-Ty\| \leq \frac{1+\lambda}{\lambda}\|x-y\| = L_*\|x-y\|, \quad \forall x,y \in D(T), \qquad (1.6)$$

where $L_* = \frac{1+\lambda}{\lambda}$, hence T is Lipschitzian in Hilbert space. It also follows from (1.5) that

$$\|Tx-Ty\| \leq \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x-y\| = L_*\|x-y\|,$$

where $L_* = \frac{1+\sqrt{k}}{1-\sqrt{k}}$.

Note that the class of strict pseudocontraction mappings strictly includes the class of nonexpansive mappings. Clearly, T is nonexpansive if and only if $k=0$ in (1.5).

Many iteration methods have been proposed to approximate fixed points of strictly pseudocontractive mappings by many authors (see, e.g., [9,10,13,15]). Recently, Wang [19] introduced the following hybird iteration method for nonexpansive mappings and obtained some strong and weak convergence theorems of fixed points of nonexpansive mappings.

For arbitrarily given $x_1 \in H$, the iterative scheme $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}}x_n, \quad n \geq 1,$$

where $T^{\lambda_{n+1}}x_n = Tx_n - \lambda_{n+1}\mu A(Tx_n)$, $T: H \to H$ is a nonexpansive mapping with $F(T) \neq \varnothing$, and $A: H \to H$ an $\eta-$strongly monotone and $L-$Lipschitizian mapping, $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [0,1)$.

Later, Osilike, Isiogugu and Nwokoro [15] used this method to approximate fixed points of $k-$strictly pseudocontractive mappings and obtained some strong and weak convergence theorems. Their results improved and extended the results of Wang [19].

Motivated and inspired by the recent work of Wang [19], Osilike, Osiogugu and Nwokoro [15], Combettes and Hirstoaga [5], Takahashi and Takahashi [18], etc., we introduce an iteration method, which does not involve in projection operators, to approximate a common element of the set of solutions of $EP(1.1)$ and the set of fixed points of a strictly pseudocontractive mapping of Browder$-$Petryshyn type in Hilbert spaces. The iterative method reduces the burden of computation task in iterative process and the results presented in this paper improve and extend the recent corresponding results.

2. Preliminaries

Throughout this paper, H is a Hilbert space with inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$, K is a nonempty closed convex subset of H.

Let bifunction $G : K \times K \to R$ satisfy the following conditions:

(A1) $G(x, x) = 0, \quad \forall x \in K$;
(A2) $G(x, y) + G(y, x) \leq 0, \quad \forall x, y \in K$;
(A3) For all $x, y, z \in K$, $\lim_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
(A4) For each $x \in K$, the function $y \longmapsto G(x, y)$ is convex and lower semi-continuous.

By using the conditions above, Combettes and Hirstoaga [5] obtained the following results.

Lemma 2.1. ([5]) *Let K be a nonempty closed convex subset of Hilbert space H and let $G : K \times K \to R$ be a bifunction satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that*

$$G(z, y) + \frac{1}{r} < y - z, z - x > \geq 0, \quad \forall y \in K.$$

Lemma 2.2. ([5]) *Assume that $G : K \times K \to R$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r(x) : H \to H$ as follows:*

$$T_r(x) = \{z \in K : G(z, y) + \frac{1}{r} < y - z, z - x > \geq 0, \quad \forall y \in K\}, \quad \forall x \in H.$$

Then,

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, that is, for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq < T_r x - T_r y, x - y >;$$

(3) *$G(T_r) = EP(G)$;*
(4) *$EP(G)$ is nonempty, closed and convex.*

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x. It is well known that every Hilbert space satisfies Opial's condition.

A mapping $T : K \to E$ is said to be semi−compact, if for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0, (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{x_j}\}$ converges strongly to $x^* \in K$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demi-closed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

T is said to satisfy condition (A) if $F(T) \neq \varnothing$ and there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) \geq 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in D(T)$, where $d(x, F(T)) := \inf\{\|x - p\| : p \in F(T)\}$.

Lemma 2.3. ([10]) *Let H be a Hilbert space.The following identities hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2 < x - y, y >, \quad \forall x, y \in H$;

(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall t \in [0, 1], \quad \forall x, y \in H$.

Lemma 2.4. ([13]) *Let E be a real $q-$uniformly smooth Banach which also uniformly convex and let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a strictly pseudocontractive mapping of Browder−Petryshyn type. Then $(I - T)$ is demiclosed on K, where I is the identity mapping.*

Lemma 2.5. ([14]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real number satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, n \geq 1$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $lim_{n\to\infty}a_n = 0$.

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $G : K \times K \to R$ be a bifunction satisfying $(A1) - (A4)$, and $T : K \to K$ be a strictly pseudocontractive mapping of Browder−Petryshyn type for some $0 \leq k < 1$ such that $F(T) \cap EP(G) \neq \varnothing$, and $A : K \to K$ be an $L-$Lipschitzian mapping. For any given $x_1 \in K, \{x_n\}$ is defined by*

$$\begin{cases} G(u_n, y) + \frac{1}{r_n} < y - u_n, u_n - x_n > \geq 0, \quad \forall y \in K, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T^{\lambda_n} u_n, \quad n \geq 1, \end{cases}$$

where $T^{\lambda_n}x = Tx - \lambda_n \mu A(Tx)$ *for all* $x \in K, \mu > 0$, *and* $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $k < \alpha \leq \alpha_n \leq \beta < 1$ *for some* $\alpha, \beta \in (0,1)$*;*
(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$*;*
(iii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then

(1) $\lim_{n\to\infty} \|x_n - q\|$ *exists for each* $q \in F(T) \cap EP(G)$*;*
(2) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$*;*
(3) $\{x_n\}$ *converges weakly to a common element of* $F(T)$ *and* $EP(G)$*;*
(4) $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a common element of* $F(T)$ *and* $EP(G)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F(T) \cap EP(G)) = 0$, *where* $d(x_n, F(T) \cap EP(G))$ *denotes the metric distance from* x_n *to* $F(T) \cap EP(G)$.

Proof. (1) It follows from Lemma 2.2 that $u_n = T_{r_n} x_n$. For any $q \in F(T) \cap EP(G)$ and any positive integer n, we have

$$\|u_n - q\| \leq \|T_{r_n} x_n - T_{r_n} q\| \leq \|x_n - q\|.$$

In addition (by Lemma 2.3)

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\alpha_n(u_n - q) + (1-\alpha_n)(T^{\lambda_n} u_n - q)\|^2 \\
&= \|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q) - (1-\alpha_n)\lambda_n \mu A(Tu_n)\|^2 \\
&= \|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q)\|^2 + (1-\alpha_n)^2 \lambda_n^2 \mu^2 \|A(Tu_n)\|^2 \\
&\quad -2(1-\alpha_n)\lambda_n \mu < A(Tu_n), \alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q) > \\
&\leq \alpha_n \|u_n - q\|^2 + (1-\alpha_n)\|Tu_n - q\|^2 - \alpha_n(1-\alpha_n)\|u_n - Tu_n\|^2 \\
&\quad +2(1-\alpha_n)\lambda_n \mu \|A(Tu_n)\| \cdot \|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q)\| \\
&\quad +(1-\alpha_n)^2 \lambda_n^2 \mu^2 \|A(Tu_n)\|^2 \\
&\leq \alpha_n \|u_n - q\|^2 + (1-\alpha_n)[\|u_n - q\|^2 + k\|u_n - Tu_n\|^2] \\
&\quad -\alpha_n(1-\alpha_n)\|u_n - Tu_n\|^2 + (1-\alpha_n)^2 \lambda_n^2 \mu^2 \|A(Tu_n)\|^2 \\
&\quad +2(1-\alpha_n)\lambda_n \mu \|A(Tu_n)\| \cdot \|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q)\| \\
&= \|u_n - q\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - Tu_n\|^2 \\
&\quad +2(1-\alpha_n)\lambda_n \mu \|A(Tu_n)\| \cdot \|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q)\| \\
&\quad +(1-\alpha_n)^2 \lambda_n^2 \mu^2 \|A(Tu_n)\|^2. \qquad (3.1)
\end{aligned}$$

It follows from (1.6) that

$$\|A(Tu_n)\| \leq LL_* \|u_n - q\| + \|A(q)\|. \qquad (3.2)$$

and

$$\|\alpha_n(u_n - q) + (1-\alpha_n)(Tu_n - q)\| \leq (1+L_*)\|u_n - q\|. \tag{3.3}$$

Thus, it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\leq \|u_n - q\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - Tu_n\|^2 \\
&\quad +(1-\alpha_n)^2\lambda_n^2\mu^2[L^2L_*^2\|u_n - q\|^2 + 2LL_*\|u_n - q\|.\|A(q)\| + \|A(q)\|^2] \\
&\quad +2(1-\alpha_n)\lambda_n\mu[LL_*\|u_n - q\| + \|A(q)\|](1+L_*)\|u_n - q\| \\
&= \|u_n - q\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - Tu_n\|^2 \\
&\quad +(1-\alpha_n)^2\lambda_n^2\mu^2L^2L_*^2\|u_n - q\|^2 + (1-\alpha_n)^2\lambda_n^2\mu^2\|A(q)\|^2 \\
&\quad +2LL_*(1-\alpha_n)^2\lambda_n^2\mu^2\|u_n - q\|.\|A(q)\| \\
&\quad +2(1-\alpha_n)\lambda_n\mu LL_*(1+L_*)\|u_n - q\|^2 \\
&\quad +2(1-\alpha_n)\lambda_n\mu(1+L_*)\|u_n - q\|.\|A(q)\| \\
&\leq \|u_n - q\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - Tu_n\|^2 \\
&\quad +(1-\alpha_n)^2\lambda_n^2\mu^2L^2L_*^2\|u_n - q\|^2 + (1-\alpha_n)^2\lambda_n^2\mu^2\|A(q)\|^2 \\
&\quad +LL_*(1-\alpha_n)^2\lambda_n^2\mu^2[\|u_n - q\|^2 + \|A(q)\|^2] \\
&\quad +2(1-\alpha_n)\lambda_n\mu LL_*(1+L_*)\|u_n - q\|^2 \\
&\quad +(1-\alpha_n)\lambda_n\mu(1+L_*)[\|u_n - q\|^2 + \|A(q)\|^2] \\
&= [1 + (1-\alpha_n)^2\lambda_n^2\mu^2LL_*(1+LL_*) + (1-\alpha_n)\lambda_n\mu(1+L_*)(1 \\
&\quad +2LL_*)]\|u_n - q\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - Tu_n\|^2 \\
&\quad +[(1-\alpha_n)^2\lambda_n^2\mu^2(1+LL_*) + (1-\alpha_n)\lambda_n\mu(1+L_*)]\|A(q)\|^2 \\
&\leq (1+a_n)\|u_n - q\|^2 - (1-\beta)(\alpha - k)\|u_n - Tu_n\|^2 + b_n \\
&\leq (1+a_n)\|x_n - q\|^2 - (1-\beta)(\alpha - k)\|u_n - Tu_n\|^2 + b_n,
\end{aligned} \tag{3.4}$$

where

$$a_n = (1-\alpha_n)^2\lambda_n^2\mu^2LL_*(1+LL_*) + (1-\alpha_n)\lambda_n\mu(1+L_*)(1+2LL_*),$$

and

$$b_n = (1-\alpha_n)^2\lambda_n^2\mu^2(1+LL_*) + (1-\alpha_n)\lambda_n\mu(1+L_*)\|A(q)\|^2.$$

Obviously, $\sum_{n=1}^{\infty} a_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$. It follows from Lemma 2.5 that $\lim_{n\to\infty}\|x_n - q\|$ exists for each $q \in F(T) \cap EP(G)$. This implies that $\{x_n\}$ is bounded, so are $\{Tx_n\}, \{A(Tx_n)\}$. In addition, $\{Tu_n\}$ and $\{A(Tu_n)\}$ are bounded, too, since $\|u_n - q\| \leq \|x_n - q\|$. This completes the proof of (1).

(2) Since $\{\|x_n - q\|\}$ is bounded, there exist $M > 0$ such that $\|x_n - q\|^2 \leq M$ for all $n \geq 1$. Thus it follows from (3.4) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - (1-\beta)(\alpha - k)\|u_n - Tu_n\|^2 + \sigma_n, \tag{3.5}$$

where $\sigma_n = b_n + Ma_n$, so that

$$\sum_{j=1}^{\infty}(1-\beta)(\alpha-k)\|u_{j+1}-Tu_{j+1}\|^2 \le \|x_1-q\|^2+\sum_{j=1}^{\infty}\sigma_j,$$

and since $\sum_{n=1}^{\infty}\sigma_n<\infty$, it follows that

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{3.6}$$

Since

$$\begin{aligned}\|T^{\lambda_n}u_n-u_n\| &= \|Tu_n-\lambda_n\mu A(Tu_n)-u_n\|\\ &\le \|Tu_n-u_n\|+\lambda_n\mu\|A(Tu_n)\|,\end{aligned}$$

it follows from (3.6) that

$$\lim_{n\to\infty}\|T^{\lambda_n}u_n-u_n\|=0. \tag{3.7}$$

In addition, since $x_{n+1}-u_n=(1-\alpha_n)(T^{\lambda_n}u_n-u_n)$, from (3.7), we know that

$$\lim_{n\to\infty}\|x_{n+1}-u_n\|=0. \tag{3.8}$$

Thus, it follows from (3.6) and (3.8) that

$$\lim_{n\to\infty}\|x_{n+1}-Tu_n\|=0. \tag{3.9}$$

On the other hand, since

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \|x_{n+1}-Tu_n\|+\|Tu_n-Tx_{n+1}\|\\ &\le \|x_{n+1}-Tu_n\|+L_*\|u_n-x_{n+1}\|,\end{aligned}$$

by (3.8) and (3.9), we have

$$\lim_{n\to\infty}\|x_{n+1}-Tx_{n+1}\|=0. \tag{3.10}$$

This completes the proof of (2).

(3) For any $q\in F(T)\cap EP(G)$. Since

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1+a_n)\|u_n-q\|^2-(1-\beta)(\alpha-k)\|u_n-Tu_n\|^2+b_n\\ &\le \|u_n-q\|^2+\sigma_n,\end{aligned}$$

where $\sigma_n=b_n+Ma_n$ and

$$\begin{aligned}\|u_n-q\|^2 &= \|T_{r_n}x_n-T_{r_n}q\|^2\\ &\le <x_n-q,u_n-q>\\ &= \frac{1}{2}(\|x_n-q\|^2+\|u_n-q\|^2-\|x_n-u_n\|^2),\end{aligned}$$

then

$$\|x_{n+1}-q\|^2\le\|x_n-q\|^2-\|x_n-u_n\|^2+\sigma_n.$$

So

$$\|x_n - u_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \sigma_n.$$

Thus, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

Since H is a Hilbert space and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a $p \in K$. Thus, it follows from (3.10) and Lemma 2.4 that $p \in F(T)$. In addition, it follows from (3.11) that $\{u_{n_j}\}$ converges weakly to $p \in K$, too.

We now show that $p \in EP(G)$. Since $u_n = T_{r_n} x_n$, we have

$$G(u_n, y) + \frac{1}{r_n} < y - u_n, u_n - x_n > \geq 0, \quad \forall y \in K.$$

From (A2), we obtain

$$\frac{1}{r_n} < y - u_n, u_n - x_n > \geq G(y, u_n).$$

Thus,

$$\frac{1}{r_{n_j}} < y - u_{n_j}, u_{n_j} - x_{n_j} > \geq G(y, u_{n_j}).$$

It follows from $(A4)$ and (3.11) that

$$G(y, p) \leq 0, \quad \forall y \in K. \tag{3.12}$$

For $t \in (0, 1]$ and $\forall y \in K$. Setting $z_t = ty + (1 - t)p$, we know that $z_t \in K$. It follows from (3.12) that $G(z_t, p) \leq 0$. From $(A1)$ and $(A4)$, we have

$$0 = G(z_t, z_t) \leq tG(z_t, y) + (1 - t)G(z_t, p) \leq tG(z_t, y).$$

From $(A3)$, we know that $G(p, y) \geq 0$, so $p \in EP(G)$. Hence $p \in F(T) \cap EP(G)$

We now prove that $\{x_n\}$ converges weakly to a common element of $F(T)$ and $EP(G)$. First of all, (3.10) and Lemma 2.4 guarantee that each weakly subsequential limit of $\{x_n\}$ is a fixed point of T. At the same time, using the proof of above, it is easily proved that each weakly subsequential limit of $\{x_n\}$ also is a solution of $EP(1.1)$.

Secondly, since every Hilbert space satisfies Opial's condition, and Opial's condition guarantees that the weakly subsequential limit of $\{x_n\}$ is unique, hence $\{x_n\}$ converges weakly to a common element of $F(T)$ and $EP(G)$. This completes the proof of (3).

(4) Suppose that $\{x_n\}$ converges strongly to a common element q of $F(T)$ and $EP(G)$. Then $\lim_{n\to\infty} \|x_n - q\| = 0$. Since

$$0 \leq d(x_n, F(T) \cap EP(G)) \leq \|x_n - q\|,$$

we have $\liminf_{n\to\infty} d(x_n, F(T) \cap EP(G)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)\cap EP(G)) = 0$, then it follows from (3.4) and Lemma 2.5 that $\lim_{n\to\infty} d(x_n, F(T)\cap EP(G)) = 0$. Thus for arbitrary $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, F(T)\cap EP(G)) < \frac{\varepsilon}{4}, \forall n \geq N_1$, Furthermore, $\sum_{n=1}^{\infty} \sigma_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty} \sigma_j < \frac{\varepsilon^2}{16}, \forall n \geq N_2$. Put $N = max\{N_1, N_2\}$, then $d(x_N, F(T)\cap EP(G) < \frac{\varepsilon}{4}$ and $\sum_{j=N}^{\infty} \sigma_j < \frac{\varepsilon^2}{16}$. It follows from (3.5) that for any $n, m \geq N$ and for all $q \in F(T)\bigcap EP(G)$,

$$\begin{aligned}
\|x_n - x_m\|^2 &\leq [\|x_n - q\| + \|x_m - q\|]^2 \\
&\leq 2[\|x_n - q\|^2 + \|x_m - q\|^2] \\
&\leq 2[\|x_N - q\|^2 + \sum_{j=N}^{n} \sigma_j + \|x_N - q\|^2 + \sum_{j=N}^{m} \sigma_j] \\
&\leq 4\|x_N - q\|^2 + 4\sum_{j=N}^{\infty} \sigma_j.
\end{aligned}$$

Thus

$$\|x_n - x_m\| \leq 2\|x_N - q\| + 2(\sum_{j=N}^{\infty} \sigma_j)^{\frac{1}{2}}.$$

Taking the infinimum in the inequality above for all $q \in F(T)\cap EP(G)$, we obtain

$$\|x_n - x_m\| \leq 2d(x_N, F(T)\cap EP(G)) + 2(\sum_{j=N}^{\infty} \sigma_j)^{\frac{1}{2}} < \varepsilon, \quad \forall n, m \geq N.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $p \in K$ such that $\{x_n\}$ converges strongly to p. Then

$$d(p, F(T)\cap EP(G)) = \lim_{n\to\infty} d(x_n, F(T)\cap EP(G)) = 0.$$

Since $F(T)$ is closed and convex [10], and $EP(G)$ also is closed and convex [5], we have $F(T)\cap EP(G)$ is closed and convex,too. Hence $p \in F(T)\cap EP(G)$. In addition, since $\|u_n - p\| \leq \|x_n - p\|$, we know that $\{u_n\}$ converges strongly to p, too. The proof is completed □

Theorem 3.2. *Under the conditions of theorem 3.1, if T is semi−compact, then $\{x_n\}$ converges strongly to a common element of $F(T)$ and $EP(G)$.*

Proof. It follows from Theorem 3.1 that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists, for each $q \in F(T)\cap EP(G)$. Therefore there exists a subsequence

$\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in K$. Using the same argument as in the proof in theorem 3.1, we have $p \in F(T) \cap EP(G)$. Thus, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Furthermore, $\{x_n\}$ converges strongly to p since $\lim_{n\to\infty} \|x_{n_j} - p\| = 0$. In addition, since $\|u_n - q\| \leq \|x_n - q\|$ for each $q \in F(T) \cap EP(G)$, so $\{u_n\}$ also converges strongly to a common element of $F(T)$ and $EP(G)$. The proof is completed. □

Theorem 3.3. *Under the conditions of Theorem 3.1 , if T satisfies condition (A), then $\{x_n\}$ converges strongly to a common element of $F(T)$ and $EP(G)$.*

Proof. It follows from Theorem 3.1 that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies condition (A) and f is a nondecreasing function with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T) \cap EP(G)) = 0$, Thus, it follows form Theorem 3.1 that $\{x_n\}$ converges strongly to a common element of $F(T)$ and $EP(G)$. In addition, Since$\|u_n - q\| \leq \|x_n - q\|$ for each $q \in F(T) \cap EP(G)$, so $\{u_n\}$ also converges strongly to a common element of $F(T)$ and $EP(G)$. The proof is completed. □

Remark 3.4. *The iterative method of Theorem 3.1 reduces to the iterative method in [4] when $\lambda_n = 0$ for all $n \geq 1$. So, our results improve and extend the results of Ceng [4].*

Remark 3.5. *The iterative scheme of Theorem 3.1 does not involve in projection operator, here it is unnecessary to compute projection subsets in each step iteration process.*

Remark 3.6. *The mapping A in [19] is $\eta-$strongly monotone and $L-$Lipschitzian, but the mapping A in Theorem 3.1 just is $L-$Lipschitzian, which is weaker than the one in [19].*

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