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THE QUADRATIC FUNCTIONAL EQUATION IN MENGER PROBABILISTIC NORMED SPACES

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Abstract. We prove the stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

in the setting of Menger probabilistic normed spaces. As application, we get some stability results in the setting of both classical and non-classical normed spaces.

1. Introduction

The study of stability problem for functional equations goes back to a question raised by Ulam [27] concerning the stability of group homomorphisms that affirmatively answered for Banach spaces by Hyers [9]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. The paper [21] of Rassias has provided a lot of influence in the development of

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what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers to e.g. [7, 10, 12, 23] for more information on such problems.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is said to be the quadratic functional equation. Skof [26] was the first one who investigated the stability of quadratic equation. She showed that, if f is a mapping from a normed space X into a Banach space Y satisfying

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \epsilon$$

for some $\epsilon > 0$, then there is a unique quadratic function $g : X \to Y$ such that

$$\|f(x)-g(x)\| \le \frac{\epsilon}{2}.$$

Cholewa [5] extended Skof's theorem by replacing X by an abelian group Skof's result was later generalized by Czerwik [6] in the spirit of Hyers–Ulam–Rassias. The stability problem of the quadratic equation has been extensively investigated by a number of mathematicians, see e.g. [7, 11, 13, 14, 18, 20, 22] and references therein. In addition, Mirmostafaee, Moslehian, Vishki [16, 17, 19], Alsina [1], Miheţ and Radu [15] investigated the stability of various functional equations in the settings of fuzzy, probabilistic and random normed spaces.

In the sequel, we shall adopt the usual terminology, notations and conventions of the theory of probabilistic normed spaces, as in [4, 15, 24, 25].

A function $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is non-decreasing and left continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$. The class of all distribution functions F with $F(0) = 0$ is denoted by D^+. ε_0 is the specific distribution function defined by

$$\varepsilon_0(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$$

Definition 1.1. (see [8]) *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a triangular norm (briefly, a t–norm) if T satisfies the following conditions:*

(i) T is commutative and associative;
(ii) $T(a,1) = a$ for all $a \in [0,1]$;
(iii) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ $\quad (a,b,c,d \in [0,1])$.

Three typical examples of continuous t–norms are $T_P(a,b) = ab$, $T_L(a,b) = \max(a+b-1,0)$ and $T_M(a,b) = \min(a,b)$.

Recall that, if T is a t–norm and $\{a_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n a_i$ is defined recursively by $T_{i=1}^1 a_i = a_1$ and $T_{i=1}^n a_i = T(T_{i=1}^{n-1} a_i, a_n)$ for $n \ge 2$. Define $T_{i=m+1}^n a_i = T_{i=1}^{n-m} a_{m+i}$ for each given number $m \ge 0$. In the

case where $\{a_n\}$ is the constant sequence with the term a, use the notation $T^{n-m}a = T_{i=m+1}^{n} a_i$. In general, if $a_i \geq a$, for all $i \geq 1$, then $T_{i=1}^{n} a_i \geq T^n a$.
A direct verification reveals that, for each $a \in [0,1]$, the sequence $\{T^n(a)\}$ is decreasing with $T^n(a) \geq a$ and $T^n(1) = 1$, for all n. We set $T^\infty(a) = \lim_{n\to\infty} T^n(a)$. For instance, we trivially have ${T_M}^\infty(a) = a$, for all $a \in [0,1]$ and ${T_P}^\infty(a) = 0$, for all $a \in [0,1)$.
We say that a t-norm T is of Hadžić type if the family $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$, that is,

$$\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1); a > 1-\delta \Rightarrow T^n(a) > 1-\varepsilon \quad (n \geq 1)\,.$$

T_M is a trivial example of a t-norm of Hadžić type, but T_P is not of Hadžić type.

Definition 1.2. ([4]) *A* Menger probabilistic normed space *(briefly, Menger PN-space) is a triple (X, μ, T), where X is a real vector space, T is a continuous t–norm, and μ is a mapping from X into D^+ such that the following conditions hold:*

(PN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(PN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all x in X, $\alpha \neq 0$ and $t \geq 0$;
(PN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

A classical example of a Menger PN-space is a normed space $(X, \|\cdot\|)$ endowed with $\mu_x(t) = \frac{t}{t+\|x\|}$ and T_M. A more illuminating example comes from fuzzy normed spaces. Following [3], a pair (X, N) is called a fuzzy normed space, in which X is a real linear space and $N\colon X \times \mathbb{R} \to [0,1]$ is a a mapping (the so-called fuzzy subset) satisfying

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
Then X equipped with $\mu_x(t) = N(x,t)$ and T_M can be regarded as a Menger PN-space.

If a t-norm T is such that $\sup_{a<1} T(a,a) = 1$ (and from now on we assume that our t-norms are not only continuous but also have a such property) then every Menger PN-space (X, μ, T) is metrizable (see [24]) whose topology τ induced by the base $\{\mathcal{U}(\epsilon,\lambda);\ \epsilon > 0,\ 0 < \lambda < 1\}$ of neighborhoods of 0, where $\mathcal{U}(\epsilon,\lambda) = \{x \in X;\ \mu_x(\epsilon) > 1-\lambda\}$.

Definition 1.3. *Let (X, μ, T) be a Menger PN-space.*

(i) A sequence $\{x_n\}$ *in* X *is said to be* convergent *to* x *in* X *in the topology* τ *if for every* $t > 0$ *and* $\varepsilon > 0$, *there exists positive integer* N *such that* $\mu_{x_n-x}(t) > 1-\varepsilon$ *whenever* $n \geq N$.

(ii) A sequence $\{x_n\}$ *in* X *is called* Cauchy *in the topology* τ *if, for every* $t > 0$ *and* $\varepsilon > 0$, *there exists positive integer* N *such that* $\mu_{x_n-x_m}(t) > 1-\varepsilon$ *whenever* $n \geq m \geq N$.

(iii) A Menger PN-space (X,μ,T) *is said to be* complete *if and only if every Cauchy sequence in* X *is convergent to a point in* X.

Theorem 1.4. ([24]) *For a sequence* $\{x_n\}$ *in the Menger PN-space* (X,μ,T), *if* $x_n \to x$ *then* $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$, *for all* $t \in \mathbb{R}$.

In this paper, we establish the stability of the quadratic functional equation in the setting of Menger probabilistic normed spaces.

2. Main results

We commence with the following theorem which is our main result.

Theorem 2.1. *Let* X *be a linear space,* (Z,ξ,T') *be a Menger PN-space, let* $\varphi : X\times X \to Z$ *be a mapping such that for some* $\alpha < 4$,

$$\xi_{\varphi(2x,2y)}(t) \geq \xi_{\alpha\varphi(x,y)}(t) \quad (x,y\in X, t>0) \tag{2.1}$$

and let (Y,μ,T) *be a complete Menger PN-space in which* T *is of Hadžić type. If* $f : X \to Y$ *is a mapping such that* $f(0)=0$ *and*

$$\mu_{f(x+y)+f(x-y)-2f(x)-2f(y)}(t) \geq \xi_{\varphi(x,y)}(t) \quad (x,y\in X, t>0), \tag{2.2}$$

then there exists a unique quadratic mapping $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq T^{\infty}(\xi_{\varphi(x,x)}((4-\alpha)t)). \tag{2.3}$$

Proof. Putting $y = x$ in (2.2) we get

$$\mu_{f(2x)-4f(x)}(t) \geq \xi_{\varphi(x,x)}(t) \quad (x\in X, t>0), \tag{2.4}$$

that is

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \xi_{\varphi(x,x)}(4t)\,.$$

Replacing x by $2^{k-1}x$ in (2.4), we obtain

$$\mu_{f(2^kx)-4f(2^{k-1}x)}(t) \geq \xi_{\varphi(2^{k-1}x,2^{k-1}x)}(t)$$

whence, by (2.1) and PN(2),

$$\mu_{\frac{f(2^kx)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}}\left(\frac{t}{4^k}\right) \geq \xi_{\varphi(x,x)}\left(\frac{t}{\alpha^{k-1}}\right)$$

so

$$\mu_{\frac{f(2^kx)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}}\left(\frac{1}{4}\left(\frac{\alpha}{4}\right)^{k-1}t\right) \geq \xi_{\varphi(x,x)}(t) \quad (x\in X, t>0, k\geq 1).$$

We also have

$$
\begin{aligned}
\mu_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}} & \left(\frac{1}{4}\sum_{k=m+1}^{n}(\frac{\alpha}{4})^{k-1}t\right) \\
&= \mu_{\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}\right)}\left(\sum_{k=m+1}^{n}(\frac{\alpha}{4})^{k-1}\frac{t}{4}\right) \\
&\geq T_{k=m+1}^{n}\left(\mu_{\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}}((\frac{\alpha}{4})^{k-1}\frac{t}{4})\right) \\
&\geq T^{n-m}\left(\xi_{\varphi(x,x)}(t)\right).
\end{aligned} \tag{2.5}
$$

It follows from (2.5) that

$$
\mu_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}(t) \geq T^{n-m}\left(\xi_{\varphi(x,x)}\left(\frac{4t}{\sum_{k=m+1}^{n}(\frac{\alpha}{4})^{k-1}}\right)\right). \tag{2.6}
$$

Let $\varepsilon > 0$, $t > 0$ be given. Since T is of Hadžić type, so there is $\delta > 0$ such that for each a with $a > 1-\delta$ we have that

$$
T^n(a) > 1-\varepsilon \quad (n \geq 1). \tag{2.7}
$$

Since $\sum_{k=1}^{\infty}(\frac{\alpha}{4})^{k-1} < \infty$, it follows from $\xi_{\varphi(x,x)} \in D^+$ that there is $N_0 > 0$ such that for each $m, n > N_0$,

$$
\xi_{\varphi(x,x)}\left(\frac{4t}{\sum_{k=m+1}^{n}(\frac{\alpha}{4})^{k-1}}\right) > 1-\delta\,,
$$

whence, by (2.6) and (2.7), we have

$$
\mu_{\frac{f(2^m x)}{4^m}-\frac{f(2^n x)}{4^n}}(t) > 1-\varepsilon.
$$

Thus $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence. Since (Y, μ, T) is complete we can set $Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^{2n}}$ for all $x \in X$. Putting $m = 0$ in (2.6) we get

$$
\begin{aligned}
\mu_{\frac{f(2^n x)}{4^n}-f(x)}(t) &\geq T^n\left(\xi_{\varphi(x,x)}(\frac{4t}{\sum_{k=1}^{n}(\frac{\alpha}{4})^{k-1}})\right) \\
&\geq T^n\left(\xi_{\varphi(x,x)}(\frac{4t}{\sum_{k=1}^{\infty}(\frac{\alpha}{4})^{k-1}})\right) \\
&= T^n\big(\xi_{\varphi(x,x)}((4-\alpha)t)\big).
\end{aligned}
$$

Taking the limit as $n \to \infty$, then Theorem 1.4 implies that

$$
\mu_{Q(x)-f(x)}(t) \geq T^{\infty}(\xi_{\varphi(x,x)}((4-\alpha)t)).
$$

Now, we show that Q is a quadratic mapping. Replacing x, y with $2^n x$ and $2^n y$, respectively, in (2.2) to get

$$\begin{aligned} \mu_{\frac{f(2^n x+2^n y)}{4^n}+\frac{f(2^n x-2^n y)}{4^n}-2\frac{f(2^n x)}{4^n}-2\frac{f(2^n y)}{4^n}}(t) &\geq \xi_{\varphi(2^n x, 2^n y)}(4^n t) \\ &\geq \xi_{\varphi(x,y)}((\frac{4}{\alpha})^n t)\,. \end{aligned} \tag{2.8}$$

Since $\lim_{n\to\infty} \xi_{\varphi(x,y)}((\frac{4}{\alpha})^n t) = 1$ we conclude from (2.8) that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y).$$

To prove the uniqueness of the quadratic function Q, let us assume that there exists a quadratic function Q' satisfying (2.3). Obviously we have $Q(2^m x) = 4^m Q(x)$ and $Q'(2^m x) = 4^m Q'(x)$ for all $x \in X$ and $m \in \mathbb{N}$. It follows from (2.3) that for each $x \in X$,

$$\begin{aligned} \mu_{Q(x)-Q'(x)}(t) &= \mu_{Q(2^m x)-Q'(2^m x)}(2^{2m} t) \\ &\geq T(\mu_{Q(2^m x)-f(2^m x)}(2^{2m-1} t), \mu_{f(2^m x)-Q'(2^m x)}(2^{2m-1} t)) \\ &\geq T(T^\infty(\xi_{\varphi(x,x)}(2^{2m-1}(4-\alpha)t)), T^\infty(\xi_{\varphi(x,x)}(2^{2m-1}(4-\alpha)t)). \end{aligned}$$

Taking limit as $m \to \infty$ we conclude that $\mu_{Q(x)-Q'(x)}(t) = 1$, namely $Q(x) = Q'(x)$. $\square$

We conclude the paper with some applications of Theorem 2.1 in the setting of fuzzy normed spaces. Recall that every fuzzy normed space (X, N) can be regarded as a Menger PN-space (X, μ, T_M) with $\mu_x(t) = N(x, t)$. Now Theorem 2.1 yields the following result, which can be compared with other results of [17] in this framework.

Corollary 2.2. *Let X be a linear space, (Z, N) be a fuzzy normed space, $\varphi : X \times X \to Z$ be a mapping such that for some $\alpha < 4$,*

$$N(\varphi(2x, 2y), t) \geq N(\alpha\varphi(x, y), t) \quad (x, y \in X, t > 0)$$

and let (Y, M) be a fuzzy Banach space. If $f : X \to Y$ is a mapping such that $f(0) = 0$ and

$$M(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \geq N(\varphi(x, y), t) \quad (x, y \in X, t > 0),$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$M(f(x) - Q(x), t) \geq N(\varphi(x, x), (4-\alpha)t)).$$

As a consequence of the latter corollary we have the following result which gives a better estimation than [17, Theorem 2.3] (see also [10]).

Corollary 2.3. *Let f be a mapping from a normed space $(X, \|\cdot\|)$ into a Banach space $(Y, |||\cdot|||)$ such that $f(0) = 0$. Let for some $0 < p < 2$,*

$$|||f(x+y) + f(x-y) - 2f(x) - 2f(y)||| \leq \|x\|^p + \|y\|^p \quad (x, y \in X).$$

Then there is a unique quadratic function $Q : X \to Y$ such that

$$|||Q(x) - f(x)||| \leq \frac{1}{2 - 2^{p-1}} ||x||^p \qquad (x \in X).$$

Proof. Define $\varphi : X \times X \to \mathbb{R}$ by $\varphi(x, y) = \|x\|^p + \|y\|^p$. Now use Corollary 2.2 for the latter φ and $\alpha = 2^p$ in the setting of the fuzzy normed spaces $(\mathbb{R}, N)$ and (Y, M) in which N and M are given by $N(x, t) = \frac{t}{t+|x|}$ and $M(x, t) = \frac{t}{t+|||x|||}$. □

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