Nonlinear Functional Analysis and Applications
Vol. 16, No. 3 (2011), pp. 285-296

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

EIGENVALUE PROBLEMS FOR SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

Fenghua Yang[1], Lijun Wang[2] and Zengqin Zhao[3]

[1]College of Mathematics Sciences, Qufu Normal University
Qufu, Shandong 273165, People's Republic of China
e-mail: yang0738@163.com

[2]Computer Science College, Qufu Normal University
Rizhao, Shandong 276826, People's Republic of China
e-mail: Wanglijunsd@126.com

[3]College of Mathematics Sciences, Qufu Normal University
Qufu, Shandong 273165, People's Republic of China
e-mail: zqzhao@qfnu.edu.cn

Abstract. This paper is concerned with the existence and nonexistence of positive solutions for a class of the Sturm-Liouville boundary value problem. Some results of existence and uniqueness for positive solution are established. In particular, such a positive solution of BVP depends on the parameter.

1. Introduction

Sturm-Liouville boundary value problems play a very important role in both theory and application, which have been widely studied by many authors (see $[1-5]$ and references therein). For example, Ge and Ren [1] have established the existence of one positive solution depend on parameter by using the fixed point theorems. Sun and Zhang [2] have applied the fixed point index theorem and the first eigenvalue to establish the existence of positive solutions when it has no parameter.

[0]Received March 25, 2010. Revised May 12, 2011.
[0]2000 Mathematics Subject Classification: 34B16, 34B18, 34B24.
[0]Keywords: Singular Sturm-Liouville boundary value problem, impulsive, the fixed point index, positive solution.
[0]The project is supported financially by the Natural Science Foundation of Shandong Province of China(ZR2010AM005).

In this paper, we consider the following Sturm-Liouville boundary value problem with parameter

$$\begin{cases} (p(t)u'(t))' + \lambda a(t)f(u(t)) = 0, t \in J, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0, \end{cases} \tag{1.1}$$

where $J = (0,1), \alpha_i \geq 0, \beta_i \geq 0 (i = 1,2)$, $p(t) \in C([0,1], \mathbb{R}^+) \bigcap C^1(J, \mathbb{R}^+)$, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(t) \in C(J, \mathbb{R}^+)$ is allowed to be singular at $t = 0$ or $t = 1$, $\mathbb{R}^+ = [0, +\infty)$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$, $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 \int_0^1 \frac{ds}{p(s)} > 0$.

By applying the fixed point index theorem, we shall establish the existence of at least two, one and zero positive solutions for the above problem, which improve and generalize the corresponding results of papers [1, 2]. For the convenience, we make the following assumptions:

(H_1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing.

(H_2) $f(0) = c > 0$.

(H_3) $\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty$.

(H_4) $a(t) \in C((0,1), \mathbb{R}), a(t) \not\equiv 0$ in $(0,1)$ and

$$\int_0^1 \frac{1}{\rho}\left(\beta_1 + \alpha_1 \int_0^s \frac{d\tau}{p(\tau)}\right)\left(\beta_2 + \alpha_2 \int_s^1 \frac{d\tau}{p(\tau)}\right) a(s)ds < +\infty.$$

2. Preliminaries and some lemmas

In this section, we will introduce several definitions and give some lemmas (see [3, 13]).

Definition 2.1. *A function* $u(t) \in C([0,1], \mathbb{R}) \cap C^2((0,1), \mathbb{R}), p(t)u'(t) \in C^1([0,1], \mathbb{R})$, *is called a solution of (1.1) if it satisfies (1.1).*

Now we denote the Green's functions for the following boundary value problems

$$\begin{cases} (p(t)u'(t))' = 0, \quad 0 \leq t \leq 1, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0, \end{cases}$$

by $G(t,s)$. It is well known that $G(t,s)$ can be written by

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\beta_1 + \alpha_1 B(0,s))(\beta_2 + \alpha_2 B(t,1)), & 0 \leq s \leq t \leq 1, \\ (\beta_1 + \alpha_1 B(0,t))(\beta_2 + \alpha_2 B(s,1)), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.1}$$

where $B(t,s) = \int_t^s \frac{d\tau}{p(\tau)}$, $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0,1)$. It is easy to verify the following properties of $G(t,s)$:

(**I**) $G(t,s) \le G(s,s) \le \frac{1}{\rho}\left(\beta_1 + \alpha_1 B(0,1)\right)\left(\beta_2 + \alpha_2 B(0,1)\right) < +\infty$,

(**II**) $G(t,s) \ge \sigma G(s,s)$, for any $t \in [a,b] \subset (0,1), s \in [0,1]$, where

$$0 < \sigma = \min\left\{\frac{\beta_2 + \alpha_2 B(b,1)}{\beta_2 + \alpha_2 B(0,1)}, \frac{\beta_1 + \alpha_1 B(0,a)}{\beta_1 + \alpha_1 B(0,1)}\right\} < 1. \tag{2.2}$$

Let $X = C([0,1],\mathbb{R})$ and

$$K = \{u \in C([0,1],\mathbb{R}) : u(t) \ge 0, t \in [0,1], \text{and } \ u(t) \ge \sigma\|u\|, t \in [a,b]\}.$$

Clearly, K is a cone of X. Next, let us define an operator $\Phi_\lambda : X \to X$ by

$$(\Phi_\lambda u)(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds, \quad t \in [0,1].$$

Clearly, by (H_1) and (H_4), we know that the operator Φ_λ is well defined, and u is a positive solution of the BVP (1.1) if and only if u is a positive fixed point of the operator Φ_λ.

Lemma 2.1. $\Phi_\lambda(K) \subset K$.

Proof. We show that for any $u \in K$

$$(\Phi_\lambda u)(t) \ge \sigma\|(\Phi_\lambda u)(t)\|, t \in [a,b].$$

For any $u \in K$, from the property (**I**) of $G(t,s)$, we know

$$\|\Phi_\lambda\| \le \lambda \int_0^1 G(s,s)a(s)f(u(s))ds. \tag{2.3}$$

On the other hand, by the property (**II**) of $G(t,s)$, for any $t \in [a,b]$, we have

$$(\Phi_\lambda u)(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \ge \sigma\lambda \int_0^1 G(s,s)a(s)f(u(s))ds. \tag{2.4}$$

It follows from (2.3) and (2.4) that for any $u \in K$,

$$(\Phi_\lambda u)(t) \ge \sigma\|(\Phi_\lambda u)(t)\|, t \in [a,b].$$

Thus, $\Phi_\lambda u \in K$.Therefore, $\Phi_\lambda(K) \subset K$. □

Lemma 2.2. $\Phi_\lambda : K \to K$ is a completely continuous operator.

Proof. For any $n \ge 2$, we defined a continuous function a_n by

$$a_n(t) = \begin{cases} \inf\left\{a(t), a(\frac{1}{n})\right\}, & 0 < t \le \frac{1}{n}, \\ a(t), & \frac{1}{n} \le t \le 1 - \frac{1}{n}, \\ \inf\left\{a(t), a(1-\frac{1}{n})\right\}, & 1 - \frac{1}{n} \le t \le 1. \end{cases}$$

Next, for $n \ge 2$, we define an operator $\Phi_{\lambda n} : K \to K$ by

$$(\Phi_{\lambda n} u)(t) = \lambda \int_0^1 G(t,s)a_n(s)f(u(s))ds, \quad t \in [0,1].$$

Obviously, for any $n \geq 2$, $\Phi_{\lambda n}$is completely continuous on K by an application of the Ascoli-Arzela theorem (see [14]). Then $\|\Phi_{\lambda n} - \Phi_\lambda\| \to 0$, as $n \to +\infty$. In fact, for any $u \in B_1 = \{u \in K : \|u\| \leq 1\}$, from (H_1), (H_4) and the property (**I**) of $G(t,s)$, we obtain

$$\begin{aligned}
\|\Phi_{\lambda n} u - \Phi_\lambda u\| &= \max_{t\in[0,1]} \left| \lambda \int_0^1 G(t,s)[a(s) - a_n(s)] f(u(s)) ds \right| \\
&\leq \lambda \int_0^{\frac{1}{n}} G(s,s)|a(s) - a_n(s)| f(u(s)) ds \\
&\quad + \lambda \int_{1-\frac{1}{n}}^1 G(s,s)|a(s) - a_n(s)| f(u(s)) ds \\
&\leq \lambda M \int_0^{\frac{1}{n}} G(s,s)|a(s) - a_n(s)| ds \\
&\quad + \lambda M \int_{1-\frac{1}{n}}^1 G(s,s)|a(s) - a_n(s)| ds \\
&\to 0,\ n \to +\infty,
\end{aligned}$$

where $M = \max_{0\leq x\leq 1} f(x)$. Hence $\|\Phi_{\lambda n} - \Phi_\lambda\| \to 0$, as $n \to +\infty$. Therefore, Φ_λ is completely continuous. This completes the proof of Lemma 2.1. □

Let X be a Banach space, $K \subset X$ be a cone in X. For $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$. The following Lemma is needed in this paper.

Lemma 2.3.([13]) Let $\Phi : K \to K$ be a completely continuous operator, Assume $\Phi x \neq x$ for every $x \in \partial K_r$.Then the following conclusions hold.

(i) If $\|x\| \leq \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;

(ii) if $\|x\| \geq \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

3. Existence of positive solutions

Lemma 3.1. Assume that $(H_1) - (H_4)$ hold. Then there exists a $\lambda^* > 0$ such that the operator Φ_{λ^*} has a fixed point $u^* \in K$.

Proof. Set

$$e(t) = \int_0^1 G(t,s)a(s)ds. \tag{3.1}$$

It follows from (H_4) that e is well defined and $e > 0$. Let $\lambda^* = M_{f_e}^{-1}$, where $M_{f_e} = \max_{s\in[0,1]} f(e(s)) > 0$, and

$$(\Phi_{\lambda^*} u)(t) = \lambda^* \int_0^1 G(t,s)a(s)f(u(s))ds, \quad t \in [0,1].$$

Since $M_{f_e} \geq c > 0$, for any $t \in [0,1]$, we have

$$e(t) = \int_0^1 G(t,s)a(s)ds \geq \lambda^* \int_0^1 G(t,s)a(s)f(e(s))ds.$$

Let $u_0(t) = e(t), u_n(t) = (\Phi_{\lambda^*} u_{n-1})(t)$ $(n = 1, 2, \cdots), t \in [0,1]$. Then

$$u_0(t) = e(t) \geq u_1(t) \geq \cdots \geq u_n(t) \geq \cdots \geq c\lambda^* e(t).$$

This together with Lemma 2.2 implies that there exists $u^* \in K$ such that $u_n \to u^*$ in K and u^* is a fixed point of the operator Φ_{λ^*}. The proof is completed. □

Lemma 3.2. Suppose that (H_1)-(H_3) hold . Then there exists a constant $C_1 > 0$ such that $\|u\| < C_1$ for all $\lambda \in I = [c, \infty)(c > 0)$ and all possible fixed points u of Φ_λ at λ.

Proof. Set

$$S_\lambda = \{u \in K : \Phi_\lambda u = u, \lambda \in I\}.$$

We need to prove that there exists a constant $C_1 > 0$ such that $\|u\| < C_1$ for all $u \in S_\lambda$. If the number of elements of S_λ is finite, then the result is obvious. If not, without loss of generality, we assume that there exists a sequence $\{u_n\}_{n=0}^{\infty}$ such that $\lim_{n\to+\infty} \|u_n\| = +\infty$, where $u_n \in K$ is the fixed point of the operator Φ_λ for $\lambda = \lambda_n \in I$ $(n = 1, 2, \cdots)$, then

$$u_n(t) \geq \sigma\|u_n\|, \quad t \in J' = [a, b].$$

We choose a constant $J_1 > 0$ such that

$$J_1 c\sigma^2 \int_a^b G(s,s)a(s)ds > 1.$$

From (H_3), there exists a constant $L_1 > 0$ such that

$$f(u) \geq J_1 u, \quad u \geq L_1.$$

And by $\lim_{u\to\infty} \|u_n\| = +\infty$, there exists a nature number N_0 sufficiently large such that $\|u_{N_0}\| > L_1/\sigma > L_1$. Hence, for any $t \in [a, b]$, we have $u_{N_0}(t) \geq \sigma\|u_{N_0}\| > L_1$.

Therefore, for any $t \in [a,b]$, we have

$$\begin{aligned}\|u_{N_0}\| \geq u_{N_0}(t) = (\Phi_{\lambda_{N_0}} u_{N_0})(t) &= \lambda_{N_0} \int_0^1 G(t,s)a(s)f(u_{N_0}(s))ds \\ &\geq \lambda_{N_0} \int_a^b G(t,s)a(s)f(u_{N_0}(s))ds \\ &\geq J_1 c \int_a^b G(t,s)a(s)u_{N_0}(s)ds \\ &\geq J_1 c \int_a^b \frac{G(t,s)}{G(s,s)} G(s,s)a(s)\sigma\|u_{N_0}\|ds \\ &\geq J_1 c\sigma^2 \int_a^b G(s,s)a(s)ds\|u_{N_0}\| \\ &> \|u_{N_0}\|,\end{aligned}$$

which is a contradiction. Thus the proof is completed. □

Lemma 3.3. Suppose that (H_1) and (H_2) hold, and that the operator Φ_λ has a positive fixed point in K at $\widetilde{\lambda}$. Then for every $\lambda_* \in (0,\widetilde{\lambda})$ the operator Φ_λ has a fixed point $u_* \in K$ at λ_*.

Proof. Let $\widetilde{u}(t)$ be a positive fixed point of Φ_λ at $\widetilde{\lambda}$. Then

$$\widetilde{u}(t) = \widetilde{\lambda} \int_0^1 G(t,s)a(s)f(\widetilde{u}(s))ds \geq \lambda_* \int_0^1 G(t,s)a(s)f(\widetilde{u}(s))ds,$$

where $0 < \lambda_* < \widetilde{\lambda}$. Let

$$(\Phi_{\lambda_*} u)(t) = \lambda_* \int_0^1 G(t,s)a(s)f(u(s))ds,$$

$u_0(t) = \widetilde{u}(t), u_n(t) = (\Phi_{\lambda_*} u_{n-1})(t) = (\Phi^n_{\lambda_*} u_0)(t)$. Then

$$c\lambda_* e(t) \leq u_{n+1}(t) \leq u_n(t) \leq \cdots \leq u_1(t) \leq u_0(t).$$

Similar to the proof of Lemma 3.1, which implies that $\{\Phi^n_{\lambda_*} u_0\}_{n=0}^\infty$ converges to a fixed point $u_* \in K$ of the operator Φ_{λ_*}. So the proof is completed. □

Lemma 3.4. Suppose that (H_1)-(H_3) hold. Let

$$\Lambda = \{\lambda > 0 : \Phi_\lambda \text{ has at least one fixed point at } \lambda\}.$$

Then Λ is bounded set.

Proof. Suppose to the contrary that there exists a fixed point sequence $\{u_n\} \subset K$ of Φ_λ at λ_n such that $\lim_{n\to\infty} \lambda_n = +\infty$. Then we need to consider two cases:

(i) there exists a constant $H > 0$ such that $\|u_n\| \leq H, n = 0,1,2\cdots$.

(ii) there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty} \subset \{u_n\}$ such that $\lim_{k\to\infty} \|u_{n_k}\| = +\infty$ which is impossible by Lemma 3.2.

Only (i) is considered. Assume that the case (i) holds. We can choose $L_0 > 0$ enough small such that $f(0) = C > L_0 H$, and further $f(u_n) \geq L_0 H$ $(n = 1, 2, \cdots)$. For any $t \in [0, 1]$, we have

$$u_n(t) = \lambda_n \int_0^1 G(t,s)a(s)f(u_n(s))ds. \tag{3.2}$$

Then for $t \in J'$

$$\begin{aligned} H \geq \|u_n\| &\geq \lambda_n \int_0^1 G(t,s)a(s)L_0 H ds \\ &\geq \lambda_n L_0 H \sigma \int_a^b G(s,s)a(s)ds. \end{aligned}$$

Leads to $1 \geq \lambda_n L_0 \sigma \int_a^b G(s,s)a(s)ds$, which is a contradiction with $\lim_{n\to\infty} \lambda_n = \infty$. The proof is completed. □

Lemma 3.5. Let $\widetilde{\lambda} = \sup \Lambda$. Then $\Lambda = (0, \widetilde{\lambda}]$, where Λ is defined just as in Lemma 3.4.

Proof. In view of Lemma 3.3, it follows that $(0, \widetilde{\lambda}) \subset \Lambda$. We only need to prove $\widetilde{\lambda} \in \Lambda$. In fact ,by the definition of $\widetilde{\lambda}$, we may choose a distinct nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \Lambda$ such that $\lim_{n\to\infty} \lambda_n = \widetilde{\lambda}$. Let $u_n \in K$ be the positive fixed point of Φ_λ at $\lambda_n, n = 1, 2, \cdots$. By Lemma 3.2, $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded, so it has a subsequence denoted by $\{u_n\}_{n=1}^{\infty}$, converging to $\widetilde{u} \in K$. Note that

$$u_n = \lambda_n \int_0^1 G(t,s)a(s)f(u_n(s))ds. \tag{3.3}$$

Taking the limitation $n \to \infty$ to both sides of 3.3, then

$$\widetilde{u} = \widetilde{\lambda} \int_0^1 G(t,s)a(s)f(\widetilde{u}(s))ds,$$

which shows that $\widetilde{u}$ is a fixed point of Φ_λ at $\lambda = \widetilde{\lambda}$. The proof is completed. □

Theorem 3.1. Suppose that (H_1)-(H_3) hold. Then there exists a $\lambda^* > 0$ such that (1.1) has at least two, one and no positive solutions for $0 < \lambda < \lambda^*, \lambda = \lambda^*, \lambda > \lambda^*$, respectively.

Proof. Suppose that (H_1)-(H_3) hold. Then there exists a $\lambda^* > 0$ such that Φ_λ has a fixed point $u_{\lambda^*} \in K$ at $\lambda = \lambda^*$ by Lemma 3.1. In view of Lemma 3.3, Φ_λ also has a fixed point $u_{\underline{\lambda}} < u_{\lambda^*}, u_{\underline{\lambda}} \in K$ and $0 < \underline{\lambda} < \lambda^*$. Note that $f(u)$

is uniformly continuous in u on a compact subset of $\mathbb{R}$. For $0 < \underline{\lambda} < \lambda^*$, there exists a constant $\delta_0 > 0$ such that

$$f((u_{\lambda^*} + \delta)(s)) - f(u_{\lambda^*}(s)) \leq f(0)\left(\frac{\lambda^*}{\underline{\lambda}} - 1\right) \tag{3.4}$$

for $s \in [0,1], 0 < \delta \leq \delta_0$. Hence, by (3.4), we know

$$\begin{aligned}
&\underline{\lambda}\int_0^1 G(t,s)a(s)f((u_{\lambda^*}+\delta)(s))ds - \lambda^*\int_0^1 G(t,s)a(s)f(u_{\lambda^*}(s))ds\\
\leq\ & \underline{\lambda}\int_0^1 G(t,s)a(s)[f((u_{\lambda^*}+\delta)(s)) - f(u_{\lambda^*}(s))]ds\\
& -(\lambda^* - \underline{\lambda})\int_0^1 G(t,s)a(s)f(u_{\lambda^*}(s))ds\\
\leq\ & (\lambda^* - \underline{\lambda})\int_0^1 G(t,s)a(s)f(0)ds - (\lambda^* - \underline{\lambda})\int_0^1 G(t,s)a(s)f(u_{\lambda^*}(s))ds\\
=\ & (\lambda^* - \underline{\lambda})\int_0^1 G(t,s)a(s)[f(0) - f(u_{\lambda^*}(s))]ds\\
\leq\ & 0,
\end{aligned}$$

then

$$\Phi_{\underline{\lambda}}(u_{\lambda^*} + \delta) \leq \Phi_{\lambda^*}(u_{\lambda^*}) = u_{\lambda^*} < u_{\lambda^*} + \delta. \tag{3.5}$$

Set $D_{u_{\lambda^*}} = \{u \in C[0,1] : -\delta < u(t) < u_{\lambda^*} + \delta\}$. Then $D_{u_{\lambda^*}}$ is a bounded open subset of $C[0,1]$, $\theta \in D_{u_{\lambda^*}}$ and $\Phi_{\underline{\lambda}} : K \cap D_{u_{\lambda^*}} \to K$ is a completely continuous operator. Furthermore, $\Phi_{\underline{\lambda}} u \neq u$ for $u \in K \cap \partial D_{u_{\lambda^*}}$. Indeed set $u \in K \cap \partial D_{u_{\lambda^*}}$. Then there exists $t_0 \in [0,1]$ such that $u(t_0) = \|u\| = \|u_{\lambda^*} + \delta\|$ and by (3.5), we obtain

$$\begin{aligned}
\Phi_{\underline{\lambda}} u(t_0) &= \underline{\lambda}\int_0^1 G(t_0,s)a(s)f(u(s))ds\\
&\leq \underline{\lambda}\int_0^1 G(t_0,s)a(s)f((u_{\lambda^*}+\delta)(s))ds = \Phi_{\underline{\lambda}}(u_{\lambda^*}+\delta)\\
&< u_{\lambda^*}(t_0) + \delta = u(t_0) \leq u(t_0).
\end{aligned}$$

By Lemma 2.3, we have $i(\Phi_{\underline{\lambda}}, K \cap D_{u_{\lambda^*}}, K) = 1$.

Choosing $J_3 > 0$ such that $J_3\underline{\lambda}\sigma^2\int_a^b G(s,s)a(s)ds > 1$. By (H_3), there exists a constant $L_2 > u(t_0)$ such that $f(u) \geq J_3 u$ for any $u > L_2$. Set $R = L_2/\sigma$ and $K_R = \{u \in K : \|u\| < R\}$. Then $\Phi_{\underline{\lambda}} : \overline{K_R} \to K$ is completely continuous. It is easy to obtain that

$$\|(\Phi_{\underline{\lambda}} u)\| \geq J_3\underline{\lambda}\sigma^2\|u\|\int_a^b G(s,s)a(s)ds > \|u\|,$$

for $u \in \partial K_R$. By Lemma 2.3, $i(\Phi_{\underline{\lambda}}, K_R, K) = 0$. By the additivity of fixed pint index,

$$i(\Phi_{\underline{\lambda}}, K_R \backslash \overline{K \cap D_{u_{\lambda^*}}}, K) = i(\Phi_{\underline{\lambda}}, K_R, K) - i(\Phi_{\underline{\lambda}}, K \cap D_{u_{\lambda^*}}, K) = -1.$$

So, $\Phi_{\underline{\lambda}}$ has a fixed point in $\{K \cap D_{u_{\lambda^*}}\}\backslash\{\emptyset\}$ and another fixed point in $K_R \backslash \overline{K \cap D_{u_{\lambda^*}}}$ by choosing $\lambda^* = \widetilde{\lambda}$. The proof is completed. □

4. Uniqueness and dependence on the parameter

In the previous section,we have obtained existence and nonexistence results for (1.1), In this section we consider the uniqueness of positive solutions and dependence of solutions on the parameter λ. So we need to impose an additional condition on f:

(H_5) $f(\rho u) \geq \rho^\alpha f(u)$ for any $0 < \rho < 1$, where $\alpha \in (0,1)$ is independent of u and ρ.

Define an operator $\Psi : K \to K$ by

$$(\Psi u)(t) = \int_0^1 G(t,s)a(s)f(u(s))ds, t \in [0,1].$$

Lemma 4.1. Assume that (H_1) (H_2) and (H_5) hold. Then for any $u \in K$ there exist real numbers $W_u \geq w_u > 0$ such that

$$w_u e(t) \leq (\Psi u)(t) \leq W_u e(t), t \in [0,1].$$

Proof. For any $u \in K, t \in [0,1]$

$$\begin{aligned}(\Psi u)(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\leq f(\|u\|)\int_0^1 G(t,s)a(s)ds = W_u e(t),\end{aligned}$$

where $W_u = f(\|u\|)$. Note that, for any $u \in K\backslash\{\emptyset\}$, let $p = \sigma\|u\| > 0$. Then $u(t) \geq p$ for any $t \in [a,b]$. In addition, by (H_5) there exist $s_0 > c$ and $u^0 \in (0,+\infty)$ such that $f(u^0) \geq s_0$. If $p \geq u^0$, then for any $t \in [a,b]$

$$f(u(t)) \geq f(p) \geq f(u^0) \geq s_0.$$

If $p < u^0$, then for any $t \in [a,b]$

$$f(u(t)) \geq f(p) = f\left(\frac{p}{u^0}u^0\right) \geq \left(\frac{p}{u^0}\right)^\alpha f(u^0) \geq \left(\frac{p}{u^0}\right)^\alpha s_0.$$

Then

$$\begin{aligned}(\Psi u)(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\geq \int_a^b G(t,s)a(s)f(u(s))ds \\ &\geq \min\left\{c, \left(\frac{p}{u^0}\right)^\alpha s_0\right\} \int_0^1 G(t,s)a(s)ds = w_u e(t),\end{aligned}$$

where $w_u = \min\left\{c, \left(\frac{p}{u^0}\right)^\alpha s_0\right\}$. The proof is completed. □

Theorem 4.1. Assume that (H_1), (H_2), (H_5) hold and $\lambda = 1$. Then

(i) BVP(1.1) has a unique positive solution $u^* \in K$ satisfying

$$m_q e \leq u^* \leq M_q e,$$

where $0 < m_q < M_q$ are constants.

(ii) For any $u_0(t) \in K$, the sequence

$$u_n(t) = \int_0^1 G(t,s)a(s)f(u_{n-1}(s))ds, n = 1, 2, \cdots$$

converges uniformly to the unique solution $u^* \in C[0,1]$ in $[0,1]$, and the rate of the convergence is determined by

$$\|u_n - u^*\| = O(1 - d^{\alpha^n}),$$

where $0 < d < 1$ is a positive number.

Proof. In view of $(H_1), (H_2), (H_5)$, $\Psi : K \to K$ is nondecreasing operator and satisfies $\Psi(\rho u) \geq \rho^\alpha \Psi(u)$ for $u \in K$. Since $f(u)$ is nondecreasing in u, then for $u_* \leq u_{**}$ we have

$$\begin{aligned}(\Psi u_*)(t) &= \int_0^1 G(t,s)a(s)f(u_*(s))ds \\ &\leq \int_0^1 G(t,s)a(s)f(u_{**}(s))ds = (\Psi u_{**})(t).\end{aligned} \tag{4.1}$$

By Lemma 4.1, for $e(t)$ defined by (3.1), there exist $W_e \geq w_e > 0$ such that

$$w_e e(t) \leq (\Psi e)(t) \leq W_e e(t).$$

Set

$$w = \sup\{w_e : w_e e \leq (\Psi e)\}, \quad W = \inf\{W_e : \Psi e \leq W_e e\}$$

Take w_m, W_M such that

$$0 < w_m < \min\{1, w^{\frac{1}{1-\alpha}}\}, \quad \max\{1, W^{\frac{1}{1-\alpha}}\} < W_M < +\infty.$$

Let $u_0 = w_m e, v_0 = W_M e, u_n = \Psi u_{n-1}, v_n = \Psi v_{n-1}, n = 1, 2, \cdots$. By (4.1) we have

$$w_m e = u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0 = W_M e. \tag{4.2}$$

Let $d = w_m/W_M$. Then we have

$$0 < d < 1, u_n \ge d^{\alpha^n} v_n, n = 1, 2, \cdots . \tag{4.3}$$

In fact, $u_0 = dv_0$ is obvious. Assume that (4.3) holds for $n = m$ (m is a positive integer),i.e. $u_m \ge d^{\alpha^m} v_m$. Then

$$u_{m+1} = \Psi u_m \ge \Psi(d^{\alpha^m} v_m) \ge (d^{\alpha^m})^\alpha \Psi v_m = d^{\alpha^{m+1}} \Psi v_m = d^{\alpha^{m+1}} v_{m+1}.$$

By mathematical induction, it is easy to see that (4.3) holds for any nature number n. Furthermore, in view of (4.1)-(4.3), for any nature numbers n and m, we have

$$0 \le u_{n+m} - u_n \le v_n - u_n \le (1 - d^{\alpha^n}) v_0 = (1 - d^{\alpha^n}) W_M e,$$

and

$$\|u_{n+m} - u_n\| \le \|v_n - u_n\| \le (1 - d^{\alpha^n}) W_M \|e\|. \tag{4.4}$$

Thus, there exists $u^* \in K$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = u^*.$$

Then u^* is a fixed point of Ψ satisfying

$$w_m e \le u^* \le W_M e.$$

Next we show that u^* is the unique positive fixed point of Ψ in K. Note that W_M, w_m are arbitrary. By virtue of (4.2), (4.4) and Lemma 4.1, we see that (ii) holds. The proof is completed. □

Theorem 4.2. Suppose that (H_1),(H_2) and (H_5) hold. Then (1.1) has a unique positive solution $u_\lambda(t)$ for any $\lambda > 0$. Furthermore, the solution $u_\lambda(t)$ satisfies the following properties:

(i) $u_\lambda(t)$ is nondecreasing on $\lambda \in (0, +\infty)$.

(ii) $\lim_{\lambda\to 0} \|u_\lambda\| = 0$ and $\lim_{\lambda\to\infty} \|u_\lambda\| = +\infty$.

(iii) $u_\lambda(t)$ is continuous with respect to λ, i.e. $\lambda \to \lambda_0 > 0$ implies $\|u_\lambda - u_{\lambda_0}\| \to 0$.

Proof. Let

$$\Phi_\lambda u = \lambda \Psi u$$

. Then Φ_λ satisfies all conditions of Ψ. Similar to the proof of Theorem 4.1, we obtain that Φ_λ has a unique positive fixed point $u_\lambda \in K$ that is, $(\lambda\Psi)u_\lambda = u_\lambda$. So u_λ is a unique positive solution of (1.1). Note that the unique positive solution u_λ is in the cone K. In view of condition (H_5), we can prove terms (i)-(iii). □

We remark that the function f satisfying the conditions of Theorem 4.2 can be easily found. For example

$$f(u) = u^{\alpha_1} + u^{\alpha_2} + \cdots + u^{\alpha_m} + \frac{1}{2},$$

where $\alpha_i > 0$, m is a positive integer.

References

[1] Weigao Ge and Jingli Ren, *New existence theorem of positive solutions for Sturm-Liouville boundary value problems*, Appl. Math. Comput., **148** (2004), 631–644.

[2] Jingxian Sun and Guowei Zhang, *Positive solutions of singular nonlinear Sturm-Liouville problems*, Aata Math. Sinica (in Chinese), **48** (2005), 1196–1104.

[3] Sun Yan, Beilong Xu and Lishan Liu, *Positive solutions of singular boundary value problems for Sturm-Liouville equations*, J. Sys. Sci. Math. Scis. (in Chinese), **25** (2005), 69–77.

[4] Junyu Wang, Wenjie Gao and Zhongxin Zhang, *Nonexistence existence and multiplicity results for Sturm-Liouville boundary value problem*, Aata Math. Sinica (in Chinese), **48** (2005), 739–746.

[5] Hairong Lian and Weigao Ge, *Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line*, J. Math. Anal. Appl., **321** (2006), 781–792.

[6] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Science Oress, Singapore, 1994.

[7] S. D. Taliaferro, *A nonlinear singular boundary value problem.* Nonliear Anal., **3** (1997), 897–904.

[8] Xi'an Xu, *Positive solutions of generalized Emden-Fowler equation*, Nonlinear Anal., **53** (2003), 23–44.

[9] Xi'an Xu, *Positive solutions for semiposition implusive differential boundary value problems*, Chinese Annals of Mathematics (in Chinese), **25A6** (2004), 717–724.

[10] Xiaoning Lin and Daqing Jiang, *Multiple positive solutions of Dirichlet boundary value problems for second order implusive differential equations*, J. Math. Anal. Appl., **321** (2006), 501–514.

[11] Eun Kyoung Lee and Yong-Hoon Lee, *Multiple positive solutions of singular two point boundary value problems for second order implusive differential equations*, Appl. Math. Comput., **156** (2004), 745–759.

[12] Dajun Guo, *Nonlinear Functional Analysis*, Shandong Sci. Tech. (in Chinese), Jinan, 2004.

[13] Dajun Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[14] K. Yosida, *Functional Analysis*, fourth ed., Springer-Verlag, Berlin, 1978.