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SHRINKING HYBRID DESCENT-LIKE METHODS FOR NONEXPANSIVE MAPPINGS AND SEMIGROUPS

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Abstract. In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming and the descent-like method for finding a fixed point of a nonexpansive mapping and a common fixed point of a nonexpansive semigroup in Hilbert spaces. The main results in this paper modify and improve some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle .,. \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote by $P_C x$ the metric projection of an element $x \in H$ onto C. It is well-known that P_C is a nonexpansive mapping on H for any closed convex subset C in H. Recall that a mapping T is said to be nonexpansive on C, if $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [2].

Let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C, that is,
(1) for each $t > 0, T(t)$ is a nonexpansive mapping on C;

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(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s+t) = T(s) \circ T(t)$ for all $s, t > 0$; and
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Assume that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. We know that $\mathcal{F}$ is a closed convex subset [8] and that $\mathcal{F} \neq \emptyset$, if C is bounded [6].

For finding a fixed point of a nonexpansive mapping T on C, Alber [1] proposed the following descent-like method:

$$x_{n+1} = P_C(x_n - \mu_n(I - T)x_n), n \geq 0, x_0 \in C, \tag{1.1}$$

where I denotes the identity mapping in H, and proved that if the sequence of positive real numbers $\{\mu_n\}$ is chosen such that $\mu_n \to 0$ as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;
(ii) all weak accumulation points of $\{x_n\}$ belong to $F(T)$;
(iii) if $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_n\}$ converges weakly to $\tilde{x}$.

Motivated by Solodov and Svaiter's algorithm [11] , Nakajo and Takahashi [8] introduced the following strongly convergence iteration procedures:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - x_0, z - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), n \geq 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, for finding a fixed point of a nonexpansive mapping T on C, and

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - x_0, z - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), n \geq 0, \end{aligned} \tag{1.3}$$

where where T_n is defined by

$$T_n y = \frac{1}{\lambda_n} \int_0^{\lambda_n} T(s) y ds,$$

for each $y \in C$, $\alpha_n \in$ [0,a] for some $a \in$ [0,1) and $\{\lambda_n\}$ is a positive real number divergent sequence, for finding a common fixed point of a nonexpansive semigroup $\{T(t) : t > 0\}$.

Further, in 2008, Takahashi et al. [12] proposed a simple variant of (1.3) that has the following form:

$$\begin{aligned} x_0 &\in H, C_1 = C, x_1 = P_{C_1}x_0,\\ y_n &= \alpha_n x_n + (1-\alpha_n)T_n x_n,\\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{C_{n+1}}x_0, n \ge 1. \end{aligned} \tag{1.4}$$

They showed (Theorem 4.4 in [12]) that if $0 \le \alpha_n \le a < 1, 0 < \lambda_n < \infty$ for all $n \ge 1$ and $\lambda_n \to \infty$, then $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}}x_0$. At the time, Saejung [9] considered the following analogue without Bochner integral:

$$\begin{aligned} x_0 &\in H, C_1 = C, x_1 = P_{C_1}x_0,\\ y_n &= \alpha_n x_n + (1-\alpha_n)T(t_n)x_n,\\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{C_{n+1}}x_0, n \ge 1, \end{aligned} \tag{1.5}$$

where $0 \le \alpha_n \le a < 1, \liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$. Then $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}}x_0$.

If $C \equiv H$, then C_n and Q_n in (1.2)-(1.5) are two halfspaces. So, the projection x_{n+1} onto $C_n \cap Q_n$ or C_{n+1} in these methods can be found by an explicit formula [11]. Clearly, if C is a proper subset of H, then C_n and Q_n in these algorithms are not two halfspaces. Then, the following problem is posed: how to construct the closed convex subsets C_n and Q_n and if we can express x_{n+1} of the above algoritms in a similar form as in [11]? This problem is solved very recently in [3-5]. In the works, C_n and Q_n in (1.2)-(1.3) are replaced by two halfspaces and y_n is the right hand side of (1.1) with a modification. In this paper, using the idea, we present a new variant for (1.4)-(1.5) where C_{n+1} becomes a halfspace H_{n+1} defined below. More precisely, we consider the following algorithms:

$$\begin{aligned} x_0 &\in H = H_0, y_n = x_n - \mu_n(I - TP_C)x_n,\\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0, \end{aligned} \tag{1.6}$$

for finding an element in $F(T)$;

$$\begin{aligned} x_0 &\in H = H_0, y_n = x_n - \mu_n(I - T_nP_C)x_n),\\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0; \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} x_0 \in H = H_0, y_n &= x_n - \mu_n(I - T(t_n)P_C)x_n,\\ H_{n+1} &= \{z \in H_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{H_{n+1}}x_0, n \ge 0, \end{aligned} \tag{1.8}$$

for finding an element in $\mathcal{F}$.

We shall prove that iteration processes (1.6) and (1.7), (1.8) converge strongly to a fixed point of T and a common fixed point of $\{T(t) : t > 0\}$ in sections 2 and 3, respectively.

The symbols $\rightharpoonup$ and $\to$ denote weak and strong convergences, respectively.

2. Strong convergence to a fixed point of nonexpansive mappings

We formulate the following facts needed in the proof of our results.

Lemma 2.1 [7]. *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z - x\| \le \|y - x\|$ for all $y \in C$, and $z = P_C x$ if and only if $\langle z - x, y - z\rangle \ge 0$ for all $y \in C$.*

Theorem 2.2. *Let C be a nonempty closed convex subset in a real Hilbert space H and let T be a nonexpansive mapping on C such that $F(T) \ne \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1)$ for some $a \in (0, 1]$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by (1.6), converge strongly to the same point $u_0 = P_{F(T)}x_0$.*

Proof. First, note that $\|y_n - z\| \le \|x_n - z\|$ is equivalent to

$$\langle y_n - x_n, x_n - z\rangle \le -\frac{1}{2}\|y_n - x_n\|^2.$$

Thus, H_n is a halfspace. Next, we show that $F(T) \subset H_n$ for all $n \ge 0$. It is clear that $F(T) = F(TP_C) := \{p \in H : TP_C p = p\}$ for any mapping T from C into C. So, we have for each $p \in F(T)$ that

$$\begin{aligned} \|y_n - p\| &= \|(1 - \mu_n)x_n + \mu_n TP_C x_n - p\| \\ &= \|(1 - \mu_n)(x_n - p) + \mu_n(TP_C x_n - TP_C p)\| \\ &\le (1 - \mu_n)\|x_n - p\| + \mu_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

Therefore, $p \in H_n$ for all $n \ge 0$.

Further, since $F(T)$ is a nonempty closed convex subset of H, by Lemma 2.1, there exists a unique element $u_0 \in F(T)$ such that $u_0 = P_{F(T)}x_0$. From

$x_{n+1} = P_{H_{n+1}} x_0$, we obtain that

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in H_{n+1}$. As $u_0 \in F(T) \subset H_{n+1}$, we get

$$\|x_{n+1} - x_0\| \leq \|u_0 - x_0\| \quad \forall n \geq 0. \tag{2.1}$$

Now, we show that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0, \tag{2.2}$$

for each fixed integer $m > 0$. Indeed, from the definition of H_{n+1}, it implies that $H_{n+1} \subseteq H_n$ and hence we have that

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \geq 0.$$

Therefore, there exists $\lim_n \|x_n - x_0\| = c$. Next, by Lemma 2.1, $x_{n+m} \in H_n$ and $x_n = P_{H_n} x_0$, we get that

$$\langle x_n - x_0, x_{n+m} - x_n \rangle \geq 0.$$

Thus,

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+m} - x_n\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \end{aligned}$$

from that and $\lim_n \|x_n - x_0\| = c$, (2.2) is implied. So, $\{x_n\}$ is a Cauchy sequence. We assume that $x_n \to p \in H$. On the other hand, from (2.2) and the following inequalities

$$\begin{aligned} \|x_n - TP_C x_n\| &= \frac{1}{\mu_n}\|y_n - x_n\| \\ &\leq \frac{1}{a}(\|y_n - x_{n+m}\| + \|x_{n+m} - x_n\|) \\ &\leq \frac{2}{a}\|x_{n+m} - x_n\|, \end{aligned}$$

we get

$$\lim_{n\to\infty} \|x_n - TP_C x_n\| = 0.$$

So, $p = TP_C p$. It means that $p \in F(T)$. Now, from (2.1) and Lemma 2.1, it implies that $p = u_0$. The strong convergence of the sequence $\{y_n\}$ to u_0 is followed from

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \mu_n \|x_n - TP_C x_n\| = 0$$

and $x_n \to u_0$. This completes the proof. □

3. Strong convergence to a common fixed point of nonexpansive semigroups

Lemma 3.1 *[10]. Let C be a nonempty bounded closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C. Then, for any $h > 0$*

$$\lim_{t\to\infty}\sup\sup_{y\in C}\left\|T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)-\frac{1}{t}\int_0^t T(s)yds\right\|=0.$$

Theorem 3.2. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0}F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{\lambda_n\}$ is a positive real number divergent sequence. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.7), converge strongly to the same point $u_0 = P_{\mathcal{F}}x_0$.*

Proof. For each $p \in \mathcal{F} \subseteq C$, we have from (1.7) and $p = P_Cp$ that

$$\begin{aligned}
\|y_n - p\| &= \left\|(1-\mu_n)(x_n-p)+\mu_n\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds - p\right)\right\| \\
&\leq (1-\mu_n)\|x_n-p\| + \mu_n\left\|\frac{1}{\lambda_n}\int_0^{\lambda_n}(T(s)P_Cx_n - T(s)P_Cp)ds\right\| \\
&\leq (1-\mu_n)\|x_n-p\| + \mu_n\frac{1}{\lambda_n}\int_0^{\lambda_n}\|x_n-p\|ds \\
&= \|x_n-p\|.
\end{aligned}$$

Therefore, $p \in H_n$. It means that $\mathcal{F} \subset H_n$ for all $n \geq 0$. As in the proof of Theorem 2.2, we get that $\{x_n\}$ is well defined, it converges strongly to an element $p \in H$, and

$$\|x_{n+1} - x_0\| \leq \|u_0 - x_0\|, \quad \lim_{n\to\infty}\left\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds\right\| = 0, \tag{3.1}$$

where $u_0 = P_{\mathcal{F}}x_0$. Since

$$\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds \in C$$

and P_C is a nonexpansive mapping, we have that

$$\begin{aligned}
\left\|P_Cx_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds\right\| &= \left\|P_Cx_n - P_C\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds\right\| \\
&\leq \left\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_nds\right\|.
\end{aligned}$$

So, we obtain from (3.1) that

$$\lim_{n\to\infty}\left\|P_Cx_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| = 0. \tag{3.2}$$

This together with (3.1) and $x_n \to p$ implies that the sequence $\{P_Cx_n\}$ also converges strongly to p. Since C is closed, we get $p \in C$.

On the other hand, we have for each $h > 0$ that

$$\begin{aligned}\|T(h)P_Cx_n - P_Cx_n\| \le{}& \left\|T(h)P_Cx_n - T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right)\right\| \\ &+ \left\|T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right) - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\| \\ &+ \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds - P_Cx_n\right\| \\ \le{}& 2\left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds - P_Cx_n\right\| \\ &+ \left\|T(h)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right) - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)P_Cx_n ds\right\|.\end{aligned} \tag{3.3}$$

Let $C_0 = \{z \in C : \|z - u_0\| \le 2\|x_0 - u_0\|\}$. Since $u_0 = P_{\mathcal{F}}x_0 \in C$, we have from (3.1) and

$$\begin{aligned}\|P_Cx_n - u_0\| &= \|P_Cx_n - P_Cu_0\| \\ &\le \|x_n - u_0\| \\ &\le \|x_n - x_0\| + \|x_0 - u_0\| \\ &\le 2\|x_0 - u_0\|.\end{aligned}$$

So, C_0 is a nonempty bounded closed convex subset. It is easy to verify that $\{T(t) : t > 0\}$ also is a nonexpansive semigroup on C_0. By Lemma 3.1, (3.3) and $P_Cx_n \to p$, we get $p = T(h)p$ for each $h > 0$. So, $p \in \mathcal{F}$. Again, from (3.1) and $p \in \mathcal{F}$, it implies that $p = u_0$ and $y_n \to u_0$ as $n \to \infty$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0}F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{t_n\}$ is a sequence of positive real numbers satisfying the condition $\liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.8), converge strongly to the same point $u_0 = P_{\mathcal{F}}x_0$.*

Proof. As in the proof of Theorems 2.2 and 3.2, we get

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\|, \quad \lim_{n\to\infty}\|x_n - T(t_n)P_Cx_n\| = 0, \tag{3.4}$$

$$\lim_{n\to\infty} \|P_C x_n - T(t_n)P_C x_n\| = 0, \tag{3.5}$$

and the sequence $\{x_n\}$ and $\{P_C x_n\}$ also converge strongly to $p \in C$.

Without loss of generality, as in [7], let

$$\lim_{j\to\infty} t_{n_j} = \lim_{j\to\infty} \frac{\|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\|}{t_{n_j}} = 0. \tag{3.6}$$

Now, we prove that $p = T(t)p$ for a fixed $t > 0$. It is easy to see that

$$\begin{aligned}\|P_C x_{n_j} - T(t)p\| \le & \sum_{l=0}^{[t-t_{n_j}]-1} \|T(lt_{n_j})P_C x_{n_j} - T((l+1)t_{n_j})P_C x_{k_j}\| \\ & + \left\|T\Big(\Big[\frac{t}{t_{n_j}}\Big]\Big)P_C z_{n_j} - T\Big(\Big[\frac{t}{t_{n_j}}\Big]\Big)p\right\| \\ & + \left\|T\Big(\Big[\frac{t}{t_{k_j}}\Big]\Big)p - T(t)p\right\| \\ \le & \frac{t}{t_{n_j}}\|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\| + \|P_C x_{n_j} - p\| \\ & + \left\|T\Big(t - \Big[\frac{t}{t_{n_j}}\Big]t_{n_j}\Big)p - p\right\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|P_C x_{n_j} - T(t)p\| \le & \frac{t}{t_{n_j}}\|P_C x_{n_j} - T(t_{n_j})P_C x_{n_j}\| \\ & + \|P_C x_{n_j} - p\| + \sup\{\|T(s)p - p\| : 0 \le s \le t_{n_j}\}.\end{aligned}$$

This fact, together with (3.6) and property (4) for the semigroup, implies that

$$\lim_{j\to\infty} \|P_C x_{n_j} - T(t)p\| = 0.$$

Therefore, $p \in \mathcal{F}$. So, from (3.4), we have that the sequence $\{x_n\}$ converges strongly to u_0 as $n \to \infty$. The strong convergence of the sequence $\{y_n\}$ to u_0 is followed from (1.8), (3.4), $\mu_n \in (a, 1]$ and $x_n \to u_0$ as $n \to \infty$. The theorem is proved. □

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