Nonlinear Functional Analysis and Applications
Vol. 16, No. 3 (2011), pp. 297-304

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

THE COMPLETENESS OF CONE METRIC SPACE AND FIXED POINT THEOREMS IN IT

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Abstract. In this article, we consider the completeness of cone metric space, and prove some fixed point theorems in cone metric space.

1. Introduction

In [2] Huang and Zhang introduce cone metric spaces and prove some fixed point theorems for contractive mappings that generalize some results of fixed point in metric spaces.

We recall the definition of cone metric spaces and some properties of it.

Definition 1.1. ([2]) *Let E be a real Banach space and P a subset of E. Then, P is called a cone if and only if*
(1) P is closed, nonempty and $P \neq \{0\}$;
(2) $a, b \in R, a, b \geq 0; x, y \in P \Rightarrow ax + by \in P$;
(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

[0]Received October 25, 2010. Revised March 11, 2011.
[0]2000 Mathematics Subject Classification: 45E50, 47H10.
[0]Keywords: Cone metric space, completeness, fixed point.
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Definition 1.2. ([2]) *The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P.*

In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.3. ([2]) *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*
(1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.4. *Let $E = R^2$ and $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset R^2$, $X = R^2$. And suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P).*

Definition 1.5. ([2]) *Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $c \gg 0$, there is N such that for all $n > N, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.(i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$).*

Definition 1.6. ([2]) *Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X, if for any $c \in E$ with $c \gg 0$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.*

Definition 1.7. ([2]) *Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

2. Completeness of cone metric space

Lemma 2.1. ([2]) *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then*
(1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta (n \to \infty)$.
(2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta (n, m \to \infty)$.

Lemma 2.2. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. If for Cauchy sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x. Then $\{x_n\}$ also converges to x.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that $n_k > N, d(x_{n_k}, x) \ll \frac{c}{2}$. We have $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$. And $\{x_n\}$ is a Cauchy sequence, we have for all $n, n_k > N$, that $d(x_n, x_m) \ll \frac{c}{2}$. So we obtain $d(x_n, x) \ll c$. Therefore $\{x_n\}$ also converges to x. $\square$

Theorem 2.3. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $B(x_n, r_n) = \{x \in X : d(x, x_n) \leq r_n\}$ and $B(x_n, r_n) \supset B(x_{n+1}, r_{n+1})(\forall n \in N)$. Then X is complete if and only if as $r_n \to 0$, we can obtain $\bigcap_{n=1}^{\infty} B(x_n, r_n) \neq \emptyset$ (it is a single point set).*

Proof. Suppose X is complete. For all $m > n(m, n \in N), B(x_m, r_m) \subset B(x_n, r_n)$, we can see $d(x_n, x_m) < r_n \to 0$. Choose $c \in E$ with $0 \ll c$, there is N, for all $m, n > N, d(x_n, x_m) \ll c$, then $\{x_n\}$ is Cauchy sequence. And X is complete, so we can see $\{x_n\}$ is convergent. So there exists $x_0 \in X$ and that $\lim_{n\to\infty} x_n = x_0(x_0 \in \bigcap_{n=1}^{\infty} B(x_n, r_n))$. Conversely, if there exists $n_0 \in N$ and $x_0 \notin B(x_{n_0}, r_{n_0})$, then we can see $d = d(x_0, B(x_{n_0}, r_{n_0})) > 0$. So $d(x_0, x_n) \geq d(x_0, B(x_n, r_n)) \geq d(x_0, B(x_{n_0}, r_{n_0})) = d > 0$, it is contradictory with $\lim_{n\to\infty} x_n = x_0$. So $x_0 \in \bigcap_{n=1}^{\infty} B(x_n, r_n)$. At the same time,if there is another $\bar{x}_0 \neq x_0$, we also have $\bar{x}_0 \in \bigcap_{n=1}^{\infty} B(x_n, r_n)$, then we can obtain $0 < d(x_0, \bar{x}_0) \leq r_n(\forall n \in N)$, it is contradictory with $r_n \to 0(n \to 0)$. Then we can obtain $\bigcap_{n=1}^{\infty} B(x_n, r_n)$ is a single point set.

Conversely, suppose if $r_n \to 0$, then $\bigcap_{n=1}^{\infty} B(x_n, r_n) \neq \emptyset$. Arbitrary Cauchy sequence $\{x_n\}$, we can choose subsequence $\{x_{n_k}\}$ and $d(x_{n_k}, x_{n_{k+1}}) \ll \frac{c}{2^k}$, and we can see that $B(x_{n_k}, \frac{c}{2^{k-1}}) \subset B(x_{n_{k-1}}, \frac{c}{2^{k-2}})$ and $\frac{c}{2^{k-1}} \to 0$, so we can obtain that $\bigcap_{n=1}^{\infty} B(x_{n_k}, \frac{c}{2^{k-1}}) \neq \emptyset$. We can choose $x_0 \in \bigcap_{n=1}^{\infty} B(x_{n_k}, \frac{c}{2^{k-1}})$, it is easy to obtain $\lim_{k\to\infty} x_{n_k} = x_0$. From Lemma2.2 we can obtain that $\lim_{n\to\infty} x_n = x_0$. So X is complete. $\square$

Corollary 2.4. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $F_n = B(x_n, r_n) \in (X, d)$, arbitrary $\{F_n\}(F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots) \in (X, d)$. If $D_n = \sup_{x_1, x_2 \in F_n} d(x_1, x_2) \to 0(n \to \infty)$, then there exists a $x_0 \in X$ s.t. $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.*

3. Fixed point theorems

In 1982, Brian Fisher proved a fixed point theorem[1]. On the basis of it, we prove our main results, some fixed point theorems in the cone metric space.

Theorem 3.1. *Let $(X,d),(Y,p)$ be a complete cone metric space, P be a normal cone with normal constant K, the continuous mapping $T : X \to Y$ and the mapping $S : Y \to X$ satisfies:*

$$d(STx, STx') \le C \max\{d(x,x'), \frac{1}{2}[d(x,STx) + d(x',STx')], p(Tx,Tx')\};$$

$$p(TSy, TSy') \le C \max\{p(y,y'), \frac{1}{2}[p(y,TSy) + p(y',TSy')], d(Sy,Sy')\}$$

$\forall x,x' \in X, y,y' \in Y$, $0 < C < 1$. *Then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. And $Tz = w, Sw = z$.*

Proof. Choose $x \in X$. Set $x_n = (ST)^n x, y_n = T(ST)^{n-1}x, (n = 1,2,\cdots)$, we can see sequence $\{x_n\} \subset X, \{y_n\} \subset Y$. We have

$$\begin{aligned} &d(x_n, x_{n+1}) \\ &= d(ST(ST)^{n-1}x, ST(ST)^n x) \\ &\le C \max\{d(x_{n-1},x_n), \frac{1}{2}[d(x_{n-1},x_n) + d(x_n,x_{n+1})], p(y_n,y_{n+1})\} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &p(y_n, y_{n+1}) \\ &= p(TST(ST)^{n-2}x, TST(ST)^{n-1}x) \\ &\le C \max\{p(y_{n-1},y_n), \frac{1}{2}[p(y_{n-1},y_n) + p(y_n,y_{n+1})], d(x_{n-1},x_n)\}. \end{aligned} \tag{3.2}$$

We now have four cases. Firstly we consider:
(1) if $d(x_n,x_{n+1}) \le d(x_{n-1},x_n), p(y_n,y_{n+1}) \le p(y_{n-1},y_n)$, by the inequality (3.1),(3.2) we can obtain

$$d(x_n,x_{n+1}) \le C \max\{d(x_{n-1},x_n), p(y_n,y_{n+1})\}, \tag{3.3}$$

$$p(y_n,y_{n+1}) \le C \max\{p(y_{n-1},y_n), d(x_{n-1},x_n)\}. \tag{3.4}$$

Then we consider the other three cases,
(2) if $d(x_n,x_{n+1}) > d(x_{n-1},x_n), p(y_n,y_{n+1}) > p(y_{n-1},y_n)$, by the inequality (3.1),(3.2) we can obtain

$$\begin{aligned} d(x_n,x_{n+1}) &\le C \max\{d(x_n,x_{n+1}), p(y_n,y_{n+1})\} = Cp(y_n,y_{n+1}) \\ &\le C \max\{d(x_{n-1},x_n), p(y_n,y_{n+1})\}, \\ p(y_n,y_{n+1}) &\le C \max\{d(x_{n-1},x_n), p(y_n,y_{n+1})\} = Cd(x_{n-1},x_n) \\ &\le C \max\{d(x_{n-1},x_n), p(y_{n-1},y_n)\}, \end{aligned}$$

(3) if $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), p(y_n, y_{n+1}) > p(y_{n-1}, y_n)$, by the inequality (3.1),(3.2) we can obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq C \max\{d(x_{n-1}, x_n), p(y_n, y_{n+1})\}, \\ p(y_n, y_{n+1}) &\leq C \max\{d(x_{n-1}, x_n), p(y_n, y_{n+1})\} = Cd(x_{n-1}, x_n) \\ &\leq C \max\{d(x_{n-1}, x_n), p(y_{n-1}, y_n)\}, \end{aligned}$$

(4) if $d(x_n, x_{n+1}) > d(x_{n-1}, x_n), p(y_n, y_{n+1}) \leq p(y_{n-1}, y_n)$, by the inequality (3.1),(3.2) we can obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq C \max\{d(x_n, x_{n+1}), p(y_n, y_{n+1})\} = Cp(y_n, y_{n+1}) \\ &\leq C \max\{d(x_{n-1}, x_n), p(y_n, y_{n+1})\}, \\ p(y_n, y_{n+1}) &\leq C \max\{d(x_{n-1}, x_n), p(y_{n-1}, y_n)\}. \end{aligned}$$

We can obtain the same result on three cases above with the similar method, it is the inequality (3.3),(3.4).

By the mathematical induction and the inequality (3.3),(3.4), we can obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq C^n \max\{d(x, x_1), p(y_1, y_2)\}; \\ p(y_n, y_{n+1}) &\leq C^{n-1} \max\{d(x, x_1), p(y_1, y_2)\}. \end{aligned}$$

By $0 < C < 1$, we can see
$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq \frac{C^m}{1-C} d(x_1, x_0)$. We get $\|d(x_n, x_m)\| \leq \frac{C^m}{1-C} K \|d(x_1, x_0)\|$. This implies $d(x_n, x_m) \to \infty (n, m \to \infty)$. Hence $\{x_n\} \subset X, \{y_n\} \subset Y$ is Cauchy sequence. By the completeness of X, Y, let $x_n \to z \in X, y_n \to w \in Y$, and the mapping T is continues, we can see

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{n-1} = Tz = w$$

and

$$d(STz, x_n) \leq C \max\{d(z, x_{n-1}), \frac{1}{2}[d(z, STz) + d(x_{n-1}, x_n)], p(Tz, Tx_{n-1})\}.$$

Let $n \to \infty$, we can see
$d(STz, z) \leq C \max\{d(z, z), \frac{1}{2}[d(z, STz) + d(z, z)], p(Tz, Tz)\} = \frac{1}{2} Cd(z, STz)$. And $0 < C < 1$, so we can obtain $z = STz$. That is $STz = Sw = z$, so z is the fixed point of ST.

Now we assume ST has another fixed point z', then

$$\begin{aligned} d(z, z') &= d(STz, STz') \\ &\leq C \max\{d(z, z'), \frac{1}{2}[d(z, STz) + d(z', STz')], p(Tz, Tz')\} \\ &= C \max\{d(z, z'), p(Tz, Tz')\} = Cp(Tz, Tz'); \end{aligned}$$

$$\begin{aligned} but \quad & p(Tz, Tz') = p(TSTz, TSTz') \\ & \leq C\max\{p(Tz, Tz'), \frac{1}{2}[p(Tz, TSTz) + p(Tz', TSTz')], d(STz, STz')\} \\ & = Cd(z, z'), \end{aligned}$$

so we can obtain $d(z, z') \leq C^2 d(z, z')$, and $0 < C < 1$, we can see $z = z'$.

With the same method, we can obtain w is the unique fixed point of TS. □

Theorem 3.2. *Let $(X, d), (Y, p)$ be a complete cone metric space, P be a normal cone with normal constant K, the continuous mapping $T : X \to Y$ and the mapping $S : Y \to X$ satisfies:*

$$\begin{aligned} d(STx, STx') \leq C\max\{& d(x, x'), d(x, STx), d(x', STx'), \\ & \frac{1}{2}[d(x, STx') + d(x', STx)], p(Tx, Tx')\}; \end{aligned}$$

$$\begin{aligned} p(TSy, TSy') \leq C\max\{& p(y, y'), p(y, TSy), p(y', TSy'), \\ & \frac{1}{2}[p(y, TSy') + p(y', TSy)], d(Sy, Sy')\} \end{aligned}$$

$\forall x, x' \in X, y, y' \in Y$, $0 < C < 1$. Then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. And $Tz = w, Sw = z$.

Proof. Choose $x \in X$. Set $x_n = (ST)^n x, y_n = T(ST)^{n-1}x, (n = 1, 2, \cdots)$, we can see sequence $\{x_n\} \subset X, \{y_n\} \subset Y$. We have

$$\begin{aligned} & d(x_n, x_{n+1}) \\ & = d(ST(ST)^{n-1}x, ST(ST)^n x) \\ & \leq C\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ & \qquad \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)], p(y_n, y_{n+1})\} \\ & \leq C\max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], p(y_n, y_{n+1})\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & p(y_n, y_{n+1}) \\ & = p(TST(ST)^{n-2}x, TST(ST)^{n-1}x) \\ & \leq C\max\{p(y_{n-1}, y_n), \frac{1}{2}[p(y_{n-1}, y_n) + p(y_n, y_{n+1})], d(x_{n-1}, x_n)\}. \end{aligned} \tag{3.6}$$

With the same prove method as the Theorem 3.1, we can obtain the inequality as the same as (3.3),(3.4).

$$d(x_n, x_{n+1}) \leq C\max\{d(x_{n-1}, x_n), p(y_n, y_{n+1})\}, \tag{3.7}$$

$$p(y_n, y_{n+1}) \leq C\max\{p(y_{n-1}, y_n), d(x_{n-1}, x_n)\}. \tag{3.8}$$

By $0 < C < 1$, we can see
$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq \frac{C^m}{1-C} d(x_1, x_0)$.
We get $\|d(x_n, x_m)\| \leq \frac{C^m}{1-C} K \|d(x_1, x_0)\|$. This implies $d(x_n, x_m) \to 0 (n, m \to \infty)$. Hence $\{x_n\} \subset X, \{y_n\} \subset Y$ is Cauchy sequence. By the completeness of X, Y, let $x_n \to z \in X, y_n \to w \in Y$, and the mapping T is continues, we can see

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{n-1} = Tz = w$$

$$\begin{aligned} & d(STz, x_n) \\ \leq & C \max\{d(z, x_{n-1}), d(z, STz), d(x_{n-1}, x_n), \frac{1}{2}[d(z, x_n) + d(x_{n-1}, STz)], \\ & \quad p(Tz, Tx_{n-1})\}. \end{aligned}$$

Let $n \to \infty$. Then , we can see

$$\begin{aligned} & d(STz, z) \\ & \leq C \max\{d(z, z), d(z, STz), d(z, z), \frac{1}{2}[d(z, STz) + d(z, z)], p(Tz, Tz)\} \\ & = Cd(z, STz). \end{aligned}$$

And $0 < C < 1$, so we can obtain $z = STz$. That is $STz = Sw = z$, so z is the fixed point of ST.

Now we assume ST has another fixed point z', then

$$\begin{aligned} & d(z, z') \\ & = d(STz, STz') \\ & \leq C \max\{d(z, z'), d(z, STz), d(z', STz'), \frac{1}{2}[d(z, STz') + d(z', STz)], p(Tz, Tz')\} \\ & = C \max\{d(z, z'), p(Tz, Tz')\} = Cp(Tz, Tz'); \end{aligned}$$

but

$$\begin{aligned} & p(Tz, Tz') = p(TSTz, TSTz') \\ & \leq C \max\{p(Tz, Tz'), p(Tz, TSTz), p(Tz', TSTz'), \\ & \qquad \frac{1}{2}[p(Tz, TSTz') + p(Tz', TSTz)], d(STz, STz')\} \\ & = Cd(z, z'), \end{aligned}$$

so we can obtain $d(z, z') \leq C^2 d(z, z')$, and $0 < C < 1$, we can see $z = z'$. With the same method, we can obtain w is the unique fixed point of TS. □

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