Nonlinear Functional Analysis and Applications
Vol. 16, No. 3 (2011), pp. 313-321

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

CONVERGENCE TO ATTRACTING SETS FOR DISCRETE DISPERSIVE DYNAMICAL SYSTEMS

Alexander J. Zaslavski

Department of Mathematics,
The Technion-Israel Institute of Technology
32000 Haifa, Israel
e-mail: ajzasl@tx.technion.ac.il

Abstract. In this paper we study convergence of trajectories of discrete dispersive dynamical systems generated by set-valued mappings to their attracting sets.

1. Introduction

Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. A discrete-time dynamical system is described by a space of states and a transition operator which can be set-valued. Usually in the dynamical systems theory a transition operator is single-valued. In the present paper we study a class of dynamical systems introduced in [5] and studied in [6, 7, 9] with a compact metric space of states and a set-valued transition operator. Such dynamical systems describe economical models [2, 6, 8]. Note that convergence of trajectories of dynamical systems with a complete metric space of states which is not necessarily compact and with contractive and nonexpansive set-valued transition operators were studied in [1, 3, 4].

Let (X, ρ) be a compact metric space and let $a : X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in X \times X : \ y \in a(x)\}$$

[0]Received July 18, 2010. Revised October 29, 2010.
[0]2000 Mathematics Subject Classification: 37B99, 93C25.
[0]Keywords: Attracting set, compact metric space, set-valued mapping, trajectory.

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \ x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we study convergence of trajectories of the dynamical system generated by the set-valued mapping a. Following [5-7], [9] this system is called a discrete dispersive dynamical system.

First we define a trajectory of this system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Let T_1, T_2 be integers such that $0 \leq T_1 \leq T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t = T_1, \dots, T_2 - 1$.

Put

$$\Omega(a) = \{z \in X : \ \text{for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \text{ such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \epsilon\}. \tag{1.1}$$

Clearly, $\Omega(a)$ is closed subset of (X, ρ). In [9] the set $\Omega(a)$ was called a global attractor of a. Note that in [5-7] $\Omega(a)$ was called a turnpike set of a. This terminology was motivated by mathematical economics [2], [6], [8].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : \ y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have $\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0$.

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

In [9] we studied convergence of trajectories to the global attractor $\Omega(a)$ and established the following two results.

Proposition 1.1. *Let $\epsilon > 0$. Then there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$*

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T(\epsilon)\} \leq \epsilon.$$

Proposition 1.2. *Let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \leq \delta$ for each integer $t \geq 0$ the following inequality holds:*

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T(\epsilon)\} \leq \epsilon.$$

A point $x \in X$ is called stable (with respect to a) if there exists a trajectory $\{x_t\}_{t=0}^{\infty}$ such that

$$x_0 = x \text{ and } \liminf_{t\to\infty} \rho(x_t, x) = 0.$$

Denote by $\Pi(a)$ the set of all stable points. Clearly, $\Pi(a) \subset \Omega(a)$.

Denote by $S(X)$ the set of all nonempty closed subsets of (X, ρ) equipped with the Hausdorff metric

$$H(A, B) = \max\{\sup_{x\in A} \rho(x, B),\ \sup_{y\in B} \rho(y, A)\} \tag{1.2}$$

which is defined for each pair of nonempty sets $A, B \subset X$. It is well-known that $(S(X), H)$ is a complete metric space.

We assume that the mapping $a : X \to S(X)$ is continuous.

In this paper we study the convergence of trajectories to the set $\Pi(a)$. We will establish the following results.

Theorem 1.3. *For each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^{\infty}$ such that*

$$\liminf_{t\to\infty} \rho(x_t, \Pi(a)) = 0.$$

Theorem 1.3 is proved in Section 2.

Theorem 1.4. *Assume that F is a nonempty subset of X and that for each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^{\infty}$ such that*

$$x_0 = x \text{ and } \liminf_{t\to\infty} \rho(x_t, F) = 0.$$

Then for each $\epsilon > 0$ there exists a natural number q such that for each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^{q}$ such that

$$x_0 = x \text{ and } \min\{\rho(x_t, F) :\ t = 1, \dots, q\} \le \epsilon.$$

Theorem 1.4 is proved in Section 3.

Theorems 1.3 and 1.4 imply the following result.

Theorem 1.5. *For each $\epsilon > 0$ there exists a natural number q such that for each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^{q}$ such that*

$$x_0 = x \text{ and } \min\{\rho(x_t, \Pi(a)) :\ t = 1, \dots, q\} \le \epsilon.$$

Theorem 1.6. *Assume that F is a nonempty subset of X and that for each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^{\infty}$ such that*

$$x_0 = x \text{ and } \liminf_{t\to\infty} \rho(x_t, F) = 0.$$

Let $\epsilon > 0$. Then there exist a natural number q and a positive number δ such that for each sequence of mappings $a_t : X \to 2^X \setminus \{\emptyset\}$, $t = 0, \dots, q-1$ which satisfy

$$H(a_t(x), a(x)) \le \delta \text{ for all } x \in X \text{ and all } t = 0, \dots, q-1$$

and each $x \in X$ there exists a sequence $\{x_t\}_{t=0}^q \subset X$ such that

$$x_0 = x,\ x_{t+1} \in a_t(x_t),\ t = 0, \dots, q-1$$

and

$$\min\{\rho(x_t, F) :\ t = 1, \dots, q\} \le \epsilon.$$

Theorem 1.6 is proved in Section 4.

Theorems 1.3 and 1.6 imply the following result.

Theorem 1.7. *Let $\epsilon > 0$. Then there exist a natural number q and a positive number δ such that for each sequence of mappings $a_t : X \to 2^X \setminus \{\emptyset\}$, $t = 0, \dots, q-1$ which satisfy*

$$H(a_t(x), a(x)) \le \delta \text{ for all } x \in X \text{ and all } t = 0, \dots, q-1$$

and each $x \in X$ there exists a sequence $\{x_t\}_{t=0}^q \subset X$ such that

$$x_0 = x,\ x_{t+1} \in a_t(x_t),\ t = 0, \dots, q-1$$

and

$$\min\{\rho(x_t, \Pi(a)) :\ t = 1, \dots, q\} \le \epsilon.$$

Theorem 1.6 implies the following result.

Theorem 1.8. *Assume that F is a nonempty subset of X and that for each $x \in X$ there exists a trajectory $\{x_t\}_{t=0}^\infty$ such that*

$$x_0 = x \text{ and } \liminf_{t \to \infty} \rho(x_t, F) = 0.$$

Let $\{\delta_t\}_{t=0}^\infty$ be a sequence of positive numbers such that

$$\lim_{t \to \infty} \delta_t = 0$$

and let $a_t : X \to 2^X \setminus \{\emptyset\}$, $t = 0, 1, \dots$ be a sequence of mappings which satisfy

$$H(a_t(x), a(x)) \le \delta_t \text{ for all } x \in X \text{ and all integers } t \ge 0.$$

Then each $x \in X$ there exists a sequence $\{x_t\}_{t=0}^\infty \subset X$ such that

$$x_0 = x,\ x_{t+1} \in a_t(x_t),\ t = 0, 1, \dots$$

and

$$\liminf_{t \to \infty} \rho(x_t, F) = 0.$$

Theorems 1.3 and 1.8 imply the following result.

Theorem 1.9. *Let* $\{\delta_t\}_{t=0}^{\infty}$ *be a sequence of positive numbers such that*

$$\lim_{t\to\infty} \delta_t = 0$$

and let $a_t : X \to 2^X \setminus \{\emptyset\}$, $t = 0, 1, \dots$ *be a sequence of mappings which satisfy*

$$H(a_t(x), a(x)) \le \delta_t \text{ for all } x \in X \text{ and all integers } t \ge 0.$$

Then each $x \in X$ *there exists a sequence* $\{x_t\}_{t=0}^{\infty} \subset X$ *such that*

$$x_0 = x,\ x_{t+1} \in a_t(x_t),\ t = 0, 1, \dots$$

and

$$\liminf_{t\to\infty} \rho(x_t, \Pi(a)) = 0.$$

2. Proof of Theorem 1.3

Denote by $\mathcal{M}$ the collection of all nonempty closed sets $D \subset X$ such that $a(D) \subset D$. Clearly, $\mathcal{M} \ne \emptyset$ because $X \in \mathcal{M}$.

Let $D_1, D_2 \in \mathcal{M}$. We say that $D_1 \le D_2$ if $D_1 \subset D_2$.

The Zorn's lemma implies the following result.

Lemma 2.1. *For each* $D \in \mathcal{M}$ *there is a minimal element* D_0 *of* $\mathcal{M}$ *such that* $D_0 \subset D$.

For each $x \in X$ denote by $E(x)$ the closure of $\cup_{i=0}^{\infty} a^i(x)$.

The next lemma follows from the continuity of a.

Lemma 2.2. *Let* $x \in X$. *Then* $E(x) \in \mathcal{M}$.

Lemmas 2.1 and 2.2 imply the following result.

Lemma 2.3. *Let* D *be a minimal element of* $\mathcal{M}$. *Then for each* $x \in D$, $D = E(x)$.

Corollary 2.4. *Let* D *be a minimal element of* $\mathcal{M}$, $x, y \in D$ *and* $\epsilon > 0$. *Then there exist an integer* $q \ge 1$ *and a trajectory* $\{x_t\}_{t=0}^{q}$ *such that* $x_0 = x$ *and* $\rho(x_q, y) \le \epsilon$.

Corollary 2.5. *Let* D *be a minimal element of* $\mathcal{M}$. *Then* $D \subset \Pi(a)$.

If we have the following lemma, then the Theorem 1.3 is proved.

Lemma 2.6. *Let* $x \in X$ *and* $\epsilon > 0$. *Then there exist an integer* $q \ge 1$ *and a trajectory* $\{x_t\}_{t=0}^{q} \subset X$ *such that* $x_0 = x$ *and* $\rho(x_q, \Pi(a)) \le \epsilon$.

Proof. Fix

$$x_1 \in a(x). \tag{2.1}$$

By Lemmas 2.1 and 2.2 there is a minimal element $D_* \in \mathcal{M}$ such that

$$D_* \subset E(x_1). \tag{2.2}$$

Fix

$$z \in D_*. \tag{2.3}$$

By (2.1)-(2.3) there exist a natural number q and a trajectory $\{x_t\}_{t=0}^q$ such that

$$x_0 = x \text{ and } \rho(x_q, z) \leq \epsilon. \tag{2.4}$$

In view of (2.3), (2.4) and Lemma 2.2

$$\rho(x_q, \Pi(a)) \leq \epsilon.$$

Lemma 2.6 is proved. □

Hence, Theorem 1.3 follows from Lemma 2.6.

3. Proof of Theorem 1.4

Let $\epsilon > 0$ and $x \in X$. There exist an integer $q \geq 1$ and a trajectory $\{x_t\}_{t=0}^q$ such that

$$x_0 = x \text{ and } \rho(x_q, F) < \epsilon/4. \tag{3.1}$$

Put

$$\delta_q = \epsilon/8. \tag{3.2}$$

By induction it is not difficult to show the existence of a sequence of positive numbers $\{\delta_i\}_{i=0}^q$ such that for each integer $i \in [0, q-1]$

$$\delta_i < 2^{-1}\delta_{i+1} \tag{3.3}$$

and

$$H(a(x_i), a(z)) < \delta_{i+1} \text{ for each } z \in X \text{ satisfying } \rho(x_i, z) \leq \delta_i. \tag{3.4}$$

Let

$$y \in X \text{ and } \rho(x, y) < \delta_0. \tag{3.5}$$

Set

$$y_0 = y. \tag{3.6}$$

By (3.1), (3.4), (3.5), (3.6) and the inclusion $x_1 \in a(x_0)$ there is

$$y_1 \in a(y_0) \tag{3.7}$$

such that

$$\rho(y_1, x_1) < \delta_1. \tag{3.8}$$

Assume that an integer k satisfies $1 \leq k \leq q$ and we defined $y_0, \ldots, y_k \in X$ such that

$$y_{i+1} \in a(y_i) \text{ for all integers } i \text{ satisfying } 0 \leq i < k \tag{3.9}$$

and

$$\rho(x_i, y_i) \leq \delta_i, \; i = 0, \ldots, k. \tag{3.10}$$

(In view of (3.5), (3.7) and (3.8) our assumption holds for $k = 1$.)

If $k = q$, then our construction is completed.

Assume that $k < q$. By (3.4) and (3.10)

$$H(a(x_k), a(y_k)) < \delta_{k+1}.$$

Combined with the inclusion $x_{k+1} \in a(x_k)$ this implies that there is $y_{k+1} \in a(y_k)$ such that $\rho(x_{k+1}, y_{k+1}) < \delta_{k+1}$ and the assumption made for k holds also for $k+1$. Therefore by induction we constructed a sequence $\{y_i\}_{i=0}^{q}$ such that

$$y_0 = y,$$

$$y_{i+1} \in a(y_i), \; i = 0, \ldots, q-1,$$

$$\rho(x_q, y_q) < \delta_q. \tag{3.11}$$

By (3.1), (3.2) and (3.11)

$$\rho(y_q, F) \leq \rho(x_q, y_q) + \rho(x_q, F) < \epsilon.$$

Thus we have shown that for each $x \in X$ there exist an open neighborhood V_x of x in X and a natural number $q(x)$ such that for each $y \in V_x$ there is a trajectory $\{y_i\}_{i=0}^{q(x)}$ such that

$$y_0 = y \text{ and } \rho(y_{q(x)}, F) < \epsilon. \tag{3.12}$$

Clearly, $X \subset \cup_{x \in X} V_x$. Since X is compact there exists a finite sequence $x_1, \ldots, x_p \in X$ such that

$$\cup\{V_{x_i} : \; i = 1, \ldots, p\} = X. \tag{3.13}$$

Put

$$q = \max\{q(x_i) : \; i = 1, \ldots, p\}. \tag{3.14}$$

Let $x \in X$. By (3.13) there is a natural number $i \leq p$ such that

$$x \in V_{x_i}. \tag{3.15}$$

By (3.15) and the choice of V_{x_i} and $q(x_i)$ there exists a trajectory $\{x_j\}_{j=0}^{q(x_i)}$ such that

$$x_0 = x, \; \rho(x_{q(x_i)}, F) < \epsilon.$$

This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.6

Let $\epsilon > 0$. By Theorem 1.4 there exists a natural number q such that the following property holds:

(P1) For each $x \in X$ there exists a trajectory $\{x_i\}_{i=0}^q$ such that

$$x_0 = x \text{ and } \min\{\rho(x_t, F) : \ t = 1, \dots, q\} \le \epsilon/8.$$

Put

$$\delta_q = \epsilon/8. \tag{4.1}$$

Since X is compact and the mapping a is continuous there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^q$ such that for each integer $i \in [1, q]$

$$\delta_{i-1} < 2^{-1}\delta_i \tag{4.2}$$

and

$$H(a(y), a(z)) \le 2^{-1}\delta_i \text{ for each } y, z \in X \text{ satisfying } \rho(y, z) < \delta_{i-1}. \tag{4.3}$$

Put

$$\delta = \delta_0/4. \tag{4.4}$$

Let

$$a_i : X \to 2^X \setminus \{\emptyset\}, \ i = 0, \dots, q-1, \tag{4.5}$$

$$H(a_i(x), a(x)) \le \delta \text{ for all } x \in X \text{ and all } t = 0, \dots, q-1 \tag{4.6}$$

and $x \in X$. By (P1) there exists a sequence $\{y_i\}_{i=0}^q \subset X$ such that

$$y_0 = x, \ y_{i+1} \in a(y_i), \ i = 0, \dots, q-1 \tag{4.7}$$

and

$$\min\{\rho(y_i, F) : \ i = 1, \dots, q\} \le \epsilon/8. \tag{4.8}$$

Assume that an integer $j \in [0, q-1]$ and we defined $x_0, \dots, x_j \in X$ such that

$$x_0 = x, \tag{4.9}$$

$$x_{i+1} \in a_i(x_i) \text{ for all integers } i \text{ satisfying } 0 \le i \le j-1, \tag{4.10}$$

and

$$\rho(x_i, y_i) \le \delta_i, \ i = 0, \dots, j. \tag{4.11}$$

(Clearly, for $j = 0$ the assumption holds.) By (4.11),

$$\rho(x_j, y_j) \le \delta_j.$$

Combined with (4.3) this implies

$$H(a(x_j), a(y_j)) < 2^{-1}\delta_{j+1}.$$

Together with (4.7) this implies that

$$\rho(y_{j+1}, a(x_j)) < 2^{-1}\delta_{j+1}. \tag{4.12}$$

By (4.2), (4.4) and (4.6)

$$H(a_j(x_j), a(x_j)) < 4^{-1}\delta_{j+1}.$$

Combined with (4.12) this implies that

$$\rho(y_{j+1}, a_j(x_j)) < (3/4)\delta_{j+1}$$

and there is $x_{j+1} \in a_j(x_j)$ such that $\rho(x_{j+1}, y_{j+1}) < \delta_{j+1}$. Therefore the assumption made for j holds also for $j+1$. Thus by induction we constructed a sequence $\{x_i\}_{i=0}^{q} \subset X$ such that

$$x_0 = x,$$

$$x_{i+1} \in a_i(x_i),\ i = 0, \dots, q-1, \tag{4.13}$$

$$\rho(x_i, y_i) \leq \epsilon/8,\ i = 0, \dots, q. \tag{4.14}$$

By (4.8) there is $j \in \{1, \dots, q\}$ such that $\rho(y_j, F) \leq \epsilon/8$. Combined with (4.14) this implies that $\rho(x_j, F) \leq \epsilon/2$. This completes the proof of Theorem 1.6.

References

[1] F.S. de Blasi, J. Myjak, S. Reich and A.J. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued and Variational Analysis, **17** (2009), 97–112.

[2] V.L. Makarov and A.M. Rubinov, *Mathematical theory of economic dynamics and equilibria*, Springer-Verlag, New York, 1977.

[3] S. Reich and A.J. Zaslavski, *Convergence of inexact iterative schemes for nonexpansive set-valued mappings*, Fixed Point Theory Appl., **2010** (2010).

[4] S. Reich and A.J. Zaslavski, *Approximating fixed points of contractive set-valued mappings*, Communications in Mathematical Analysis, **8** (2010), 70–78.

[5] A.M. Rubinov, *Turnpike sets in discrete disperse dynamical systems*, Sib. Math. J., **21** (1980), 136–146.

[6] A.M. Rubinov, *Multivalued mappings and their applications in economic mathematical problems*, Nauka, Leningrad, 1980.

[7] A.J. Zaslavski, *Turnpike sets of continuous transformations in compact metric spaces*, Sib. Math. J., **23** (1982), 198–203.

[8] A.J. Zaslavski, *Turnpike properties in the calculus of variations and optimal control*, Springer, New York, 2006.

[9] A.J. Zaslavski, *Uniform convergence to global attractors for discrete disperse dynamical systems*, Nonlinear Dynamics and System Theory, **7** (2007), 315–325.