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# A NOTE ON (C, 1)(E, q) PRODUCT SUMMABILITY

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**Abstract.** In this paper, two new theorems on (C, 1)(E, q) product summability of Fourier series and its conjugate series have been established.

## 1. Introduction

Several researchers like Singh [7], Khare [3], Mittal and Kumar [5], Singh and Singh [8] , Pandey [6] and Jadia [2] have studied $(N, p_n)$, (N, p, q), almost (N, p, q) and matrix summability methods of Fourier Series and its conjugate series using different conditions. But nothing seems to have been done so far to study (C, 1)(E, q) product summability of Fourier series and its conjugate series. Therefore, in this paper, two theorems on (C, 1)(E, q) summability of Fourier series and its conjugate series under a general condition have been proved.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its $n^{th}$ partial sum $\{s_n\}$.

The(C, 1) transform is defined as the $n^{th}$ partial sum of (C, 1) summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + ......s_n}{n+1}$$

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$$= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \; as \; n \to \infty \tag{1.1}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C, 1) method.

If

$$(E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n} \binom{n}{k} q^{n-k} \; s_k \to s \text{ as } n \to \infty \tag{1.2}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ with partial sum $s_n$ is said to be summable by $(E,q)$ method to a definite number s (Hardy[1]).

The (C,1) transform of the (E,q) transform defines (C,1) (E,q) transform and it can be denoted $C_n^1 E_n^q$.

Thus if

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} E_k^q \to s \; as \; n \to \infty \tag{1.3}$$

where $E_n^q$ denotes the (E,q) transform of $s_n$ and $C_n^1$ denotes the (C,1) transform of $s_n$. Then the series $\sum_{n=0}^{\infty} \mathrm{u}_n$ is said to be summable by $(E,q)\,(C,1)$ means or summable $(E,q)\,(C,1)$ to a definite number s. Therefore, we can write $C_n^1 E_n^q \to s$ as $n \to \infty$.

The method (C,1)(E,q) is regular and this case is supposed throughout this paper.

Let $f(x)$ be a $2\pi$-periodic function and integrable over $[-\pi,\pi]$ in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f\,(x) \backsim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n\,(x) \tag{1.4}$$

The conjugate series of Fourier series (1.4) is given by

$$\sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n\,(x) \tag{1.5}$$

We shall call (1.5) as conjugate Fourier series throughout this paper.

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$
$$\psi(t) = f(x+t) + f(x-t)$$
$$K_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[ \frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right]$$

$\tau = \left[\frac{1}{t}\right]$ where $\tau$ denotes the greatest integer not greater than $\frac{1}{t}$.

## 2. Main Theorems

We prove the following theorems:

**Theorem 2.1.** *Let $\{p_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that*

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \ \ as\ n \to \infty.$$

*If*

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).P_\tau}\right], \ \ as\ t \to +0, \tag{2.1}$$

*where $\alpha(t)$ is a positive, monotonic and non-increasing function of $t$ and*

$$\log(n+1) = O\left[\{\alpha(n+1)\}.P_{n+1}\right], \ \ as\ n \to \infty, \tag{2.2}$$

*then the Fourier series (1.4) is summable (C,1)(E,q) to f(x) .*

**Theorem 2.2.** *Let $\{p_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that*

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \ \ as\ n \to \infty.$$

*If*

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).P_\tau}\right] \ \ as\ t \to +0, \tag{2.3}$$

*where $\alpha(t)$ is a positive, monotonic and non-increasing function of $t$,*

$$(1+q)^\tau \sum_{k=\tau}^{n} (1+q)^{-k} = O(n+1) \tag{2.4}$$

*and condition (2.2) holds then the conjugate series (1.5) is summable (C,1)(E,q) to*

$$\overline{f}(x) = -\frac{1}{2\pi} \int_0^{2\pi} \psi(t)\ \cot\left(\frac{t}{2}\right)\ dt$$

*at every point where this integral exists.*

## 3. Lemmas

For the proof of our theorems, following lemmas are required:

**Lemma 3.1.** *For* $0 \le t \le \frac{1}{n+1}$

$$|K_n(t)| = O(n+1).$$

*Proof.* For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^{n} \left[ \frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \right] \right| \\
&\le \frac{1}{2\pi\ (n+1)} \left| \sum_{k=0}^{n} \left[ \frac{1}{(1+q)^k} (2k+1) \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \right] \right| \\
&= \frac{1}{2\pi\ (n+1)} \sum_{k=0}^{n} (2k+1) \\
&= O(n+1).
\end{aligned}$$

□

**Lemma 3.2.** *For* $\frac{1}{n+1} \le t \le \pi$,

$$|K_n(t)| = O\left(\frac{1}{t}\right).$$

*Proof.* For $\frac{1}{n} \le t \le \pi$, by applying Jordan's Lemma $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}, \sin nt \le 1$

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{2\pi\ (n+1)} \left| \sum_{k=0}^{n} \left[ \frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \left\{ \binom{k}{\nu} q^{k-\nu} \frac{1}{\left(\frac{t}{\pi}\right)} \right\} \right] \right| \\
&= \frac{1}{2\ t\ (n+1)} \sum_{k=0}^{n} \left[ \frac{1}{(1+q)^k} (1+q)^k \right] \\
&= \frac{1}{2\ t\,(n+1)} \sum_{k=0}^{n} 1 \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

**Lemma 3.3.** *For* $0 \leq t \leq \frac{1}{n+1}$,

$$\bar{K}_n(t) = O\left(\frac{1}{t}\right).$$

*Proof.* For $0 \leq t \leq \frac{1}{n+1}$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $|\cos nt| \leq 1$.

$$\begin{aligned}
\left|\bar{K}_n(t)\right| &\leq \frac{1}{2\pi\ (n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu} q^{k-\nu}\ \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}\right]\right| \\
&= \frac{1}{2\pi\ (n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}\ \frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin\left(\frac{t}{2}\right)\right|} \\
&= \frac{1}{2\ t\ (n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu} \\
&= \frac{1}{2\ t\ (n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}(1+q)^k \\
&= \frac{1}{2\ t\ (n+1)}\sum_{k=0}^{n}1 \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

**Lemma 3.4.** *For* $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ *and any* $n$, *we have*

$$\bar{K}_n(t) = O\left[\frac{\tau^2}{(n+1)}\right] + O\left[\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right].$$

*Proof.* For $0 \leq \frac{1}{n+1} \leq t \leq \pi$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$,

$$\begin{aligned}
\left|\bar{K}_n(t)\right| &\leq \frac{1}{2\pi\ (n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}\ \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]\right| \\
&= \frac{1}{2\pi\ (n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|
\end{aligned}$$

$$\leq \frac{1}{2t\ (n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|\left|e^{i\frac{t}{2}}\right|$$

$$\leq \frac{1}{2t\ (n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\ (n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$+\frac{1}{2t\ (n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|. \tag{3.1}$$

Now considering first term of (3.1)

$$\frac{1}{2t\ (n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\ (n+1)}\left|\sum_{k=0}^{\tau-1}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right|\left|e^{i\nu t}\right|$$

$$\leq \frac{1}{2t\ (n+1)}\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right]$$

$$= \frac{1}{2t\ (n+1)}\sum_{k=0}^{\tau-1}1$$

$$= \frac{\tau}{2t\ (n+1)}$$

$$= O\left\{\frac{\tau^2}{(n+1)}\right\}. \tag{3.2}$$

Considering second term of (3.1) and using Abel's lemma

$$\frac{1}{2t\ (n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^k}\ Re\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\ (n+1)}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}\ \underset{0\leq m\leq k}{max}\ \left|\sum_{\nu=0}^{m}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right|$$

$$\leq \frac{1}{2t\ (n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}$$

$$= O\left[\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right]. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we get

$$\bar{K}_n(t) = O\left[\frac{\tau^2}{(n+1)}\right] + O\left[\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right].$$

□

## 4. Proof of Theorem 2.1.

Following Titchmarsh [9] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (1.4) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Therefore using (1.4), the $(E,q)$ transform $E_n^q$ of $s_n(f;x)$ is given by

$$E_n^q - f(x) = \frac{1}{2\pi\ (1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin\left(\frac{t}{2}\right)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin\left(k+\frac{1}{2}\right)t\right\}dt.$$

Now denoting $(C,1)(E,q)$ transform of $s_n(f;x)$ as $C_n^1E_n^q$, we write

$$\begin{aligned} C_n^1E_n^q - f(x) &= \frac{1}{2\pi\ (n+1)}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin\left(\frac{t}{2}\right)}\right. \\ &\qquad\left.\times\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right] \\ &= \int_0^{\pi}\phi(t)\ K_n(t)\,dt. \end{aligned}$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^{\pi}\phi(t)\ K_n(t)\,dt = o(1) \text{ as } n\to\infty.$$

We have, for $0<\delta<\pi$,

$$\begin{aligned} \int_0^{\pi}\phi(t)\ K_n(t)\,dt &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} + \int_{\delta}^{\pi}\right]\phi(t)\ K_n(t)\,dt \\ &= I_{1.1} + I_{1.2} + I_{1.3}\ \ \text{(say)}. \end{aligned} \tag{4.1}$$

We consider,

$$\begin{aligned}
|I_{1.1}| &\leq \int_0^{\frac{1}{n+1}} |\phi(t)|\, |K_n(t)|\, dt \\
&= O(n+1)\left[\int_0^{\frac{1}{n+1}} |\phi(t)|\ \ dt\right] \text{ (using Lemma 1)} \\
&= O(n+1)\left[o\left\{\frac{1}{(n+1)\ \ \alpha(n+1).P_\tau}\right\}\right] \quad \text{by (2.1)} \\
&= O\left[\frac{1}{\alpha(n+1)\,P_{n+1}}\right] \\
&= O\left\{\frac{1}{\log(n+1)}\right\} \text{ using (2.2)} \\
&= O(1), \text{ as } n \to \infty. \qquad (4.2)
\end{aligned}$$

Now we consider,

$$|I_{1.2}| \leq \int_{\frac{1}{n+1}}^{\delta} |\phi(t)|\, |K_n(t)|\, dt.$$

Using Lemma 2, we get,

$$\begin{aligned}
|I_{1.2}| &= O\left[\int_{\frac{1}{n+1}}^{\delta} |\phi(t)|\ \left(\frac{1}{t}\right) dt\right] \\
&= O\left[\left\{\frac{1}{t}\ \Phi(t)\ \right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2}\ \Phi(t)\ \ dt\right] \\
&= O\left[O\left\{\frac{1}{\alpha(t)\,P_\tau}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} O\left\{\frac{1}{t\ \alpha(t)\,P_\tau}\right\} dt\right] \text{ by (2.1)} \\
&= O\left[O\left\{\frac{1}{\alpha(n+1)\,P_{n+1}}\right\} + \int_{\frac{1}{\delta}}^{n+1} O\left\{\frac{1}{u\ \alpha(u)\,P_u}\right\} du\right] \\
&= O\left\{\frac{1}{\alpha(n+1)\,P_{n+1}}\right\} + O\left\{\frac{1}{(n+1)\,\alpha(n+1)\,P_{n+1}}\right\} \int_{\frac{1}{\delta}}^{n+1} 1.du \\
&= O\left\{\frac{1}{\log(n+1)}\right\} + O\left\{\frac{1}{\log(n+1)}\right\} \text{ by (2.2)} \\
&= O(1) + O(1), \text{ as } n \to \infty \\
&= O(1), \quad \text{as } n \to \infty. \qquad (4.3)
\end{aligned}$$

Now by Riemann-Lebesgue theorem and by regularity condition of the method of summability, we have

$$|I_{1.3}| \leq \int_{\delta}^{\pi} |\phi(t)| \; |K_n(t)| \; dt$$
$$= o(1), \text{ as } n \to \infty. \tag{4.4}$$

Combining (4.1) to (4.4), we get

$$C_n^1 E_n^q - f(x) = o(1), \text{ as } \; n \to \infty.$$

This completes the proof of Theorem 2.1.

## 5. Proof of Theorem 2.2.

Let $\bar{s}_n(f;x)$ denotes the partial sum of series (1.5). Then following Lal [4] and Riemaann- Lebesgue theorem $\bar{s}_n(f;x)$ of (1.5) is given by

$$\bar{s}_n(f;x) - \bar{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \; \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \; dt.$$

Therefore using (1.2), the $(E,q)$ transform $E_n^q$ of $\bar{s}_n(f;x)$ is given by

$$\overline{E_n^q} - \bar{f}(x) = \frac{1}{2\pi \; (1+q)^n} \int_0^{\pi} \frac{\psi(t)}{\sin\left(\frac{t}{2}\right)} \left[ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \cos\left(k+\frac{1}{2}\right)t \right] dt.$$

Now denoting $\overline{(C,1)(E,q)}$ transform of $\bar{s}_n$ by $\overline{C_n^1 E_n^q}$, we write

$$\overline{C_n^1 E_n^q} - \bar{f}(x) = \frac{1}{2\pi \; (n+1)} \sum_{k=0}^{n} \left[ \left\{ \frac{1}{(1+q)^k} \right\} \int_0^{\pi} \frac{\psi(t)}{\sin\left(\frac{t}{2}\right)} \right.$$
$$\left. \times \left\{ \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \cos\left(\nu+\frac{1}{2}\right)t \right\} dt \right]$$
$$= \int_0^{\pi} \psi(t) \; \bar{K}_n(t) \; dt.$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^{\pi} \psi(t) \; \bar{K}_n(t) \; dt = o(1) \quad as \quad n \to \infty.$$

We have, for $0 < \delta < \pi$

$$\int_0^{\pi} \psi(t) \; \bar{K}_n(t) \; dt = \left[ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} + \int_{\delta}^{\pi} \right] \psi(t) \; \bar{K}_n(t) \; dt$$
$$= I_{2.1} + I_{2.2} + I_{2.3} \quad \text{(say)}. \tag{5.1}$$

Now consider,

$$\begin{aligned}
|I_{2.1}| &\leq \int_0^{\frac{1}{n+1}} |\psi(t)| \ \left|\bar{K}_n(t)\right| dt \\
&= O\left[\int_0^{\frac{1}{n+1}} \frac{1}{t} |\psi(t)| \ dt\right] \ \text{(using Lemma 3)} \\
&= O(n+1)\left[\int_0^{\frac{1}{n+1}} |\psi(t)| \ dt\right] \\
&= O(n+1)\left[O\left\{\frac{1}{(n+1) \ \alpha(n+1) P_{n+1}}\right\}\right] \quad \text{by} \ \ (2.3) \\
&= O\left\{\frac{1}{\alpha(n+1) P_{n+1}}\right\} \\
&= O\left\{\frac{1}{\log(n+1)}\right\} \ \text{using (2.2)} \\
&= O(1), \quad \text{as } n \to \infty. \qquad (5.2)
\end{aligned}$$

Now we consider,

$$|I_{2.2}| = \int_{\frac{1}{n+1}}^{\delta} |\psi(t)| \ \left|\bar{K}_n(t)\right| dt.$$

Using Lemma 4, we get

$$\begin{aligned}
I_{2.2} &= O\left[\int_{\frac{1}{n+1}}^{\delta} \left(\frac{\tau^2}{(n+1)}\right) |\psi(t)| \, dt\right] \\
&\quad + O\left[\int_{\frac{1}{n+1}}^{\delta} \left\{\frac{\tau \ (1+q)^{\tau}}{(n+1)} \sum_{k=\tau}^{n} (1+q)^{-k}\right\} \ |\psi(t)| \, dt\right] \\
&= I_{2.2.1} + I_{2.2.2} \ \ \text{(say)}. \qquad (5.3)
\end{aligned}$$

Now we consider, by (2.2) and (2.3),

$$\begin{aligned}
I_{2.2.1} &= O\left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2} |\psi(t)| \, dt\right] \\
&= O\left(\frac{1}{n+1}\right)\left[\left\{\frac{1}{t^2}\Psi(t)\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{2}{t^3}\Psi(t) \ dt\right] \\
&= O\left(\frac{1}{n+1}\right)\left[\left\{\frac{1}{t^2} \ O\left(\frac{t}{\alpha\left(\frac{1}{t}\right) P_{\tau}}\right)\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{2}{t^3} \ O\left(\frac{t}{\alpha\left(\frac{1}{t}\right) P_{\tau}}\right) dt\right]
\end{aligned}$$

$$
\begin{aligned}
&= O\left(\frac{1}{n+1}\right)\left[O\left\{\left(\frac{1}{t\ \alpha\left(\frac{1}{t}\right) P_\tau}\right)^{\delta}_{\frac{1}{n+1}}\right\}\right]\\
&\quad + O\left(\frac{1}{n+1}\right)\left[O\left\{\int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2\ \alpha\left(\frac{1}{t}\right) P_\tau} dt\right\}\right]\\
&= O\left(\frac{1}{n+1}\right)\left[O\left\{\frac{(n+1)}{\alpha\,(n+1)\,P_{n+1}}\right\}\right] + O\left(\frac{1}{n+1}\right)\left[O\left\{\int_{\frac{1}{\delta}}^{n+1} \frac{1}{\alpha\,(u)\,P_u} du\right\}\right]\\
&= O\left\{\frac{1}{\alpha\,(n+1))P_{n+1}}\right\} + O\left(\frac{1}{n+1}\right) O\left\{\frac{1}{\alpha\,(n+1))P_{n+1}}\right\}\int_{\frac{1}{\delta}}^{n+1} 1.du\\
&= O\left\{\frac{1}{\log\,(n+1)}\right\} + O\left\{\frac{1}{\log\,(n+1)}\right\}\\
&= O(1) + O(1),\ as\ n \to \infty\\
&= O(1), \quad \text{as } n \to \infty. \qquad (5.4)
\end{aligned}
$$

Now considering

$$
I_{2.2.2} = O\left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\delta} \left(\frac{1}{t}\right)(1+q)^{\tau} \sum_{k=\tau}^{n} \frac{1}{(1+q)^{k}}\ \ |\psi\,(t)\ |\,dt\right].
$$

Using condition (2.2)-(2.4), we get

$$
\begin{aligned}
I_{2.2.2} &= O(1)\left[\int_{\frac{1}{n+1}}^{\delta} \frac{1}{t}\ \ |\psi\,(t)\ |\,dt\right]\\
&= O(1)\left[\left\{\frac{1}{t}\ \Psi\,(t)\right\}^{\delta}_{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta}\left\{\frac{1}{t^2}\ \Psi\,(t)\right\}\ dt\right]\\
&= O(1)\left[\left\{\frac{1}{t}\ O\left(\frac{t}{\alpha\left(\frac{1}{t}\right) P_\tau}\right)\right\}^{\delta}_{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2}\ O\left(\frac{t}{\alpha\left(\frac{1}{t}\right) P_\tau}\right)\ dt\right]\\
&= O(1)\left[O\left\{\left(\frac{1}{\alpha\left(\frac{1}{t}\right) P_\tau}\right)^{\delta}_{\frac{1}{n+1}}\right\}\right] + O(1)\left[O\left\{\int_{\frac{1}{n+1}}^{\delta} \frac{1}{t\ \alpha\left(\frac{1}{t}\right) P_\tau}\ \right\} dt\right]\\
&= O\left\{\frac{1}{\alpha\,(n+1))P_{n+1}}\right\} + O(1)\left[\int_{\frac{1}{\delta}}^{n+1} O\left\{\frac{1}{u\ \alpha\,(u)\,P_u}\right\} du\right]\\
&= O\left\{\frac{1}{\alpha\,(n+1))P_{n+1}}\right\} + O\left\{\frac{1}{(n+1)\,\alpha\,(n+1))P_{n+1}}\right\}\int_{\frac{1}{\delta}}^{n+1} 1\ du
\end{aligned}
$$

$$= O\left\{\frac{1}{\log(n+1)}\right\} + O\left\{\frac{1}{\log(n+1)}\right\}$$
$$= O(1) + O(1),\ as\ n \to \infty$$
$$= O(1), \quad \text{as } n \to \infty. \tag{5.5}$$

Now by Riemann-Lebesgue theorem and by regularity condition of the method of summability, we have

$$|I_{2.3}| \leq \int_{\delta}^{\pi} |\psi(t)| \left|\overline{K}_n(t)\right| dt$$
$$= O(1), \quad \text{as } n \to \infty. \tag{5.6}$$

Combining (5.1) to (5.6), we get

$$\overline{C_n^1 E_n^q} - \overline{f}(x) = o(1), \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 2.2.

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