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STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce modified Ishikawa iteration for finding a common element of the set of fixed points of quai-ϕ-nonexpansive mappings and the set of solutions of an equilibrium problem. Our results are new and can be viewed as direct generalizations and extensions of the corresponding results obtained in [11, 15]. And we give the problems studied in [8, 9, 10, 12] some new conditions under which their results are still true. We also provide some new estimation techniques in the proofs of the results.

1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a functional, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $p \in C$, such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Equilibrium problems provide us with a systematic framework to study a wide class of problems

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arising in finance economics, optimization and operation research etc., which motivate the extensive concern. In recent years, equilibrium problems have been deeply and thoroughly researched. See, for example, [2, 4, 13].

Let E be a real Banach space, C a nonempty closed convex subset of E and $S : C \to C$ a mapping. $F(S)$ denotes the fixed point of S. Recall that S is nonexpansive if

$$\| Sx - Sy \| \leq \| x - y \| \quad \forall x, y \in C.$$

$S$is said to be quasi-nonexpansive if $F(S)$ is nonempty and

$$\| Sx - y \| \leq \| x - y \| \quad \forall x \in C, y \in F(S).$$

S is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Sx_n = y_0$, then $Sx_0 = y_0$.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [7] and is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sx_n, \quad n \in \mathbb{N} \cup \{0\}. \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is in the interval [0,1].The second iteration process is referred to as Ishikawa's iteration process [5] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) Sx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sy_n, \end{cases} \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in the interval [0,1].

Generally, not much has been known regarding the convergence of the iteration processes (1.2)-(1.3) unless the underlying space E has elegant properties.

Attempts to modify the Mann's iteration method (1.2) so that strong convergence theorems for equilibrium problems and fixed point problems have recently been made. [12] proposed the following modification of the Mann's iteration (1.2) for equilibrium problems and a single relatively nonexpansive mapping S in a Banach space

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n = 0, 1, \ldots \end{cases} \tag{1.4}$$

Then, [11] further improved the above theorem by considering equilibrium problems and a pair of quasi-ϕ-nonexpansive mappings. They consider the following iteration process:

$$\begin{cases} \forall x_0 \in C, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C \\ C_{n+1} = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 1, 2, \ldots \end{cases} \tag{1.5}$$

Finally, [11] considered the problem of finding a common element in the common fixed point set of a family of quasi -ϕ-nonexpansive mappings and in the solution set of the equilibrium problem (1.1). That is, they considered the following iteration method:

$$\begin{cases} \forall x_0 \in C, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum\limits_{i=1}^{N} \alpha_{n,i} J S_i x_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C \\ C_{n+1} = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 1, 2, \ldots \end{cases} \tag{1.6}$$

Recently, [15] adapted the iteration (1.3) in Banach space. More precisely, they introduced the following iteration process for equilibrium problem and a relatively nonexpansive mapping:

$$\begin{cases} \forall x_0 \in C, \\ z_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J S x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S z_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots \end{cases} \tag{1.7}$$

Motivated by the work of [11], the purpose of this paper is to employ the idea to modified process (1.7) to prove strong convergence theorems for equilibrium problems and quasi-ϕ-nonexpansive mappings under some appropriate conditions in Banach spaces. Our results are new and can be viewed as direct generalizations and extensions of the corresponding results obtained in [11, 15]. And we give the problems studied in [8, 9, 10, 12]et al. some new conditions under which their results are still true.We also provide some new estimation techniques in the proofs of the results.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle\cdot,\cdot\rangle$ the duality product. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \| x \|^2 = \| f^* \|^2\},$$

for $x \in E$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$, then $x_n \to x$. We know the following:

(1) if E is smooth, then J is single-valued;

(2) if E is strictly convex, then J is one-to-one, that is, if $Jx \cap Jy$ is nonempty, then $x = y$;

(3) if E is reflexive, then J is onto;

(4) if E is smooth and reflexive, then J is norm-to-weak continuous, that is, $Jx_n \rightharpoonup Jx$ whenever $x_n \to x$;

(5) if E is uniformly convex, then E has the Kadec-Klee property;

(6) the norm of E^* is Fréchet differentiable if and only if E is a strictly convex and reflexive Banach space which has the Kadec-Klee property; see [14] for more details.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \| y \|^2 - 2\langle y, Jx\rangle + \| x \|^2$$

for $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\| y \| - \| x \|)^2 \le \phi(y, x) \le (\| y \| + \| x \|)^2 \tag{2.1}$$

for all $x, y \in E$

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\| x \| = \| y \| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \| x_n - y_n \| = 0$ for any two sequences $\{x_n\}$,$\{y_n\}$ in E such that $\| x_n \| = \| y_n \| = 1$ and $\lim_{n\to\infty} \frac{\|x_n+y_n\|}{2} = 1$. Let $U = \{x \in E : \| x \| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$.

Following [1], the generalized projection Π_C from E onto C is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional

$\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x)$$

If E is a Hilbert space, then $\phi(y, x) = \| x - y \|^2$ and Π_C is the metric projection of E onto C.

We know the following lemmas for generalized projections.

Lemma 2.1. ([1]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \forall x \in C, y \in E.$$

Lemma 2.2. ([1]) *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz \rangle \leq 0, \forall y \in C.$$

[6] also proved the following result. This plays an important role in the proof of the main theorem.

Lemma 2.3. ([6]) *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\| x_n - y_n \| \to 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let S be a mapping from C into itself. We denoted by $F(S)$ the set of fixed points of S. A point $p \in C$ is said to be an asymptotic fixed point of S if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n \to \infty} \| x_n - Sx_n \| = 0$. We denote the set of all asymptotic fixed points of S by $\hat{F}(S)$. Following [19], a mapping S of C into itself is said to be relatively nonexpansive if $F(S)$ is nonempty; $\phi(u, Sx) \leq \phi(u, x)$, $\forall u \in F(S),\ x \in C$; $\hat{F}(S) = F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [8, 9]. S is said to be $\phi-$nonexpansive if $\phi(Sx, Sy) \leq \phi(x, y)$, $\forall x, y \in C$. S is said to be quasi-ϕ-nonexpansive if $F(S)$ is nonempty; $\phi(u, Sx) \leq \phi(u, x)$, $\forall u \in F(S),\ x \in C$.

Remark 2.1. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires $\hat{F}(S) = F(S)$.

Remark 2.2. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed and quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$. See [5] for more details.

The following lemma is due to [11].

Lemma 2.4. ([11]) *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E, and let S be a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(S)$ is closed and convex.*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0, \forall x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0, \forall x, y \in C$;

(A3) $\forall x, y, z \in C$,

$$\limsup_{t \to 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) $\forall x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

We have the following result:

Lemma 2.5. ([3]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a functional and satisfying (A1)-(A4), let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C.$$

The following lemma is from [11]:

Lemma 2.6. ([11]) *Let C be a closed convex subset of a uniformly convex and smooth Banach space E, and let $f : C \times C \to \mathbb{R}$ be a functional , satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$$

for all $x \in E$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex and T_r is a quasi-ϕ-nonexpansive mapping.

Lemma 2.7. ([12]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a functional ,* *satisfying (A1)-(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$$

3. Main results

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a functional, satisfying (A1)-(A4) and let S, T be two closed quasi-ϕ-nonexpansive mappings from C into itself such that $F = F(S) \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \forall x_0 \in C, \\ z_n = J^{-1}(\xi_n J x_n + \eta_n J T x_n + \delta_n J S x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T z_n + \gamma_n J S z_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n)\} \\ Q_n = \{v \in C : \langle x_n - v, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots \end{cases} \tag{3.1}$$

Where J is the duality mapping on E, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$*;*
(ii) $\xi_n + \eta_n + \delta_n = 1$*;*
(iii) $\lim_{n\to\infty} \xi_n = 1$, $\liminf_{n\to\infty} \beta_n > 0$, $\liminf_{n\to\infty} \gamma_n > 0$.

If S, T is uniformly continuous, Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. We divide the proof of this theorem to 4 steps as below.

STEP 1. We show that $C_n \cap Q_n$ is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. We prove that C_n is convex. For $v_1, v_2 \in C_n$ and $t \in (0, 1)$, put $v = tv_1 + (1 - t)v_2$. It is sufficient to show that $v \in C_n$. Next, we show

$$\phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n). \tag{3.2}$$

is equivalent to

$$\begin{aligned} &2\alpha_n \langle v, J x_n \rangle + 2(1 - \alpha_n)\langle v, J z_n \rangle - 2\langle v, J u_n \rangle \\ &\leq \alpha_n \parallel x_n \parallel^2 + (1 - \alpha_n) \parallel z_n \parallel^2 - \parallel u_n \parallel^2 . \end{aligned} \tag{3.3}$$

Indeed, from the definition of $\phi(y, x)$, one can get the above inequality.

Then, by (3.3) we have C_n is convex. So, $C_n \cap Q_n$ is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. Hence, $\Pi_{C_n \cap Q_n}$ is well defined.

STEP 2. We show that $F \subset C_n \cap Q_n$.
Let $u \in F$. Putting $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N} \cup \{0\}$, By Lemma 2.6(4), we have that T_{r_n} is quasi-ϕ-nonexpansive. Since S, T are also quasi-ϕ-nonexpansive, by the definition of quasi-ϕ-nonexpansive and the convexity of $\| \cdot \|^2$ we have

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, T_{r_n} y_n) \\
&\le \phi(u, y_n) \\
&= \phi(u, J^{-1}(\alpha_n J x_n + \beta_n J T z_n + \gamma_n J S z_n)) \\
&= \| u \|^2 - 2\langle u, \alpha_n J x_n + \beta_n J T z_n + \gamma_n J S z_n \rangle \\
&\quad + \| \alpha_n J x_n + \beta_n J T z_n + \gamma_n J S z_n \|^2 \\
&\le \| u \|^2 - 2\alpha_n \langle u, J x_n \rangle - 2\beta_n \langle u, J T z_n \rangle - 2\gamma_n \langle u, J S z_n \rangle \\
&\quad + \alpha_n \| x_n \|^2 + \beta_n \| T z_n \|^2 + \gamma_n \| S z_n \|^2 \\
&\le \alpha_n \phi(u, x_n) + \beta_n \phi(u, T z_n) + \gamma_n \phi(u, S z_n) \\
&\le \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, z_n).
\end{aligned} \tag{3.4}$$

Hence, we have $u \in C_n$. This implies that

$$F \subset C_n, \forall n \in \mathbb{N} \cup \{0\}.$$

Next we show by induction that $F \subset C_n \cap Q_n, \forall n \in \mathbb{N} \cup \{0\}$. From $Q_0 = C$, we have

$$F \subset C_0 \cap Q_0$$

Suppose that $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that

$$x_{k+1} = \Pi_{C_k \cap Q_k} x_0$$

By Lemma 2.2, we have, for all $z \in C_k \cap Q_k$,

$$\langle x_{k+1} - z, J x_0 - J x_{k+1} \rangle \ge 0$$

Since $F \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, J x_0 - J x_{k+1} \rangle \ge 0, \forall z \in F$$

and hence $z \in Q_{k+1}$. So, we have

$$F \subset C_{k+1} \cap Q_{k+1}$$

Therefore we have $F \subset C_n \cap Q_n, \forall n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well-defined. From the definition of Q_n and Lemma 2.2, we have $x_n = \Pi_{Q_n} x_0$. Using $x_n = \Pi_{Q_n} x_0$, from Lemma 2.1 we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \le \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \le \phi(u, x_0)$$

for all $u \in F \subset Q_n$.Then, $\phi(x_n, x_0)$ is bounded. Therefore, $\{x_n\}$,$\{Tx_n\}$, $\{Sx_n\}$ are bounded. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $x_n = \Pi_{Q_n} x_0$, from the definition of Π_{Q_n} we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \tag{3.5}$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. So, the limit of $\{\phi(x_n, x_0)\}$ exists. From $x_n = \Pi_{Q_n} x_0$ and Lemma 2.1, we also have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \end{aligned} \tag{3.6}$$

$\forall n \in \mathbb{N} \cup \{0\}$. By (3.5) and (3.6)

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.7}$$

It follows from Lemma 2.3 that $x_n - x_m \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to p$ as $n \to \infty$.

STEP 3. We show that $p \in F$.

Firstly, we show $p \in F(S) \cap F(T)$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, we have

$$\begin{aligned} \phi(x_{n+1}, u_n) &= \phi(x_{n+1}, T_{r_n} y_n) \leq \phi(x_{n+1}, y_n) \\ &\leq \alpha_n \phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, z_n), \forall n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Then by the convexity of $\| \cdot \|^2$. We obtain

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n)) \\ &= \| x_{n+1} \|^2 - 2\langle x_{n+1}, \xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n \rangle \\ &\quad + \| \xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n \|^2 \\ &\leq \| x_{n+1} \|^2 - 2\xi_n \langle x_{n+1}, Jx_n \rangle - 2\eta_n \langle x_{n+1}, JTx_n \rangle \\ &\quad - 2\delta_n \langle x_{n+1}, JSx_n \rangle + \xi_n \| x_n \|^2 + \eta_n \| Tx_n \|^2 + \delta_n \| Sx_n \|^2 \\ &= \xi_n \phi(x_{n+1}, x_n) + \eta_n \phi(x_{n+1}, Tx_n) + \delta_n \phi(x_{n+1}, Sx_n). \end{aligned}$$

Since $\lim_{n\to\infty} \xi_n = 1$ and (3.7), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0. \tag{3.8}$$

So, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.9}$$

From (3.7)-(3.9), by Lemma 2.3, we obtain

$$\begin{aligned} \lim_{n\to\infty} \| x_{n+1} - x_n \| &= \lim_{n\to\infty} \| x_{n+1} - z_n \| = \lim_{n\to\infty} \| x_{n+1} - y_n \| \\ &= \lim_{n\to\infty} \| x_{n+1} - u_n \| = 0 \end{aligned} \tag{3.10}$$

Since J is uniformly norm-to-norm continuous on bounded sets we have

$$\lim_{n\to\infty} \| Jx_{n+1} - Jx_n \| = \lim_{n\to\infty} \| Jx_{n+1} - Jy_n \| = 0.$$

And since

$$\| x_n - z_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - z_n \|,$$

$$\| x_n - y_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - y_n \|,$$

$$\| x_n - u_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - u_n \| .$$

It follows from (3.10) that

$$\lim_{n\to\infty} \| x_n - z_n \| = \lim_{n\to\infty} \| x_n - y_n \| = \lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{3.11}$$

Hence by $x_n \to p$, we obtain, $u_n \to p$. Noticing that

$$\begin{aligned} \| Jx_{n+1} - Jy_n \| &= \| Jx_{n+1} - (\alpha_n Jx_n + \beta_n JTz_n + \gamma_n JSz_n) \| \\ &\geq \beta_n \| Jx_{n+1} - JTz_n \| + \gamma_n \| Jx_{n+1} - JSz_n \| \\ &\quad - \alpha_n \| Jx_n - Jx_{n+1} \|, \end{aligned}$$

We have that

$$\| Jx_{n+1} - JSz_n \| \leq \frac{1}{\gamma_n} (\| Jx_{n+1} - Jy_n \| + \alpha_n \| Jx_n - Jx_{n+1} \|).$$

Since $\liminf_{n\to\infty} \gamma_n \geq 0$, it follows that

$$\lim_{n\to\infty} \| Jx_{n+1} - JSz_n \| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \| x_{n+1} - Sz_n \| = 0. \tag{3.12}$$

It follows that

$$\| x_n - Sx_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - Sz_n \| + \| Sz_n - Sx_n \| .$$

Since S is uniformly continuous. It follows from (3.10)-(3.12) that $\lim_{n\to\infty} \| Sx_n - x_n \| = 0$. Then, in a similarly way, from (3.10)-(3.12) one can obtain $\lim_{n\to\infty} \| Tx_n - x_n \| = 0$. From the closeness of S and T, one has $p \in F(T) \cap F(S)$.

Next, we show $p \in EP(f)$. Let $u \in EP(f)$, from (3.4), we have

$$\begin{aligned} \phi(u, u_n) &\leq \alpha_n \phi(u, x_n) + \beta_n \phi(u, Tz_n) + \gamma_n \phi(u, Sz_n) \\ &\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, z_n). \end{aligned} \tag{3.13}$$

And since

$$\begin{aligned}\phi(u,z_n) &= \phi(u, J^{-1}(\xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n)) \\ &= \| u \|^2 - 2\langle u, \xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n)\rangle \\ &\quad + \| \xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n) \|^2 \\ &\leq \| u \|^2 - 2\xi_n\langle u, Jx_n\rangle - 2\eta_n\langle u, JTx_n\rangle - 2\delta_n\langle u, JSx_n\rangle \\ &\quad + \xi_n \| x_n \|^2 + \eta_n \| Tx_n \|^2 + \delta_n \| Sx_n \|^2 \\ &\leq \xi_n\phi(u,x_n) + \eta_n\phi(u,Tx_n) + \delta_n\phi(u,Sx_n) \\ &\leq \phi(u,x_n)\end{aligned} \tag{3.14}$$

So, from (3.13) and (3.14), we have

$$\phi(u,u_n) \leq \phi(u,x_n).$$

Since

$$\begin{aligned}&\phi(u,x_n) - \phi(u,u_n) \\ &= \| x_n \|^2 - \| u_n \|^2 - 2\langle u, Jx_n - Ju_n\rangle \\ &\leq |\| x_n \|^2 - \| u_n \|^2| - 2\langle u, Jx_n - Ju_n\rangle \\ &\leq |\| x_n \| - \| u_n \|| (\| x_n \| + \| u_n \|) + 2 \| u \| \| Jx_n - Ju_n \| \\ &\leq \| x_n - u_n \| (\| x_n \| + \| u_n \|) + 2 \| u \| \| Jx_n - Ju_n \|\end{aligned}$$

From (3.11) we have

$$\lim_{n\to\infty}(\phi(u,x_n) - \phi(u,u_n)) = 0. \tag{3.15}$$

From $u_n = T_{r_n}y_n$, (3.4), Lemma 2.7 , we have

$$\begin{aligned}\phi(u_n,y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\leq \phi(u,y_n) - \phi(u,T_{r_n}y_n) \\ &\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,z_n) - \phi(u,T_{r_n}y_n) \\ &\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,x_n) - \phi(u,T_{r_n}y_n) \\ &= \phi(u,x_n) - \phi(u,u_n)\end{aligned}$$

So, we have

$$\lim_{n\to\infty}\phi(u_n,y_n) = 0$$

Since E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty} \| u_n - y_n \| = 0. \tag{3.16}$$

Since J is uniformly norm-to-norm continuous on bounded sets we have

$$\lim_{n\to\infty} \| Ju_n - Jy_n \| = 0.$$

From the assumption $r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\| Ju_n - Jy_n \|}{r_n} = 0. \tag{3.17}$$

From $u_n = T_{r_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

from $(A2)$, we have

$$\begin{aligned} \| y - u_n \| \frac{\| Ju_n - Jy_n \|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C. \end{aligned}$$

Letting $n \to \infty$, we have from (3.11), $u_n \to p$ and $(A4)$ that

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, we have $y_t \in C$ and hence from (A3), $f(y_t, p) \leq 0$. So, from $(A1)$ we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, p) \\ &\leq tf(y_t, y). \end{aligned}$$

Dividing by t, we have

$$f(y_t, y) \geq 0, \forall y \in C.$$

Letting $t \to 0$, from $(A3)$ we have

$$f(p, y) \geq 0, \forall y \in C.$$

So, $p \in EP(f)$. This shows that $p \in F$.

STEP 4. We show that $p = \Pi_F x_0$. From $x_n = \Pi_{Q_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \forall z \in Q_n.$$

Since $F \subset Q_n$ for each $n \geq 1$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \forall z \in F.$$

By taking the limit, one has

$$\langle p - z, Jx_0 - Jp\rangle \geq 0, \forall z \in Q_n.$$

In view of Lemma 2.2, we obtain $p = \Pi_F x_0$. This completes the proof. □

As some corollaries of Theorem3.1, we have the following results immediately.

Corollary 3.1. ([11]) *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a functional , satisfying (A1)-(A4) and let S, T be two closed quasi-ϕ-nonexpansive mappings from C into itself such that $F = F(S) \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \forall x_0 \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, x_n)\} \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots \end{cases}$$

Where J is the duality mapping on E, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:
(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\liminf_{n\to\infty} \beta_n > 0$, $\liminf_{n\to\infty} \gamma_n > 0$.
Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. In Theorem 3.1, let $\xi_n = 1$, then $z_n = x_n$ and $y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n)$. The set C_n reduced to the set C_n in [11] and since our proof is different from [11], so the condition $\liminf_{n\to\infty} \alpha_n\beta_n > 0, \liminf_{n\to\infty} \alpha_n\gamma_n > 0$ can be replaced by our condition (ii). Our Q_n can be replaced by C, without affecting the main result. □

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a functional , satisfying (A1)-(A4) and let S be a closed quasi-ϕ-nonexpansive mapping from C into itself such that $F = F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \forall x_0 \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n, \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C \\ C_n = \{v \in C : \phi(v, u_n) \leq \phi(v, x_n)\} \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots \end{cases}$$

Where J is the duality mapping on E, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$ are sequences in $[0, 1]$ satisfying the restriction: $\limsup_{n\to\infty} \alpha_n < 1$. If S is uniformly *continuous,Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.*

Proof. In Corollary 3.1, let $T = I$, the identity mapping, then combining with Theorem 3.1, we have the desired result. □

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $f : C\times C \to \mathbb{R}$ be a functional , satisfying (A1)-(A4) and let S be a closed quasi-ϕ-nonexpansive mappings from C into itself such that $F = F(S)\cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \forall x_0 \in C, \\ z_n = J^{-1}(\xi_n Jx_n + (1-\xi_n)JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSz_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0, \quad n = 0, 1, \dots \end{cases}$$

Where J is the duality mapping on E, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:

(i) $\lim_{n\to\infty} \xi_n = 1$;

(ii) $\limsup_{n\to\infty} \alpha_n < 1$.

If S is uniformly continuous.Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. In Theorem3.1, let $T = S$, then by Theorem 3.1, we have the desired result. □

Remark 3.1. Noticing that, Corollary 3.2, 3.3 generalize and extend the Theorem 3.1 of [11] and Theorem 3.2 of [12] respectively. We go from relatively nonexpansive mappings to more general quasi-ϕ-nonexpansive mapping; that is we relax the strong restriction: $\hat{F}(S) = F(S)$.

Remark 3.2. In Theorem 3.1, if we set $f(x, y) = 0, \forall x, y \in C$, and $r_n = 1$, $\forall n \geq 1$, then our Theorem offers some new conditions for the corresponding problems discussed in [8], [9] and [10].

Remark 3.3. In Theorem 3.1, if E is a Hilbert space, then $\phi(x, y) = \| x - y \|^2$. All of the above results are still true which are also generalizations and extensions of corresponding results.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $f : C \times C \to$*

$\mathbb{R}$ be a functional satisfying (A1)-(A4) and Let $S_i : C \to C$ be a closed and quasi-ϕ-nonexpansive mapping for each $i \in \{1, 2, \ldots, N\}$ such that $F = \cap_{i=1}^{N} F(S_i) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} \forall x_0 \in C, \\ z_n = J^{-1}(\beta_{n,0} Jx_n + \sum\limits_{i=1}^{N} \beta_{n,i} JS_i x_n), \\ y_n = J^{-1}(\alpha_{n,0} Jx_n + \sum\limits_{i=1}^{N} \alpha_{n,i} JS_i z_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_{n,0}\phi(v, x_n) + (1 - \alpha_{n,0})\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots, \end{cases} \tag{3.18}$$

where $\{r_n\} \subset [a, \infty)$, a is a positive real number and J is the duality mapping on E. $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ are real sequences in [0,1], satisfying the following restrictions:

(i) $\sum\limits_{i=0}^{N} \alpha_{n,i} = 1, \sum\limits_{i=0}^{N} \beta_{n,i} = 1$;

(ii) $\liminf\limits_{n\to\infty} \alpha_{n,i} > 0, \forall i \in \{1, 2, \ldots, N\}, \lim\limits_{n\to\infty} \beta_{n,0} = 1.$

If S_i is uniformly continuous$\forall i \in \{1, 2, \ldots, N\}$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. As in the proof of Theorem 3.1, we divide the proof of this theorem to 4 steps as following.

STEP 1. $C_n \cap Q_n$ is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. Similarly with the step1 of Theorem3.1, we obtain the desired result.

STEP 2. We show that $F \subset C_n \cap Q_n$.

Similarly with the step 2 of Theorem 3.1, we only need to show that$F \subset C_n$. Let $u \in F$. From $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N} \cup \{0\}$, By Lemma 2.6(4), we have that T_{r_n} is quasi-ϕ-nonexpansive. Since S_i is also quasi-ϕ-nonexpansive, by the definition of quasi-ϕ-nonexpansive and the convexity of $\| \cdot \|^2$ we have

$$\begin{aligned} \phi(u, u_n) =& \phi(u, T_{r_n} y_n) \\ \leq& \phi(u, y_n) \\ =& \phi(u, J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{N} \alpha_{n,i} JS_i z_n)) \\ =& \| u \|^2 - 2\langle u, \alpha_{n,0} Jx_n + \sum_{i=1}^{N} \alpha_{n,i} JS_i z_n\rangle \end{aligned}$$

$$\begin{aligned}
&+ \| \alpha_{n,0} Jx_n + \sum_{i=1}^{N} \alpha_{n,i} JS_i z_n \|^2 \\
&\leq \| u \|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{N} \alpha_{n,i}\langle u, JS_i z_n\rangle \\
&+ \alpha_{n,0} \| x_n \|^2 + \sum_{i=1}^{N} \alpha_{n,i} \| S_i z_n \|^2 \\
&\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(u, S_i z_n) \\
&\leq \alpha_{n,0}\phi(u, x_n) + (1 - \alpha_{n,0})\phi(u, z_n).
\end{aligned} \tag{3.19}$$

Hence, we have $u \in C_n$. This implies that

$$F \subset C_n, \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore we have $F \subset C_n \cap Q_n, \forall n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well-defined. Similarly with the step 2 of Theorem 3.1 $\{x_n\}$ and $\{S_i x_n\}$ are bounded and the limit of $\{\phi(x_n, x_0)\}$ exists. From $x_n = \Pi_{Q_n} x_0$ and Lemma 2.1, we also have

$$\begin{aligned}
\phi(x_{n+1}, x_n) =&\phi(x_{n+1}, \Pi_{Q_n} x_0) \\
\leq&\phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\
=&\phi(x_{n+1}, x_0) - \phi(x_n, x_0)
\end{aligned} \tag{3.20}$$

$\forall n \in \mathbb{N} \cup \{0\}$. This means that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.21}$$

It follows from Lemma 2.3 that $x_n - x_m \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to p$ as $n \to \infty$.

STEP 3. We show that $p \in F$.

Firstly, $p \in \cap_{i=1}^{N} F(S_i)$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$ and T_{r_n} is quasi-ϕ-nonexpansive, we have

$$\begin{aligned}
\phi(x_{n+1}, u_n) &= \phi(x_{n+1}, T_{r_n} y_n) \leq \phi(x_{n+1}, y_n) \\
&\leq \alpha_{n,0}\phi(x_{n+1}, x_n) + (1 - \alpha_{n,0})\phi(x_{n+1}, z_n), \quad \forall n \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

Then by the convexity of $\| \cdot \|^2$. We obtain

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{N} \beta_{n,i}JS_ix_n)) \\ &=\| x_{n+1} \|^2 - 2\langle x_{n+1}, \beta_{n,0}Jx_n + \sum_{i=1}^{N} \beta_{n,i}JS_ix_n\rangle \\ &\quad + \| \beta_{n,0}Jx_n + \sum_{i=1}^{N} \beta_{n,i}JS_ix_n \|^2 \\ &\leq \| x_{n+1} \|^2 - 2\beta_{n,0}\langle x_{n+1}, Jx_n\rangle - 2\sum_{i=1}^{N} \beta_{n,i}\langle x_{n+1}, JS_ix_n\rangle \\ &\quad + \beta_{n,0} \| x_n \|^2 + \sum_{i=1}^{N} \beta_{n,i} \| S_ix_n \|^2 \\ &= \beta_{n,0}\phi(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x_{n+1}, S_ix_n).\end{aligned}$$

Since $\lim_{n\to\infty} \beta_{n,0} = 1$ and (3.21), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0. \tag{3.22}$$

So, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.23}$$

From (3.21)-(3.23), by Lemma 2.3, we obtain

$$\begin{aligned}\lim_{n\to\infty} \| x_{n+1} - x_n \| &= \lim_{n\to\infty} \| x_{n+1} - z_n \| = \lim_{n\to\infty} \| x_{n+1} - y_n \| \\ &= \lim_{n\to\infty} \| x_{n+1} - u_n \| = 0.\end{aligned} \tag{3.24}$$

Since J is uniformly norm-to-norm continuous on bounded sets we have

$$\lim_{n\to\infty} \| Jx_{n+1} - Jx_n \| = \lim_{n\to\infty} \| Jx_{n+1} - Jy_n \| = 0.$$

And since

$$\begin{aligned}\| x_n - z_n \| &\leq \| x_{n+1} - x_n \| + \| x_{n+1} - z_n \|, \\ \| x_n - y_n \| &\leq \| x_{n+1} - x_n \| + \| x_{n+1} - y_n \|, \\ \| x_n - u_n \| &\leq \| x_{n+1} - x_n \| + \| x_{n+1} - u_n \| .\end{aligned}$$

It follows from (3.24) that

$$\lim_{n\to\infty} \| x_n - z_n \| = \lim_{n\to\infty} \| x_n - y_n \| = \lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{3.25}$$

Hence by $x_n \to p$, we obtain, $u_n \to p$. Noticing that

$$\begin{aligned}\| Jx_{n+1} - Jy_n \| &= \| Jx_{n+1} - (\alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i}JS_iz_n) \| \\ &= \| \alpha_{n,0}(Jx_{n+1} - Jx_n) + \sum_{i=1}^{N} \alpha_{n,i}(Jx_{n+1} - JS_iz_n) \| \\ &\geq \sum_{i=1}^{N} \alpha_{n,i} \| Jx_{n+1} - JS_iz_n \| - \alpha_{n,0} \| Jx_n - Jx_{n+1} \| .\end{aligned}$$

We have that

$$\| Jx_{n+1} - JS_iz_n \| \leq \frac{1}{\alpha_{n,i}}(\| Jx_n - Jy_n \| + \alpha_{n,0} \| Jx_n - Jx_{n+1} \|).$$

Since $\liminf_{n\to\infty} \alpha_{n,i} > 0$, it follows that

$$\lim_{n\to\infty} \| Jx_{n+1} - JS_iz_n \| = 0, \quad \forall i \in \{1, 2, \ldots, N\}.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \| x_{n+1} - S_iz_n \| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{3.26}$$

It follows that

$$\begin{aligned}\| x_n - S_ix_n \| \leq &\| x_n - x_{n+1} \| + \| x_{n+1} - S_iz_n \| \\ &+ \| S_iz_n - S_ix_n \|, \quad \forall i \in \{1, 2, \ldots, N\}.\end{aligned}$$

Since S_i is uniformly continuous. It follows from (3.24)-(3.26) that $\lim_{n\to\infty} \| S_ix_n - x_n \| = 0$. From the closeness of S_i, one has $p \in \cap_{i=1}^{N} F(S_i)$.

Next, we show $p \in EP(f)$. Putting $u_n = T_{r_n}y_n$, let $u \in EP(f)$, from (3.19), we have

$$\begin{aligned}\phi(u, u_n) &\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{N} \alpha_{n,i})\phi(u, S_iz_n) \\ &\leq \alpha_{n,0}\phi(u, x_n) + (1 - \alpha_{n,0})\phi(u, z_n).\end{aligned} \tag{3.27}$$

And since

$$\begin{aligned}
\phi(u,z_n) =& \phi(u, J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{N}\beta_{n,i}JS_ix_n)) \\
=& \| u \|^2 - 2\langle u, \beta_{n,0}Jx_n + \sum_{i=1}^{N}\beta_{n,i}JS_ix_n\rangle \\
&+ \| \beta_{n,0}Jx_n + \sum_{i=1}^{N}\beta_{n,i}JS_ix_n \|^2 \\
\leq& \| u \|^2 - 2\beta_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{N}\beta_{n,i}\langle u, JS_ix_n\rangle \\
&+ \beta_{n,0} \| x_n \|^2 + \sum_{i=1}^{N}\beta_{n,i} \| S_ix_n \|^2 \\
=& \beta_{n,0}\phi(u,x_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(u,S_ix_n) \\
\leq& \phi(u,x_n)
\end{aligned} \tag{3.28}$$

So, from (3.27) and (3.28), we have

$$\phi(u,u_n) \leq \phi(u,x_n).$$

Since

$$\begin{aligned}
&\phi(u,x_n) - \phi(u,u_n) \\
=& \| x_n \|^2 - \| u_n \|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
\leq& |\| x_n \|^2 - \| u_n \|^2| - 2\langle u, Jx_n - Ju_n\rangle \\
\leq& |\| x_n \| - \| u_n \|| (\| x_n \| + \| u_n \|) + 2 \| u \| \| Jx_n - Ju_n \| \\
\leq& \| x_n - u_n \| (\| x_n \| + \| u_n \|) + 2 \| u \| \| Jx_n - Ju_n \| .
\end{aligned}$$

From (3.25) we have

$$\lim_{n\to\infty}(\phi(u,x_n) - \phi(u,u_n)) = 0. \tag{3.29}$$

From $u_n = T_{r_n}y_n$, Lemma 2.7 , we have

$$\begin{aligned}
\phi(u_n,y_n) &= \phi(T_{r_n}y_n, y_n) \\
&\leq \phi(u,y_n) - \phi(u,T_{r_n}y_n) \\
&\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,z_n) - \phi(u,T_{r_n}y_n) \\
&\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,x_n) - \phi(u,T_{r_n}y_n) \\
&= \phi(u,x_n) - \phi(u,u_n).
\end{aligned}$$

So,we have

$$\lim_{n\to\infty} \phi(u_n, y_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty} \| u_n - y_n \|= 0. \tag{3.30}$$

Then, similarly with the proof of the step 3 of theorem 3.1, we have $p \in EP(f)$. This shows that $p \in F$.

STEP 4. We show that $p = \Pi_F x_0$. From $x_n = \Pi_{Q_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in Q_n.$$

Since $F \subset Q_n$ for each $n \geq 1$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in F.$$

By taking the limit, one has

$$\langle p - z, Jx_0 - Jp\rangle \geq 0, \quad \forall z \in Q_n.$$

In view of Lemma 2.2, we obtain $p = \Pi_F x_0$. This completes the proof. □

For a special case that $N = 2$, we can obtain the following results on a pair of quasi-ϕ-nonexpansive mappings immediately from Theorem 3.2.

Corollary 3.4. Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a functional, satisfying (A1)-(A4) and let S, T be two closed quasi-ϕ-nonexpansive mappings from C into itself such that $F = F(S) \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} \forall x_0 \in C, \\ z_n = J^{-1}(\xi_n Jx_n + \eta_n JTx_n + \delta_n JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTz_n + \gamma_n JSz_n), \\ u_n \in C, s.t., f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, \ldots. \end{cases}$$

Where J is the duality mapping on E, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\xi_n + \eta_n + \delta_n = 1$;

(iii) $\lim_{n\to\infty} \xi_n = 1, \liminf_{n\to\infty} \beta_n > 0$, $\liminf_{n\to\infty} \gamma_n > 0$.

If S, T is uniformly continuous, Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Remark 3.4. If we set $\beta_{n,0} = 1$ in Theorem 3.2 and don't consider the framework of spaces, then our result generalizes and extends Theorem 2.1 of [11]. We give new conditions which are different from [11] to get the desired result.

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