Nonlinear Functional Analysis and Applications
Vol. 16, No. 3 (2011), pp. 323-329

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ITERATIVE SOLUTION OF ASYMPTOTICALLY ACCRETIVE OPERATOR EQUATIONS

Niyati Gurudwan[1] and B. K. Sharma[2]

[1]School of Studies in Mathematics, Pt.Ravishankar Shukla University,
Raipur - 492010 (C.G.), India
e-mail: niyati.kuhu@gmail.com,

[2]School of Studies in Mathematics, Pt.Ravishankar Shukla University,
Raipur - 492010 (C.G.), India
e-mail: sharmabk07@gmail.com

Abstract. A strong convergence theorem is proved for a uniformly continuous asymptotically accretive operator using the Ishikawa iterative processes with errors in the sense of Xu [Y.Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl. 224(1998), 91-101] in a normed linear space.

1. Introduction

Let E be a real normed linear space and E^* be its dual space. Let $\langle .\,, . \rangle$ denote the normalized duality pairing between the elements of E and E^*. The duality mapping $J\colon E \to E^*$ is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x^2\|,\ \|f^*\| = \|x\|\}.$$

A mapping A with domain $D(A)$ and range $R(A)$ in E is said to be *accretive* [1] if the inequality

$$\|x - y\| \le \|x - y + r(Ax - Ay)\| \tag{1.1}$$

holds for any $x, y \in D(A)$ and for all $r > 0$.

The mapping A is said to be *strongly accretive* if there is a positive constant k such that $(A - kI)$ is accretive where I denotes the identity operator on E.

[0]Received November 20, 2010. Revised August 23, 2011.
[0]2000 Mathematics Subject Classification: 47H06, 47H10.
[0]Keywords: Modified Ishikawa iteration with errors, asymptotically accretive operator, uniformly continuous, normed linear space.

A is said to be *m-accretive* if A is accretive and the operator $(I + \lambda A)$ is surjective for all $\lambda > 0$. The accretive operators are of interest mainly because many physically significant problems are modeled by mathematical systems involving various types of accretive operators (see e.g. [5, 6, 7, 11] and the references therein).

Let K be a subset of E. Then a mapping $T\colon K \to K$ is called *strongly pseudo-contractive* [2] if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.2}$$

holds for all $x, y \in K$ and $r > 0$. If $t = 1$, then T is called *pseudo-contractive.*

Using the duality map and Kato's lemma [6], the equivalent definitions of the accretive and pseudocontractive maps are as follows:
The map A is said to be *accretive* if and only if for all $x, y \in D(A)$, $\exists\, j(x-y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0,$$

so that A is said to be strongly accretive if and only if $\exists\, k > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq k\|x - y\|^2$$

and A is strongly pseudocontractive iff

$$\langle (I - A)x - (I - A)y, j(x - y)\rangle \geq k\|x - y\|^2$$

where $k = \frac{t-1}{t}$ and $t > 1$ is the constant as in (1.2).

A close study of (1.1) and (1.2) shows that a map T is pseudo-contractive if and only if the operator $A = (I - T)$ is accretive on the domain of T [1]. Consequently, the fixed points of pseudocontractive maps yield the zeros of corresponding accretive operators.

Deimling [4]) proved that if $A\colon E \to E$ is continuous and strongly accretive, then A is surjective i.e. for a given $f \in E$, the equation

$$Ax = f \tag{1.3}$$

has a unique solution. Martin [8] also proved that if $A : E \to E$ is continuous and accretive, then A is $m - accretive$, so that the equation

$$x + Ax = f \tag{1.4}$$

has a unique solution for any $f \in E$.

Let $n \in \mathbb{N}$, then T^n denotes the n^{th} iterate of a map T. An operator A with domain $D(A)$ and range $R(A)$ in E is said to be *asymptotically accretive* [9] if there exists a real sequence $\{k_n\} \subset (1,\infty)$ such that $\lim\limits_{n\to\infty} k_n = 1$ and for all $x,y \in D(A)$, $r > 0$, $n \in \mathbb{N}$, the inequality

$$\|x-y\| \le \|(1+r)(x-y) - rk_n[(I-A^n)x - (I-A^n)y]\|$$

holds.

Moore [9] established the equivalence of different definitions of asymptotically accretive operator and proved the strong convergence of suitably defined Mann and Ishikawa iteration processes to the solution of an operator equation involving asymptotically accretive operators using the below Lemma:

Lemma 1.1. [9] *Let E be a real normed space. Then the following are equivalent:*

(i) *A is asymptotically accretive.*

(ii) *There exists a real sequence $\{t_n\}_{n\ge 0} \subset (0,1)$ such that $t_n \to 0$ as $n \to \infty$ and $\forall\, x,y \in D(A)$, $n \in \mathbb{N}$ and some $j(x-y) \in J(x-y)$, the following inequality holds:*

$$\langle A^n x - A^n y, j(x-y)\rangle \ge t_n\|x-y\|^2 \tag{1.5}$$

(iii) *There exists a real sequence $\{t_n\}_{n\ge 0} \subset (0,1)$ such that $t_n \to 0$ as $n \to \infty$ and $\forall\, x,y \in D(A)$, $r > 0$, and $n \in \mathbb{N}$ the following inequality holds:*

$$\|x-y\| \le \|x-y+r[(A^n - t_nI)x - (A^n - t_nI)y]\| \tag{1.6}$$

Theorem 1.2. [9] *Let E be an arbitrary real Banach space and let $A : E \to E$ be a uniformly L-Lipschitz asymptotically accretive operator such that the operator equation $Ax = f$ has a solution $x^* \in D(A)$. Let the sequences $\{\alpha_n\}, \{\beta_n\}, \{u_n\}, \{v_n\}$ satisfy the following conditions:*

(i) $0 \le \alpha_n; \beta_n \ll t_n;\ \forall\, n \ge 0,$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0 = \lim\limits_{n\to\infty} \beta_n,$

(iii) $\sum\limits_{n\ge 0} \alpha_n t_n = \infty,$

(iv) $\|u_n\|, \|v_n\| = o(t_n),$

where $\{t_n\}$ is as defined in Lemma 1.1. Then the sequence $\{x_n\}_{n\ge 0}$ iteratively generated from an arbitrary $x_0 \in D(A)$ and some $u_0, v_0 \in E$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n(f + x_n - A^n x_n + u_n) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n(f + y_n - A^n y_n + v_n), \qquad n \ge 0 \end{aligned} \tag{1.7}$$

converges strongly to x^*.

In this paper, we have proved the strong convergence of uniformly continuous asymptotically accretive operator equation in a normed linear space using the iteration methods in the sense of Xu [12]. Thus our paper improves the results in Theorem 2 of Moore [9], Corollary 2 of Chidume [3] and Theorem 1 of Osilike, Igbokwe [10] in the following manner:

(1) The steps during the proof of boundednes and convergence of the iterative sequence proceed with the help of uniformly Lipschitz continuity of the operator in [9] and [10], completely continuous and uniformly Lipschitz in [3], whereas we have proved it using the weaker condition i.e. uniform continuity.
(2) We have used the iteration in the sense of Xu [12] which is proved to be more satisfactory in comparison to the iterations as used in [9] and [3] due to Liu [7]; since the conditions on the error terms are incompatible with the randomness of the occurrence of errors [12].
(3) Our result is proved in a normed linear space thus leaving the completion condition of Banach space as in [9] and extending the result of [3] from p-uniformly convex Banach space to a normed linear space.
(4) No extra condition is imposed on the operator as in [10].

2. Preliminaries

In the sequel, we shall make use of the following result:

Lemma 2.1. [7] *Let $\rho_n, \sigma_n, \delta_n$ be nonnegative real sequences satisfying the inequality:-*

$$\rho_{n+1} \leq (1 - t_n)\rho_n + \rho_n + \delta_n$$

where $t_n \in [0, 1]$, $\sum t_n = \infty$, $\sigma_n = O(t_n)$ and $\sum \delta_n < \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

3. Main Results

Now we present our main result:

Theorem 3.1. *Let E be a real normed linear space and let $A : E \to E$ be uniformly continuous asymptotically accretive operator such that the operator equation $Ax = f$ has a solution $x^* \in D(A)$. Let $S : E \to E$ be defined by $S^n x = f + x - A^n x$, $f \in E$ and suppose $\{S^n x\}$, $\{S^n y\}$ be bounded. Define the sequence $\{x_n\}$ iteratively from an arbitrary $x_0, u_0, v_0 \in E$ by*

$$x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n$$
$$y_n = a_n^{'} x_n + b_n^{'} S^n x_n + c_n^{'} v_n$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in E and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, $\{c_n^{'}\}$ are real sequences in $(0, 1)$ and $\{t_n\}_{n \geq 0} \subset (0, 1)$ such that $t_n \to 0$ as $n \to \infty$, satisfying the conditions

(i) $a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'}$, $\forall$ *integers* $n \geq 0$,

(ii) $\lim b_n = \lim b_n' = \lim c_n' = 0$,
(iii) $\sum b_n t_n = \infty$,
(iv) $\sum c_n < \infty$.

Then $\{x_n\}$ *converges strongly to the unique solution* x^*.

Proof. The uniqueness of the solution follows from definition of aymptotically accretive operator as in Lemma 1.1(ii).
Let $\alpha_n := b_n + c_n$, so that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n y_n - c_n(S^n y_n - u_n)$$

from which it follows that

$$\begin{aligned} x_n =& (1 + \alpha_n)x_{n+1} + \alpha_n(I - S^n - t_n)x_{n+1} - (1 - t_n)\alpha_n x_n + (2 - t_n)\alpha_n^2(x_n - S^n y_n) \\ &+ \alpha_n(S^n x_{n+1} - S^n y_n) + c_n[1 + (2 - t_n)\alpha_n](S^n y_n - u_n) \end{aligned}$$

Since x^* is a fixed point of S^n; we observe that

$$x^* = (1 + \alpha_n)x^* + \alpha_n(I - S^n - t_n)x^* - (1 - t_n)\alpha_n x^*$$

so that,

$$\begin{aligned} x_n - x^* =& (1 + \alpha_n)(x_{n+1} - x^*) + \alpha_n[(I - S^n - t_n)x_{n+1} - (I - S^n - t_n)x^*] \\ &- (1 - t_n)\alpha_n(x_n - x^*) + (2 - t_n)\alpha_n^2(x_n - S^n y_n) \\ &+ \alpha_n(S^n x_{n+1} - S^n y_n) + c_n e_n(S^n y_n - u_n) \end{aligned}$$

where $e_n = 1 + (2 - t_n)\alpha_n \leq M_1$, for some constant $M_1 > 0$.
Hence

$$\begin{aligned} \|x_n - x^*\| \geq& (1 + \alpha_n)\|x_{n+1} - x^* + \frac{\alpha_n}{1 + \alpha_n}[(I - S^n - t_n)x_{n+1} - (I - S^n - t_n)x^*]\| \\ &- (1 - t_n)\alpha_n\|x_n - x^*\| - (2 - t_n)\alpha_n^2\|x_n - S^n y_n\| \\ &- \alpha_n\|S^n x_{n+1} - S^n y_n\| - c_n e_n\|S^n y_n - u_n\| \\ \geq& (1 + \alpha_n)\|x_{n+1} - x^*\| - (1 - t_n)\alpha_n\|x_n - x^*\| \\ &- (2 - t_n)\alpha_n^2\|x_n - S^n y_n\| - \alpha_n\|S^n x_{n+1} - S^n y_n\| - c_n e_n\|S^n y_n - u_n\| \end{aligned}$$

So that,

$$\begin{aligned} \|x_{n+1} - x^*\| \leq& [\frac{1 + (1 - t_n)\alpha_n}{1 + \alpha_n}]\|x_n - x^*\| + (2 - t_n)\alpha_n^2\|x_n - S^n y_n\| \\ &+ \alpha_n\|S^n x_{n+1} - S^n y_n\| + c_n e_n\|S^n y_n - u_n\| \\ \leq& (1 - t_n\alpha_n + t_n\alpha_n^2)\|x_n - x^*\| + (2 - t_n)\alpha_n^2\|x_n - S^n y_n\| \quad (3.1) \\ &+ \alpha_n\|S^n x_{n+1} - S^n y_n\| + c_n e_n\|S^n y_n - u_n\| \end{aligned}$$

Let n_0 be a positive integer such that $0 \le \alpha_n \le \frac{5t_n}{8(1+2(2-t_n))}$ for all $n \ge n_0$.
Define

$$D := \frac{8}{k}\left\{(\sup_{x\in E}\|S^n x - x^*\| + \|x_{n_0} - x^*\| + \sup_{n\ge 1} e_n\|S^n y_n - u_n\|)\right\}.$$

A simple induction shows that for all $n \ge n_0$,

$$\|x_n - x^*\| \le D$$

and so $\{x_n\}$ is bounded.
Also,

$$\|x_n - S^n y_n\| \le \|x_n - x^*\| + \|S^n y_n - x^*\| \le 2D.$$

Now set $\beta_n := b_n^{'} + c_n^{'}$, so that

$$\begin{aligned}\|x_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &= \|\alpha_n(S^n y_n - x_n) - c_n(S^n y_n - u_n)\| + \|\beta_n(S^n x_n - x_n) - c_n^{'}(S^n x_n - v_n)\|\end{aligned}$$

Let $\delta := \max\{\sup_{n\ge 1}\|x_n\|,\ \sup_{x\in E}\|S^n x\|,\ \sup_{n\ge 1}\|u_n\|,\ \sup_{n\ge 1}\|v_n\|\}$

Thus, $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$, which implies by uniform continuity of S^n that $\|S^n x_{n+1} - S^n y_n\| \to 0$ as $n \to \infty$.
Thus by (3.1),

$$\|x_{n+1} - x^*\| \le (1 - t_n\alpha_n)\|x_n - x^*\| + 2(2-t_n)\alpha_n^2 D + \alpha_n\|S^n x_{n+1} - S^n y_n\| + c_n D$$

Let
$\rho_n := \|x_n - x^*\|,\ k_n := t_n\alpha_n,\ \sigma_n := \alpha_n\|S^n x_{n+1} - S^n y_n\| + 2(2-t_n)\alpha_n^2 D,$
$\delta_n := c_n D$
where
$k_n \in [0,1],\ \sum k_n = +\infty,\ \sigma_n = O(k_n),\ \sum \delta_n < \infty.$
Thus,

$$\rho_{n+1} \le (1-k_n)\rho_n + \sigma_n + \delta_n, \qquad n \ge 0.$$

Hence by Lemma 2.1, $\rho_n \to 0$ as $n \to \infty$,
i.e.

$$\|x_n - x^*\| \to 0 \text{ as } n \to \infty.$$

□

Corollary 3.2. *Let E, S, and A be as in Theorem 3.1. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from $x_0, u_0 \in E$, by*

$$x_{n+1} = a_n x_n + b_n S^n x_n + c_n u_n,\ n \ge 0,$$

where $\{u_n\}$ is an arbitrary sequence in E and $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $(0,1)$ satisfying the conditions:

(i) $a_n + b_n + c_n = 1$,
(ii) $\lim b_n = 0$,

(iii) $\sum b_n = +\infty, \sum c_n < \infty$.
then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* x^*.

Corollary 3.3. *Let* E *be a real normed linear space and* K *is nonempty closed, convex subset of* E. *Let* $T : K \to K$ *is a uniformly continuous asymptotically pseudocontractive mapping with a fixed point* $x^* \in K$. *Suppose* $\{u_n\}$, $\{v_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, $\{c_n^{'}\}$ *and* $\{t_n\}$ *are as in Theorem 3.1. If* $\{T^n x\}$ *and* $\{T^n y\}$ *be bounded, then* $\{x_n\}$ *converges strongly to* x^*.

Proof. Define $A^n x := f + x - T^n x, \forall\, x \in D(T)$. Then the result follows from Theorem 3.1. □

References

[1] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach space*, Bull. Amer. Math. Soc., **73** (1967), 875–882, MR:38-581.

[2] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228, MR:36-747.

[3] C. E. Chidume, *Convergence theorems for asymptotically pseudocontractive mappings*, Nonlinear Analysis, **49** (2002), 1–11, MR:2002k-47109.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer- Verlag, Berlin, 1985, MR:86j-47001.

[5] X. P. Ding, *Iterative process with errors of nonlinear equations involving m-accretive operators*, J. Math. Anal. Appl., **209** (1997), 191–201, MR:98k-47125.

[6] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, **19** (1967), 509–520, MR:37-1820.

[7] L. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mapping in Banach space*, J. Math. Anal. Appl., **194** (1995), 114–125, MR:97g-47069.

[8] R. H. Martin,Jr., *A global existence theorem for autonomuos differential equations in Banach spaces*, Proc. Amer. Math. Soc., **26** (1970), 307–314, MR:41-8791.

[9] C. Moore, *Fixed point iterations for asymptotically pseudocontractive maps*, Preprint, Int. Centre for Theor. Phys.(ICTP), IC1996282, 1996.

[10] M. O. Osilike and D. I. Igbokwe, *Convergence theorems for asymptotically pseudocontractive maps*, Bull. Korean Math. Soc., **39** (2002), 389–399, MR:2003g-47091.

[11] S. Reich, *Constructing zeros of accretive sets I and II*, Appl. Anal., **8,9** (1979), 349-352; 159–163, MR:82d-65052.

[12] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., **224** (1998), 91–101, MR:99g-47144.