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ON EXISTENCE OF PERIODIC AND SUBHARMONIC SOLUTIONS WITH SADDLE POINT CHARACTER

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Abstract. In the present paper we consider the existences of periodic and subharmonic solutions with saddle point character for the following second order non-autonomous Hamiltonian system

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R.$$

Adopting some other reasonable assumptions for $\bigtriangledown F$, we obtain some new results for existence of solutions with saddle point character by using of the saddle point reduction methods. Recent results from the literature are generalized and significantly improved.

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1. Introduction and Main Results

Consider the second order systems

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R \tag{1.1}$$

where $F : R \times R^N \to R$ is $T-$periodic ($T > 0$) in its first variable and satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a. e. $t \in [0, T]$, and there exist $a \in L^1(R^+; R^+), b \in L^1(0, T; R^+)$, such that $|F(t, x)| \leq a(|x|)b(t), |\bigtriangledown F(t, x)| \leq a(|x|)b(t)$ for all $x \in R^N$ and a. e. $t \in R$.

A solution of problem (1.1) is called to be subharmonic if it is $kT-$periodic solution for some positive integer k.

It is well known that u is a periodic solution of problem (1.1) if and only if u is a critical point in H_T^1 of functional φ, where

$$H_T^1 = \{u : [0, T] \to\ R^N | u \ is\ absolutely\ continuous,$$

$$u(0) = u(T)\ and\ \ \dot{u} \in L^2(0, T; R^N)\}$$

is a Hilbert space with the norm defined by

$$\|u\| = [\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt]^{\frac{1}{2}}$$

for $u \in H_T^1$ and

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, u(t))dt$$

is continuously differentiable and weakly lower semi-continuous on H_T^1 (see [7]). Moreover one has

$$< \varphi'(u), v >= \int_0^T [(\dot{u}(t), \dot{v}(t)) - (\bigtriangledown F(t, u(t)), v(t))]dt$$

for all $u, v \in H_T^1$. By Proposition1.1 in [7], we know there exists a constant $c_0 > 0$ such that

$$\|u\|_\infty = \max_{0 \leq t \leq T} |u(t)| \leq c_0 ||u|| \tag{1.2}$$

for all $u \in H_T^1$. For $u \in H_T^1$, let $\overline{u} = (T)^{-1} \int_0^T u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$. Let $\widetilde{H}_T^1 = \{u \in H_T^1 | \overline{u} = 0\}$, then $H_T^1 = R^N \bigoplus \widetilde{H}_T^1$ and $\widetilde{u}(t) \in \widetilde{H}_T^1$, obviously. Then one has Sobolev's inequality

$$\|\widetilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt$$

and Wertinger's inequality

$$\int_0^T |\widetilde{u}(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt.$$

If $H_T^1 = X \bigoplus Y$, for each $x \in X$ and each $y \in Y$, let $\psi(x, y) = \varphi(x+y)$. The solution $u = x + y$ of problem (1.1) is said to be of correlated property if there exists a continuous function θ such that $y = \theta(x)$ and either $\psi(x, \theta(x)) = \min_{y \in Y} \psi(x, y)$ or $\psi(x, \theta(x)) = \sup_{y \in Y} \psi(x, y)$. The solution $u = x + y$ of problem (1.1) is said to possess saddle point character if it is correlated and is a saddle point of $\psi(x, y)$.

The existence of periodic solutions for problem (1.1) under some suitable conditions have been established and it has been proved that problem (1.1) has infinitely distinct subharmonic solutions under some suitable conditions(see, e.g., [7], [8], [5], [4], [6], [9], and references therein). Recently, Zhao and Wu [13], [14], Wu [12] consider the existence of periodic solutions with saddle point character for problem (1.1) under some linear condition. They obtained the following theorem:

Theorem A: *Suppose that $F : [0, T] \to R^N$ satisfies assumption(A). If the following conditions hold:*
(i) there exists a function $\mu \in L^1(0, T; R)$ with $\int_0^T \mu(t)dt > 0$ such that $\nabla F(t, \cdot)$ is $\mu(t)-$monotone, that is

$$(\nabla F(t, x) - \nabla F(t, y), x - y) \geq \mu(t)|x - y|^2$$

for all $x, y \in R^N$ and a. e. $t \in [0, T]$;
(ii) there exist $f, g \in L^1(0, T; R^+)$ with $\int_0^T f(t)dt \leq 12/T$ such that

$$| \bigtriangledown F(t, x)| \leq f(t)|x| + g(t)$$

for all $x \in R^N$ and a. e. $t \in [0, T]$, then problem (1.1) *has at least a solution with saddle point character in H_T^1. If in addition,*
(iii) there exists $\delta > 0$ and an integer $k > 0$ such that

$$\frac{1}{2} k^2 \omega^2 |x|^2 \leq F(t, x) \leq \frac{1}{2} (k + 1)^2 \omega^2 |x|^2$$

for all $|x| \leq \delta$ and a. e. $t \in [0, T]$, where $\omega = 2\pi/T$, then problem (1.1) *has three distinct solutions with saddle point character in H_T^1.*

On the other hand, Tang and Wu [10] consider that $\nabla F(t, x)$ is sublinear and proved that under some type of coercive property on $F(t, x)$, problem (1.1) has infinitely subharmonic solutions.

We note that Theorem A only consider the linear situation and constrain the coefficient $f(t)$ of the linearity such that $\int_0^T f(t)dt \leq 12/T$, this extremely constrict the applicability for discussing the existence of subharmonic solutions of problem (1.1). Inspired and motivated by the works mentioned above, and by virtue of methods so called saddle point reduction method proposed in [3], [1], [11], we shall extend the conditions posed on $\nabla F(t,x)$ to that so called $(\mu-\alpha)-$monotone in large region and that $\nabla F(t,x)$ is sublinear in x to obtain new criterions for guaranteeing the existence and multiplicity of periodic and subharmonic solutions with saddle point character of problem (1.1) in this paper, our methods are different from the methods in the others. Our main results are the following theorems.

Theorem 1.1. *Suppose that F satisfies assumption (A) and the following conditions:*
(i) there exists a function $\mu \in L^1(0,T;R)$ with $\int_0^T \mu(t)dt > 0$, $M > 0$ and $\alpha \in [0,1)$ such that $\nabla F(t,\cdot)$ is $(\mu-\alpha)-$monotone, that is

$$(\nabla F(t,x) - \nabla F(t,y), x-y) \geq \mu(t)|x-y|^{1+\alpha}$$

for all $x, y \in R^N$ with either $|x| \geq M$ or $|y| \geq M$ and that

$$(\nabla F(t,x) - \nabla F(t,y), x-y) \geq \mu(t)|x-y|^2$$

for all $x, y \in R^N$ with both $|x| < M$ and $|y| < M$ and a. e. $t \in [0,T]$;
(ii) there exist $f, g \in L^1(0,T;R^+)$ and $M > 0$ such that

$$|\nabla F(t,x)| \leq f(t)|x|^\alpha + g(t) \tag{1.3}$$

for all $x \in R^N$ with $|x| \geq M$ and a. e. $t \in [0,T]$, then problem (1.1) has at least a solution with saddle point character in H_T^1.
In addition, if the following condition holds,
(iii) there exist $M > \delta > 0$ and an integer $k > 0$ such that

$$\frac{1}{2}k^2\omega^2|x|^2 \leq F(t,x) \leq \frac{1}{2}(k+1)^2\omega^2|x|^2$$

for all $|x| \leq \delta$ and a. e. $t \in [0,T]$, where $\omega = 2\pi/T$, then problem (1.1) has at least three periodic distinct solutions with correlated property in H_T^1 in which there exists at least one which is nontrivial one with saddle point character.

Theorem 1.2. *Suppose that F satisfies assumption (A) and the following conditions:*
(i) there exists a function $\mu \in L^1(0,T;R)$ with $\int_0^T \mu(t)dt > 0$ and $\alpha \in [0,1)$ such that $\nabla F(t,\cdot)$ is $(\mu-\alpha)-$monotone, that is

$$(\nabla F(t,x) - \nabla F(t,y), x-y) \geq \mu(t)|x-y|^{1+\alpha}$$

for all $x, y \in R^N$ and a. e. $t \in [0,T]$;

(ii) there exist $f, g \in L^1(0,T;R^+)$ *and* $M > 0$ *such that*

$$|\bigtriangledown F(t,x)| \leq f(t)|x|^\alpha + g(t)$$

for all $x \in R^N$ *with* $|x| \geq M$ *and a. e.* $t \in [0,T]$*;*
(iii)

$$F(t,x) \to +\infty$$

as $|x| \to +\infty$ *uniformly for a. e.* $t \in [0,T]$*.*

Then problem (1.1) *has* $kT-$*periodic solutions* u_k *with saddle point character in* H^1_{kT} *for every positive integer* k *such that* $\|u_k\|_\infty \to +\infty$ *as* $k \to +\infty$*.*

Theorem 1.3. *Suppose that* F *satisfies assumption (A) and conditions (i), (ii) in Theorem 1.2 and the following condition holds:*
(iii) assume that there exists a function $\gamma \in L^1(0,T)$ *such that* $F(t,x) \geq \gamma(t)$ *for all* $x \in R^N$ *and a. e.* $t \in [0,T]$*, and that there exists a subset* E *of* $[0,T]$ *with meas*$(E) > 0$ *such that*

$$F(t,x) \to +\infty$$

as $|x| \to +\infty$ *uniformly for a. e.* $t \in E$*.*

Then problem (1.1) *has* $kT-$*periodic solutions* u_k *with saddle point character in* H^1_{kT} *for every positive integer* k *such that* $\|u_k\|_\infty \to +\infty$ *as* $k \to +\infty$*.*

Remark 1.4. Note that the coercive conditions used in [10] is that $|x|^{-2\alpha}F(t,x) \to +\infty$ as $|x| \to +\infty$ uniformly for a. e. $t \in [0,T]$ or a. e. $t \in E$. Hence the results in Theorem 1.2 and Theorem 1.3 are novel and are significant improvement, compare to results from the literature mentioned above and the other references.

2. Proofs Of Theorems

Now we give the proofs of our main results. We only give the proofs of Theorem 1.1 and Theorem 1.3, Theorem 1.2 is only a special case of Theorem 1.3.

Proof of Theorem 1.1. For each fixed $u \in \widetilde{H}^1_T$ and any $v_1, v_2 \in R^N$, one has

$$\langle \varphi'(u(t)+v_1), v_1 - v_2 \rangle = -\int_0^T (\bigtriangledown F(t, u(t)+v_1), v_1 - v_2)dt,$$

$$\langle \varphi'(u(t)+v_2), v_1 - v_2 \rangle = -\int_0^T (\bigtriangledown F(t, u(t)+v_2), v_1 - v_2)dt,$$

so by condition (i) one has:

$$\int_0^T (\nabla F(t,u(t)+v_1) - \nabla F(t,u(t)+v_2), v_1 - v_2)dt \geq |v_1 - v_2|^{1+\beta} \int_0^T \mu(t)dt$$

where β denote either α or 1 as there exists one of norms of v_1, v_2 is large or not. Consequently,

$$\langle -\varphi'(u(t)+v_1) - (-\varphi'(u(t)+v_2)), v_1 - v_2\rangle \geq |v_1 - v_2|^{1+\beta} \int_0^T \mu(t)dt,$$

which implies

$$\begin{array}{l} \langle \varphi'(u(t)+v_1) - \varphi'(u(t)+v_2), v_1 - v_2\rangle \\ \leq -|v_1 - v_2|^{1+\beta} \int_0^T \mu(t)dt \\ = -\|v_1 - v_2\|\ h(\|v_1 - v_2\|) \end{array} \tag{2.1}$$

where $h(s) = \frac{\int_0^T \mu(t)dt}{T^{\frac{1+\beta}{2}}} s^\beta$ is an strictly increasing function from R^+ to R^+ such that $h(s) \to +\infty$ as $s \to +\infty$.

For each $u \in \widetilde{H}_T^1$ define the functional $\Psi_u : R^N \to R$ by

$$\Psi_u(v) = \varphi(u+v).$$

Since $\varphi \in C^1(H_T^1, R)$, then $\Psi_u \in C^1(R^N, R)$, by (2.1) Ψ_u has at most one critical point. If $\Psi_u'(0) = 0$, then Ψ_u has the only critical point $v = 0$. If $\Psi_u'(0) \neq 0$, we claim that $-\Psi_u$ is coercive. Since

$$\begin{aligned} \Psi_u(v) &= \Psi_u(0) + \int_0^1 \langle \Psi_u'(sv), v\rangle ds \\ &\leq \Psi_u(0) + \|\Psi_u'(0)\|\ \|v\| - \int_0^1 \|v\|\ h(\|sv\|)ds. \end{aligned}$$

By the property of h, we may choose R large enough such that

$$h(\|sv\|) \geq 4\ \|\Psi_u'(0)\| \ \ uniformly\ for\ \ |v| \geq R, \quad s \in [\frac{1}{2}, 1].$$

Therefore

$$\Psi_u(v) \leq \Psi_u(0) - \|\Psi_u'(0)\|\ \|v\|$$

which implies that $\Psi_u(v) \to -\infty$ as $\|v\| \to \infty$. Hence $-\Psi_u$ is coercive. Next we shall show that $\Psi_u(v)$ is concave. For given $v_1, v_2 \in R^N$, define $\xi(s) = \Psi_u(v_1 + s(v_2 - v_1))$, for $0 < \alpha < \beta < 1$, by simple computation we have

$$\xi'(\beta) - \xi'(\alpha) < 0$$

which means that ξ is concave, and so is Ψ_u. Combining the above arguments, we see that Ψ_u has a unique maximizer $\theta(u) \in R^N$ such that $\Psi_u(\theta(u)) =$

$\sup_{v\in R^N}\varphi(u+v)$, that is, for all $u\in\widetilde{H}^1_T$, $\theta(u)$ is the unique critical point for Ψ_u in R^N such that $\Psi_u(\theta(u))=\sup_{v\in R^N}\varphi(u+v)$. Therefore one has

$$\langle\varphi'(u+\theta(u)),v\rangle=0,\qquad\forall v\in R^N. \tag{2.2}$$

We shall establish that $\theta:\widetilde{H}^1_T\to R^N$ is continuous. For this we assume on the contrary, that there exist $\epsilon_0>0$ and $u_n\to u$ as $n\to+\infty$ such that

$$\|\theta(u_n)-\theta(u)\|\geq\epsilon_0.$$

Let P be the projection from H^1_T to R^N. By (2.2) we see that $\|P\varphi'(u_n+\theta(u))\|\leq h(\epsilon_0/2)$ if n large enough. Therefore by (2.1) one has

$$\begin{aligned}
&-h(\epsilon_0)\|\theta(u_n)-\theta(u)\|\\
&\geq -h(\|\theta(u_n)-\theta(u)\|)\|\theta(u_n)-\theta(u)\|\\
&\geq\langle\varphi'(u_n+\theta(u_n))-\varphi'(u_n+\theta(u)),\theta(u_n)-\theta(u)\rangle\\
&=\langle-\varphi'(u_n+\theta(u)),\theta(u_n)-\theta(u)\rangle\\
&=\langle-P\varphi'(u_n+\theta(u)),\theta(u_n)-\theta(u)\rangle\\
&\geq-\|P\varphi'(\theta(u_n)+\theta(u))\|\|\theta(u_n)-\theta(u)\|\\
&\geq-h(\epsilon_0/2)\|\theta(u_n)-\theta(u)\|
\end{aligned}$$

which implies that $h(\epsilon_0)\leq h(\epsilon_0/2)$, a contradiction proving the assertion.

For each $u\in\widetilde{H}^1_T$, define functional J as follows:

$$J(u)=\varphi(u+\theta(u))=\sup_{v\in R^N}\varphi(u+v).$$

Using the continuity of $\theta(\cdot)$, $J(u)$ is well defined and we shall show that $J(u)=\varphi(u+\theta(u))$ is of $C^1(\widetilde{H}^1_T,R)$ and

$$\langle J'(u),w\rangle=\langle\varphi'(u+\theta(u)),w\rangle,\qquad\forall u,w\in\widetilde{H}^1_T$$

this and (2.2) will imply that an element $u\in\widetilde{H}^1_T$ is a critical point of J if and only if $u+\theta(u)$ is a critical point of φ.

Indeed, for $s>0$,

$$\begin{aligned}
\frac{J(u+sw)-J(u)}{s}&=\frac{\varphi(u+sw+\theta(u+sw))-\varphi(u+\theta(u))}{s}\\
&\geq\frac{\varphi(u+sw+\theta(u))-\varphi(u+\theta(u))}{s}\\
&=\int_0^1\langle\varphi'(u+\theta(u)+tsw),w\rangle dt
\end{aligned}$$

In a similar way, we have

$$\frac{J(u+sw)-J(u)}{s}\leq\int_0^1\langle\varphi'(u+\theta(u+sw)+tsw),w\rangle dt$$

Combining above two inequalities proving the assertions.

Moreover, as $\|u\| \to \infty$ in H_T^1 if and only if $(|\overline{u}|^2 + \int_0^T |\dot{u}|^2 dt)^{\frac{1}{2}} \to \infty$, and by Wertinger's inequality it follows that $\|\widetilde{u}\| \to \infty$ implies that $\int_0^T |\dot{u}|^2 dt \to \infty$ which yet implies that $(|\overline{u}|^2 + \int_0^T |\dot{u}|^2 dt)^{\frac{1}{2}} \to \infty$, hence for each $u \in \widetilde{H}_T^1$ with $\|u\|$ large, by condition (ii) we have

$$
\begin{aligned}
&|\int_0^T F(t,u(t))dt| = |\int_0^T [F(t,u(t)) - F(t,0)]dt + \int_0^T F(t,0)dt| \\
&\leq |\int_0^T \int_0^1 (\bigtriangledown F(t,su(t)), u(t))dsdt| + c_4 \\
&\leq \int_0^T \int_0^1 | \bigtriangledown F(t,su(t))| \; |u(t)|dsdt + c_4 \\
&\leq \int_0^T \int_0^1 [f(t)|u(t)|^{1+\alpha} s^\alpha + g(t)|u(t)|]dsdt + c_4 \\
&\leq \frac{1}{1+\alpha}\|u(t)\|_\infty^{1+\alpha} \int_0^T f(t)dt + \|u(t)\|_\infty \int_0^T g(t)dt + c_4.
\end{aligned}
$$

Hence by the definition of $J(u)$ and Wertinger's inequality and (1.2) one has

$$
\begin{aligned}
J(u) \quad &\geq \quad \varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt \\
&\geq \quad \frac{1}{2}\int_0^T |\dot{u}|^2 dt - c_2\|u\|^{1+\alpha} - c_3\|u\| - c_4 \\
&\geq \quad c_1\|u\|^2 - c_2\|u\|^{1+\alpha} - c_3\|u\| - c_4 \qquad (2.3)
\end{aligned}
$$

where $c_i (i = 1,2,3,4)$ are some positive constants. Then (2.3) implies that $J(u) \to +\infty$ as $\|u\| \to +\infty$ on $\widetilde{H}_T^1$. Consequently, there exists a point $u_0 \in \widetilde{H}_T^1$ such that $J(u_0) = \min_{\widetilde{H}_T^1} J(u)$, and hence $u = u_0 + \theta(u_0)$ is a solution with saddle point character of problem (1.1) in H_T^1.

In addition, if condition (iii) holds, since φ is weakly lower semicontinuous on H_T^1, so is J on $\widetilde{H}_T^1$. By the coerciveness and weakly lower semicontinuity of J we know that J satisfies $P.S.$ condition and is bounded below. Let X_2 be a finite dimensional subspace of $X = \widetilde{H}_T^1$ given by

$$X_2 = \{\sum_{j=1}^k (a_j cosj\omega t) + b_j sinj\omega t | a_j, b_j \in R^N, j = 1,2,\cdots,k\}$$

and let $X_1 = X_2^\perp$ the orthogonal complement of X_2 in $\widetilde{H}_T^1$. We claim that $\theta(0) = 0$. Indeed, (iii) follows that $F(t,0) = 0 = \bigtriangledown F(t,0)$ for a.e.$t \in [0,T]$.

Hence by (i) one has

$$\begin{aligned} 0 &= \langle \varphi'(\theta(0)), \theta(0)\rangle = \int_0^T (-\bigtriangledown F(t,\theta(0)), \theta(0))dt \\ &= \int_0^T (-\bigtriangledown F(t,\theta(0)) - (-\bigtriangledown F(t,0)), \theta(0))dt \\ &\leq -|\theta(0)|^{1+\beta}[\int_0^T \mu(t)dt] \leq 0 \end{aligned}$$

therefore $\theta(0) = 0$. By the continuity of θ there exists a positive number $\delta_1 < \delta/(2c_0)$ such that $|\theta(u)| < \delta/2$ as $\|u\| \leq \delta_1$. Then by (iii) we obtain

$$J(u) = \varphi(u+\theta(u)) \leq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2}k^2\omega^2 \int_0^T |u+\theta(u)|^2 dt \leq 0 \quad (2.4)$$

for all $u \in X_2$ with $\|u\| \leq \delta_1$ and

$$J(u) \geq \varphi(u) \geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2}(k+1)^2\omega^2 \int_0^T |u|^2 dt \geq 0$$

for all $u \in X_1$ with $\|u\| \leq \delta_1$.

Hence $\inf\{J(u) : u \in \widetilde{H}_T^1\} \leq 0$. If $\inf\{J(u) : u \in \widetilde{H}_T^1\} = 0$, (2.4) implies that all $u \in X_2$ with $\|u\| \leq \delta_1$ are minima of J and therefore φ has infinite critical points with saddle point character. If $\inf\{J(u) : u \in \widetilde{H}_T^1\} < 0$, then by Theorem 4 in [2] it follows that J has at least two non zero critical points, hence problem (1.1) has at least two non trivial solutions with correlated property in which there exists at least one with saddle point character. In addition, since $\nabla F(t,0) = 0$ for a. e. $t \in [0,T]$, hence 0 is also a solution of problem (1.1) and $0 = 0 + \theta(0)$ by previous confirmation. Hence problem (1.1) has at least three distinct solutions with correlated property in H_T^1 in which there exists at least one which is nontrivial one with saddle point character. We complete the proof.

Proof of Theorem 1.3. Without loss of generality, we may assume that functions b in assumption(A), f, g in (1.3) are $T-$ periodic and assumptions (A), (1.3) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

Replace T by kT in the definitions of $H_T^1, \widetilde{H}_T^1, \varphi$, and φ' in Theorem 1.1, then we obtain the corresponding spaces and functionals $H_{kT}^1, \widetilde{H}_{kT}^1, \varphi_k$, and φ_k', respectively. Then one has $H_{kT}^1 = R^N \bigoplus \widetilde{H}_{kT}^1$, obviously. Hence Theorem 1.1 implies that there exists $u_0 \in \widetilde{H}_{kT}^1$ such that $u_k = u_0 + \theta(u_0)$ is a $kT-$periodic solution with saddle point character in H_{kT}^1. Now we prove that $\|u_k\|_\infty \to +\infty$

as $k \to +\infty$. Set

$$e_k(t) = k(cosk^{-1}\omega t)x_0$$

for all $t \in R$ and some $x_0 \in R^N$ with $|x_0| = 1$, where $\omega = 2\pi/T$. Then we have

$$e_k(t) \in \widetilde{H}^1_{kT}$$

and

$$\dot{e}_k(t) = -\omega(sink^{-1}\omega t)x_0$$

which implies that

$$\|\dot{e}_k(t)\|_2^2 = \frac{1}{2}kT\omega^2.$$

By the definition of u_k, we have

$$\varphi_k(u_k) = \varphi_k(u_0 + \theta(u_0)) = J(u_0) = \min_{u \in \widetilde{H}^1_{kT}} J(u)$$

$$\leq J(e_k) = \varphi_k(e_k + \theta(e_k)) = \sup_{v \in R^N} \varphi_k(e_k + v) = \sup_{R^N + e_k} \varphi_k \tag{2.5}$$

Set $\delta = measE/2$, it follows from Lemma 1 in [10] and condition (iii) that there exists a subset E_δ of E with $meas(E \setminus E_\delta) < \delta$ such that

$$F(t, x) \to +\infty \quad as \quad |x| \to +\infty$$

uniformly for all $t \in E_\delta$, which implies that

$$measE_\delta = measE - measE_\delta > \delta > 0$$

and for every $\beta > 0$, there exists $M \geq 1$ such that

$$F(t, x) \geq \beta \tag{2.6}$$

for all $|x| \geq M$ and all $t \in E_\delta$. For fixed $x \in R^N$, set

$$A_k = \{t \in [0, kT] \mid |x + e_k(t)| \leq M\}.$$

Then Tang-Wu [10] has proved that

$$measA_k \leq \frac{k\delta}{2} \tag{2.7}$$

for all large k. Let

$$E_k = \bigcup_{j=0}^{k-1} (jT + E_\delta).$$

Then it follows from (2.7) that

$$meas(E_k \setminus A_k) \geq \frac{1}{2}k\delta$$

for large k. Hence by (iii), (2.5), (2.6) and above inequality we have

$$k^{-1}\varphi_k(x+e_k) = \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT} F(t, x+e_k(t))dt$$
$$\leq \frac{1}{4}T\omega^2 - k^{-1}\int_{[0,kT]\setminus(E_k\setminus A_k)} F(t, x+e_k)dt - k^{-1}\int_{(E_k\setminus A_k)} F(t, x+e_k)dt$$
$$\leq \frac{1}{4}T\omega^2 - k^{-1}\int_{[0,kT]\setminus(E_k\setminus A_k)} \gamma(t)dt - k^{-1}\beta meas(E_k \setminus A_k)$$
$$\leq \frac{1}{4}T\omega^2 + \int_0^T |\gamma(t)|dt - \frac{T}{2}\delta\beta$$

for all $x \in R^N$ and all large k. Then by the arbitrariness of β, following the same way in [10] we complete our proof.

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