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COMMON FIXED POINT THEOREMS IN GENERALIZED ORDERED CONE METRIC SPACES

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Abstract. In this paper, we introduce the concept of generalized cone metric space and give some propositions on the concept. Then we prove several common fixed point theorems for a pair of mappings in generalized ordered cone metric spaces.

1. Introduction

Let E be a Banach space and P a subset of E. P is called a cone if and only if

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in \mathbb{R}$ with $a, b, \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$;

(c) $P \cap (-P) = \{0\}$.

For any cone P, a partial order $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. While $x \ll y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P. A cone P is called *normal* if there exists a number $K > 0$ such that

$$0 \preceq x \preceq y \implies \|x\| \leq K\|y\|$$

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for all $x, y \in E$. The least positive number K satisfying the above condition is called the *normal constant* of P.

In 2007, Huang and Zhang [5] introduced a concept called cone metric space:

Definition 1.1 ([5]). *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

(a1) $0 \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*
(a2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(a3) $d(x, y) \preceq d(x, z) + d(y, z)$ *for all* $x, y, z \in X$*.*

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is easy to see that a cone metric is a general metric space if $E = \mathbb{R}$ and $P = \mathbb{R}^+$. Hence, the concept of a cone metric space is more general than the one of a metric space. Since the concept of cone metric was introduced by Huang and Zhang [5], many fixed point theorems have been proved for mappings on normal or non-normal cone metric spaces (see, for example, [2-7] and references contained therein). Recently, some existence theorems of fixed points in ordered cone metric space were investigated by many authors (see, for example, [8-10]). In [3], Altun and Durmaz gave the following theorem:

Theorem 1.2 ([3]). *Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a normal cone P with the normal constant K) such that (X, d) is a complete cone metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*

(a) *there exists $k \in (0, 1)$ such that $d(fx, fy) \preceq kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;*

(b) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

Let $(X, \sqsubseteq)$ be a partially ordered set and $f, g : X \to X$ be two self-maps. The pair (f, g) is said to be weakly increasing w.r.t. $\sqsubseteq$ if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ for all $x \in X$. f is called g-nondecreasing if for all $x, y \in X$ with $gx \sqsubseteq gy$, then $fx \sqsubseteq fy$. f and g are called weakly compatible if for all $x \in X$ with $fx = gx$, then $fgx = gfx$. A sequence $\{x_n\}$ in X is called nondecreasing if $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$. In [4], Altun, Damnjanović and Djorić proved the following theorem:

Theorem 1.3 ([4]). *Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a cone P with int $P \neq \emptyset$) such that (X, d) is a complete cone metric space. Let $f, g : X \to X$ be self-mappings such that (f, g) is a weakly increasing pair with respect to $\sqsubseteq$. Suppose that the following conditions hold:*

(a) *there exist* $\alpha, \beta, \gamma \geq 0$ *such that* $\alpha + 2\beta + 2\gamma < 1$ *and*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)]$$

for all comparable $x, y \in X$*;*

(b) f *or* g *is continuous or*

(b′) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $x \in X$*, then* $x_n \sqsubseteq x$ *for all* n*.*

Then f *and* g *have a common fixed point* $x^* \in X$*.*

Very recently, Kadelburg, Pavlović and Radenović [6] extended Theorem 1.3 and obtained the following result:

Theorem 1.4 ([6])**.** *Let* $(X, \sqsubseteq, d)$ *be an ordered complete cone metric space. Let* (f, g) *be weakly increasing pair of self-mappings on* X *with respect to* $\sqsubseteq$*. Suppose that the following conditions hold:*

(a) *there exist* $p, q, r, s, t \geq 0$ *satisfying* $p + q + r + s + t < 1$ *and* $q = r$ *or* $s = t$*, such that*

$$d(fx, gy) \preceq pd(x, y) + qd(x, fx) + rd(y, gy) + sd(x, gy) + td(y, fx)$$

for all comparable $x, y \in X$*;*

(b) f *or* g *is continuous or*

(b′) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $x \in X$*, then* $x_n \sqsubseteq x$ *for all* $n \geq 1$*.*

Then f *and* g *have a common fixed point* $x^* \in X$*.*

In 2006, Mustafa and Sims [10] introduced a new concept called G-metric space which is a generalization of ordinary metric space.

Definition 1.5 ([10])**.** *Let* X *be a nonempty set and* $G : X^3 \to [0, \infty)$ *be a function. Then* G *is called a* G*-metric if for all* $x, y, z, a \in X$*, the following hold:*

(g1) $0 \preceq G(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(g2) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(: symmetry in all three variables);*

(g3) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(: rectangle inequality).*

The pair (X, G) *is called a generalized metric space.*

Some existence theorems of fixed points in G-metric space were obtained (see, for example, [11-14]).

In this paper, we introduce a concept called generalized cone metric space that is a generalization of G-metric space of Mustafa and Sims [10], and prove some fixed point theorems in ordered generalized cone metric space.

2. Preliminaries

Now, in this section, we give some definitions and lemmas for the main results in this paper.

Definition 2.1. Let X be a nonempty set and P be a cone on a real Banach space E. Suppose that the mapping $D : X \times X \times X \to P$ satisfies the following conditions:

(d1) $0 \preceq D(x, y, z) = 0$ if and only if $x = y = z$;

(d2) $D(x, y, z) = D(x, z, y) = D(y, z, x) = \cdots$ (: symmetry in all three variables);

(d3) $D(x, y, z) \preceq D(x, a, a) + D(a, y, z)$ for all $x, y, z, a \in X$ (: rectangle inequality).

Then D is called a generalized cone metric and the pair (X, D) is called a generalized cone metric space.

Example 2.2. *Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}$, $X = [0, 1]$ and $D(x, y, z) = (\max\{|x - y|, |y - z|, |x - z|\}, 0)$. Then (X, D) is a generalized metric space.*

Example 2.3. *Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}$, $X = [0, 1]$ and define the function $D : X \times X \times X \to E$ by*

$$D(x, y, z) = (0, \overline{max}\{x, y, z\}), \quad \forall x, y, z \in X,$$

where

$$\overline{max}\{x, y, z\} = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{else.} \end{cases}$$

It is easy to check that D satisfies (d1)-(d3) and hence (X, D) is a generalized cone metric space.

Example 2.4. *Let (X, d) be an ordinary cone metric space. Then $(X, D)$is a generalized cone metric space, where*

$$D(x, y, z) = d(x, y) + d(x, z) + d(y, z), \quad \forall x, y, z \in X.$$

Example 2.4 shows that an ordinary cone metric space can define a generalized cone metric space. Conversely, an ordinary cone metric also can be obtained by a generalized cone metric. In fact, if (X, D) is a generalized cone metric space, then an ordinary cone metric (X, d) can be defined by

$$d(x, y) = D(x, y, y) + D(y, x, x), \quad \forall x, y \in X.$$

It is easy to check that d satisfies the definition of an ordinary cone metric.

Proposition 2.5. *Let (X, D) be a generalized cone metric. For all $x, y, z, a \in X$, the following hold:*

(p1) $D(x, y, y) \preceq 2D(y, x, x)$*;*

(p2) $D(x, y, z) \preceq D(x, a, a) + D(y, a, a) + D(z, a, a)$*;*

(p3) $D(x, y, z) \preceq 2[D(x, y, y) + D(x, z, z)]$*;*

Proof. It follows from (d2) and (d3) that

$$D(x, y, y) = D(y, x, y) \preceq D(y, x, x) + D(x, x, y) = 2D(x, x, y).$$

Hence (p1) holds. For (p2), by (d2) and (d3), we have

$$\begin{aligned} D(x, y, z) &\preceq D(x, a, a) + D(y, a, z) \\ &\preceq D(x, a, a) + D(y, a, a) + D(a, a, z) \\ &= D(x, a, a) + D(y, a, a) + D(z, a, a). \end{aligned}$$

For (p3), By (d3) and (p1), we get

$$\begin{aligned} D(x, y, z) &\preceq D(x, y, y) + D(z, y, y) \\ &\preceq D(x, y, y) + D(z, x, x) + D(x, y, y) \\ &\preceq 2[D(x, y, y) + D(x, z, z)]. \end{aligned}$$

□

Definition 2.6. Let (X, D) be a generalized cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there exists a positive integer N such that, for all $m, n > N$, $D(x, x_n, x_m) \ll c$, then $\{x_n\}$ is said to be convergent to x. Write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Lemma 2.7. *Let (X, D) be a generalized cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if and only if $D(x, x_n, x_n) \to 0$ or $D(x, x, x_n) \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ converges to x. For any real number $\epsilon > 0$, choose $c \in E$ with $0 \ll c$ and $K\|c\| < \epsilon$. Since $x_n \to x$ as $n \to \infty$, there exists a positive integer N such that, for all $n > N$, $D(x, x_n, x_n) \ll c$, which implies that $D(x, x_n, x_n) \to 0$ as $n \to \infty$. It follows from (p1) that $D(x, x, x_n) \preceq 2D(x, x_n, x_n)$. Hence $D(x, x, x_n) \to 0$ as $n \to \infty$.

Conversely, suppose that $D(x, x_n, x_n) \to 0$ as $n \to \infty$. For all $c \in E$ with $0 \ll c$, there exists $r > 0$ such that, for any y with $\|y\| < r$, one has $y \ll \frac{1}{2}c$. Since $D(x, x_n, x_n) \to 0$ as $n \to \infty$, there exists a positive integer N such that

$$\|D(x, x_n, x_n)\| < r$$

for all $n > N$, which implies

$$D(x, x_n, x_n) \ll \frac{1}{2}c$$

for all $n > N$. It follows from (p3) that

$$D(x, x_n, x_m) \preceq 2[D(x, x_n, x_n) + D(x, x_m, x_m)].$$

Hence, for all $n, m > N$, one has $D(x, x_n, x_m) \ll c$. This shows that $\{x_n\}$ converges to x. Noting that, by (p1), we have

$$D(x, x_n, x_n) \preceq 2D(x, x, x_n)$$

and hence we can conclude that $\{x_n\}$ converges to x from $D(x, x, x_n) \to 0$ as $n \to \infty$. □

Lemma 2.8. *Let (X, D) be a generalized cone metric space, P is a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent, then the limit point of $\{x_n\}$ is unique.*

Proof. Assume that there exist $x \in E$ and $y \in E$ such that $x_n \to x$ and $x_n \to y$. We prove that $x = y$. From Lemma 2.7, for any $c \in E$ with $0 \ll c$, there is a positive integer N such that, for all $n \geq N$, $D(x, x_n, x_n) \ll c$ and $D(x_n, y, y) \ll c$. It follows (d3) of Definition 2.1 that

$$D(x, y, y) \leq D(x, x_n, x_n) + D(x_n, y, y) \ll 2c.$$

Since c is arbitrary, one has $D(x, y, y) = 0$. This shows that $x = y$. □

Definition 2.9. Let (X, D) be a generalized cone metric space and $\{x_n\}$ be a sequence in X. For any $c \in E$ with $0 \ll c$, if there exists a positive integer N such that, for all $m, n, l > N$, $D(x_m, x_n, x_l) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.10. Let (X, D) be a generalized cone metric space. If every Cauchy sequence in X is convergent, then (X, D) is called a complete generalized cone metric space.

Lemma 2.11. *Let (X, D) be a generalized cone metric space and P be a cone with non-empty interior. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $D(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$.*

Proof. The necessity is obvious by the Definition 2.9. Now, we prove the sufficiency. Suppose that $0 \ll c$ is arbitrary. Since $c \in int\, P$, there exists a neighborhood of 0,

$$N_\delta(0) = \{y \in E : \|y\| < \delta\},\ \delta > 0,$$

such that $c + N_\delta(0) \subseteq int\, P$. Since $D(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$, there exists a natural number N such that

$$\| - D(x_m, x_n, x_l)\| < \delta.$$

Then $-D(x_m, x_n, x_l) \in N_\delta(0)$ for all $m, n, l > N$. Hence $c - D(x_m, x_n, x_l) \in c + N_\delta(0) \subseteq int\, P$. Thus we have

$$D(x_m, x_n, x_l) \ll c, \quad \forall\, m, n, l > N.$$

This shows that $\{x_n\}$ is a Cauchy sequence. □

3. Main results

In this section, we denote the set of all positive integers by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The following are the main results of this paper.

Theorem 3.1. *Let $(X, \sqsubseteq, D)$ be an ordered complete generalized and normal cone metric space with a cone P with non-empty interior. Let (S, T) be a weakly increasing pair of self-mappings on X with respect to $\sqsubseteq$. Suppose that there exist $a, b, c, d, e \geq 0$ with $a + b + c + d + e < \frac{1}{4}$ such that*

$$\begin{aligned} D(Sx, Ty, Ty) \;\preceq\;& aD(x, y, y) + bD(x, Sx, Sx) + cD(y, Ty, Ty) \\ &+ dD(x, Ty, Ty) + eD(y, Sx, Sx) \end{aligned} \tag{3.1}$$

for all comparable $x, y \in X$. If S or T is continuous, then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \in \mathbb{N}_0$. Using that the pair of mappings (S, T) is weakly increasing, it can be easily shown that the sequence $\{x_n\}$ is nondecreasing with respect to $\sqsubseteq$, i.e., $x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$. In particular, x_{2n} and x_{2n+1} are comparable and so, if we apply (d3) and (3.1), then we obtain

$$\begin{aligned} & D(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ = \;& D(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \\ \preceq \;& aD(x_{2n}, x_{2n+1}, x_{2n+1}) + bD(x_{2n}, Sx_{2n}, Sx_{2n}) \\ & + cD(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}) \\ & + dD(x_{2n}, Tx_{2n+1}, Tx_{2n+1}) + eD(x_{2n+1}, Sx_{2n}, Sx_{2n}) \\ = \;& aD(x_{2n}, x_{2n+1}, x_{2n+1}) + bD(x_{2n}, x_{2n+1}, x_{2n+1}) \\ & + cD(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ & + dD(x_{2n}, x_{2n+2}, x_{2n+2}) + eD(x_{2n+1}, x_{2n+1}, x_{2n+1}) \\ \preceq \;& aD(x_{2n}, x_{2n+1}, x_{2n+1}) + bD(x_{2n}, x_{2n+1}, x_{2n+1}) \\ & + cD(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ & + d[D(x_{2n}, x_{2n+1}, x_{2n+1}) + D(x_{2n+1}, x_{2n+2}, x_{2n+2})]. \end{aligned}$$

This shows

$$(1 - c - d)D(x_{2n+1}, x_{2n+2}, x_{2n+2}) \preceq (a + b + d)D(x_{2n}, x_{2n+1}, x_{2n+1}),$$

that is,,

$$D(x_{2n+1}, x_{2n+2}, x_{2n+2}) \preceq \frac{a+b+d}{1-(c+d)} D(x_{2n}, x_{2n+1}, x_{2n+1}). \tag{3.2}$$

Similarly, by (d3) and Proposition (p1), we have

$$\begin{array}{ll} & D(x_{2n+2}, x_{2n+3}, x_{2n+3}) \\ \preceq & 2D(x_{2n+3}, x_{2n+2}, x_{2n+2}) \\ = & 2D(Sx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}) \\ \preceq & 2aD(x_{2n+2}, x_{2n+1}, x_{2n+1}) + 2bD(x_{2n+2}, Sx_{2n+2}, Sx_{2n+2}) \\ & +2cD(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}) \\ & +2dD(x_{2n+2}, Tx_{2n+1}, Tx_{2n+1}) + 2eD(x_{2n+1}, Sx_{2n+2}, Sx_{2n+2}) \\ \leq & 4aD(x_{2n+1}, x_{2n+2}, x_{2n+2}) + 2bD(x_{2n+2}, x_{2n+3}, x_{2n+3})b \\ & +2cD(x_{2n+1}, x_{2n+2}, x_{2n+2}) + 2eD(x_{2n+1}, x_{2n+3}, x_{2n+3}) \\ \preceq & 4aD(x_{2n+1}, x_{2n+2}, x_{2n+2}) + 2bD(x_{2n+2}, x_{2n+3}, x_{2n+3}) \\ & +2cD(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ & +2e[D(x_{2n+1}, x_{2n+2}, x_{2n+2}) + D(x_{2n+2}, x_{2n+3}, x_{2n+3})]. \end{array}$$

This shows

$$D(x_{2n+2}, x_{2n+3}, x_{2n+3}) \preceq \frac{4a+2c+2e}{1-2(b+e)} D(x_{2n+1}, x_{2n+2}, x_{2n+2}). \tag{3.3}$$

Thus it follows from (3.2) and (3.3) that

$$D(x_{2n+2}, x_{2n+3}, x_{2n+3}) \preceq \frac{4a+2c+2e}{1-2(b+e)} \cdot \frac{a+b+d}{1-(c+d)} D(x_{2n}, x_{2n+1}, x_{2n+1}). \tag{3.4}$$

Now, by (3.2), (3.4) and induction, we get

$$\begin{array}{ll} & D(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ \preceq & \frac{a+b+d}{1-(c+d)} D(x_{2n}, x_{2n+1}, x_{2n+1}) \\ \preceq & \frac{a+b+d}{1-(c+d)} \cdot \frac{4a+2c+2e}{1-2(b+e)} \cdot \frac{a+b+d}{1-(c+d)} D(x_{2n-2}, x_{2n-1}, x_{2n-1}) \\ \preceq & \cdots \preceq \frac{a+b+d}{1-(c+d)} \left(\frac{4a+2c+2e}{1-2(b+e)} \cdot \frac{a+b+d}{1-(c+d)} \right)^n D(x_0, x_1, x_1) \end{array}$$

and

$$\begin{array}{lll} D(x_{2n+2}, x_{2n+3}, x_{2n+3}) & \preceq & \frac{4a+2c+2e}{1-2(b+e)} D(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ & \preceq & \cdots \preceq \left(\frac{4a+2c+2e}{1-2(b+e)} \cdot \frac{a+b+d}{1-(c+d)} \right)^{n+1} D(x_0, x_1, x_1). \end{array}$$

Let $A = \frac{4a+2c+2e}{1-2(b+e)}$ and $B = \frac{a+b+d}{1-(c+d)}$. Since $a+b+c+d+e < \frac{1}{4}$, one has

$$4a + 2b + 2c + 4e < 1, \quad a + b + c + 2d < 1.$$

Hence $AB < 1$. Now, for $n < m$, by (d3), we have

$$\begin{array}{rl} & D(x_{2n+1}, x_{2m+1}, x_{2m+1}) \\ \preceq & D(x_{2n+1}, x_{2n+2}, x_{2n+2}) + D(x_{2n+2}, x_{2m+1}, x_{2m+1}) \\ \preceq & D(x_{2n+1}, x_{2n+2}, x_{2n+2}) + D(x_{2n+2}, x_{2n+3}, x_{2n+3}) \\ & +D(x_{2n+3}, x_{2m+1}, x_{2m+1}) \\ \preceq & D(x_{2n+1}, x_{2n+2}, x_{2n+2}) + D(x_{2n+2}, x_{2n+3}, x_{2n+3}) \\ & +D(x_{2n+3}, x_{2m+1}, x_{2m+1}) + \cdots + D(x_{2m}, x_{2m+1}, x_{2m+1}) \\ \preceq & \left(A\sum_{i=n}^{m-1}(AB)^i + \sum_{i=n+1}^{m}(AB)^i\right)D(x_0, x_1, x_1) \\ \preceq & \left(\frac{A(AB)^n}{1-AB} + \frac{(aB)^{n+1}}{1-AB}\right)G(x_0, x_1, x_1) \\ = & (1+B)\frac{A(AB)^n}{1-AB}D(x_0, x_1, x_1). \end{array}$$

Similarly, we get

$$D(x_{2n}, x_{2m+1}, x_{2m+1}) \preceq (1+A)\frac{(AB)^n}{1-AB}D(x_0, x_1, x_1),$$

$$D(x_{2n}, x_{2m}, x_{2m}) \preceq (1+A)\frac{(AB)^n}{1-AB}D(x_0, x_1, x_1)$$

and

$$D(x_{2n+1}, x_{2m}, x_{2m}) \preceq (1+B)\frac{A(AB)^n}{1-AB}D(x_0, x_1, x_1).$$

Hence, for $n < m$, one has

$$\begin{array}{rcl} D(x_n, x_m, x_m) & \preceq & \max\left\{(1+B)\frac{A(AB)^n}{1-AB}, (1+A)\frac{(AB)^n}{1-AB}\right\} D(x_0, x_1, x_1) \\ & = & \lambda_n D(x_0, x_1, x_1), \end{array}$$

where $\lambda_n \to 0$ as $n \to \infty$, which implies

$$D(x_n, x_m, x_m) \to 0 \quad (m, n \to \infty). \tag{3.5}$$

It follows from (d3) that, for any $n, m, l \in \mathbb{N}$,

$$D(x_n, x_m, x_l) \preceq D(x_n, x_m, x_m) + D(x_l, x_m, x_m). \tag{3.6}$$

Thus, by (3.5) and (3.6), we get

$$\lim_{m,n,l\to\infty} D(x_n, x_m, x_l) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. Since (X, D) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Suppose that S is a continuous mapping. then we have that $Sx_{2n} \to Sx^*$ as $n \to \infty$. Since $Sx_{2n} = x_{2n+1} \to x^*$ as $n \to \infty$, we have $Sx^* = x^*$. Now, since $x^* \sqsubseteq x^*$, by (3.1), we have

$$\begin{array}{rcl} D(Sx^*, Tx^*, Tx^*) & \preceq & aD(x^*, x^*, x^*) + bD(x^*, Sx^*, Sx^*) + cD(x^*, Tx^*, Tx^*) \\ & & +dD(x^*, Tx^*, Tx^*) + eD(x^*, Sx^*, Sx^*). \end{array} \tag{3.7}$$

Since $Sx^* = x^*$, (3.7) implies that

$$D(x^*, Tx^*, Tx^*) \preceq (c+d)D(x^*, Tx^*, Tx^*).$$

Since $c + d < 1$, it follows that $Tx^* = x^*$. Hence x^* is a common fixed point of S and T.

The proof is similar when T is a continuous mapping. This completes the proof. □

Corollary 3.2. *Let $(X, \sqsubseteq, D)$ be an ordered complete generalized and normal cone metric space with a cone P with non-empty interior. Let $S : X \to X$ be a self-mapping such that $x \sqsubseteq Sx$ for all $x \in X$. Suppose that there exist $a, b, c, d, e \geq 0$ with $a + b + c + d + e < \frac{1}{4}$ such that*

$$\begin{aligned} D(S^m x, S^n y, S^n y) \preceq\ & aD(x, y, y) + bD(x, S^m x, S^m x) + cD(y, S^n y, S^n y) \\ & + dD(x, S^n x, S^n x) + dD(y, S^m y, S^m y) \end{aligned}$$

for some $m, n \in \mathbb{N}$ with $m \leq n$ and all comparable $x, y \in X$. If S is continuous, then S has a fixed point in X.

Proof. The desired result follows from Theorem 3.1 by putting $S^m = S$ and $S^n = T$. □

Corollary 3.3. *Let $(X, \sqsubseteq, D)$ be an ordered complete generalized and normal cone metric space with a cone P with non-empty interior. Let $S : X \to X$ be a self-mapping such that $x \sqsubseteq Sx$ for all $x \in X$. Suppose that there exist $a, b, c, d, e \geq 0$ with $a + b + c + d + e < \frac{1}{4}$ such that*

$$\begin{aligned} D(Sx, Sy, Sy) \preceq\ & aD(x, y, y) + bD(x, Sx, Sx) + cD(y, Sy, Sy) \\ & + dD(x, Sx, Sx) + dD(y, Sy, Sy) \end{aligned}$$

for all comparable $x, y \in X$. If S is continuous, then S has a fixed point in X.

Proof. The desired result follows from Corollary 3.3 by putting $m = n = 1$. □

Theorem 3.4. *Let $(X, \sqsubseteq, D)$ be an ordered complete generalized and normal cone metric space with a cone P with non-empty interior. Let (S, T) be two self-mappings on X such that there exists a point $x_0 \in X$ with $Tx_0 \sqsubseteq Sx_0$. Suppose that S and T satisfy*

$$D(Sx, Sy, Sy) \preceq kD(Tx, Ty, Ty) \tag{3.8}$$

for all $x, y \in X$ satisfying $Tx \sqsubseteq Ty$, where $k \in (0, 1)$. Assume that the following conditions hold:

(a) *$T(X)$ is closed in X;*

(b) *S is T-nondecreasing and $S(X) \subset T(X)$;*

(c) *if $\{Tx_n\}$ is a nondecreasing sequence and converges to a point Tz, then $Tx_n \sqsubseteq Tz$ and $Tz \sqsubseteq TTz$.*

Then S and T have a coincidence point. Further, if S and T are weakly compatible, then they have a common fixed point in X.

Proof. Let $x_0 \in X$ be an given point. Since $S(X) \subset T(X)$, we can construct a sequence $\{x_n\}$ satisfying $Sx_n = Tx_{n+1}$ with the initial point x_0. Since $Tx_0 \sqsubseteq Sx_0 = Tx_1$ and S is T-nondecreasing, we have

$$Sx_0 \sqsubseteq Sx_1 \sqsubseteq \cdots \sqsubseteq Sx_n \sqsubseteq Sx_{n+1} \sqsubseteq \cdots$$

and

$$Tx_1 \sqsubseteq Tx_2 \sqsubseteq \cdots \sqsubseteq Tx_{n+1} \sqsubseteq Tx_{n+2} \sqsubseteq \cdots .$$

Now, by (3.8), we obtain

$$\begin{aligned} D(Sx_n, Sx_{n+1}, Sx_{n+1}) &\preceq kD(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &= kD(Sx_{n-1}, Sx_n, Sx_n) \end{aligned}$$

and so it follows that

$$D(Sx_n, Sx_{n+1}, Sx_{n+1}) \preceq k^n D(Sx_0, Sx_1, Sx_1),$$

for all $n \geq 1$. For $m, n \in \mathbb{N}$ with $m > n$, by (d3), one has

$$\begin{aligned} D(Sx_n, Sx_m, Sx_m) &\preceq D(Sx_n, Sx_{n+1}, Sx_{n+1}) + D(Sx_{n+1}, Sx_m, Sx_m) \\ &\preceq D(Sx_n, Sx_{n+1}, Sx_{n+1}) + D(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) \\ &\quad + D(Sx_{n+2}, Sx_m, Sx_m) \\ &\preceq D(Sx_n, Sx_{n+1}, Sx_{n+1}) + D(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) \\ &\quad + \cdots + D(Sx_{m-1}, Sx_m, Sx_m) \\ &\preceq (k^n + k^{n+1} + \cdots + k^{m-1}) D(Sx_0, Sx_1, Sx_1) \\ &\preceq \tfrac{k^n}{1-k} D(Sx_0, Sx_1, Sx_1) \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

It follows from (d3) that, for all $n, m, l \in \mathbb{N}$,

$$D(Sx_n, Sx_m, Sx_l) \preceq D(Sx_n, Sx_m, Sx_m) + D(Sx_m, Sx_m, Sx_l) \to 0,$$

as $n, m, l \to \infty$. This shows that $\{Sx_n\}$ is a Cauchy sequence in X and so is $\{Tx_n\}$. Since (X, D) is complete and TX is closed, there exists $x^* \in X$ such that $Tx_n \to Tx^*$ as $n \to \infty$ and, also, $Sx_n \to Tx^*$ as $n \to \infty$.

Next, we prove that $Sx^* = Tx^*$. By (c), we have $Tx_n \sqsubseteq Tx^*$ and further from (3.8) we get

$$D(Sx_n, Sx^*, Sx^*) \preceq kD(Tx_n, Tx^*, Tx^*) \to 0 \quad (n \to \infty).$$

This shows that $Sx_n \to Sx^*$ as $n \to \infty$. Hence $Sx^* = Tx^*$, that is, the point x^* is a coincidence point of two mappings S and T.

Now, suppose that S and T are weakly compatible. Then

$$STx^* = TSx^* = SSx^* = TTx^*.$$

By the assumption (c), $Tx^* \sqsubseteq TTx^*$ and it from (3.8) follows get

$$D(Sx^*, STx^*, STx^*) \preceq kD(Tx^*, TTx^*, TTx^*) = kD(Sx^*, STx^*, STx^*).$$

This shows that $D(Sx^*, STx^*, STx^*) = 0$ and hence $Sx^* = STx^* = SSx^*$, which implies that Sx^* is a fixed point of S. Since $Tx^* = Sx^*$ and $SSx^* = TTx^*$, we have $Tx^* = TTx^*$, which implies that Tx^* is a fixed point of T. Therefore, $Sx^* = Tx^*$ is a common fixed point of S and T. This completes the proof. □

Example 3.5. *Let $X = [0,1]$ and the order relation $\sqsubseteq$ be defined by*

$$x \sqsubseteq y \iff \{(x = y) \text{ or } (x, y \in [0,1] \text{ with } x \leq y)\}.$$

Let $E = (-\infty, \infty)$, $P = [0, \infty)$ and define the mappings $S, T : X \to X$ by

$$S(x) = \frac{1}{2}x, \quad T(x) = x, \quad \forall x \in X.$$

Let $D : X \times X \times X \to E$ be a function as defined in Example 2.3. Take $k \in (\frac{1}{2}, 1)$. It is easy to check that all the conditions on S and T are satisfied. Further, $x^* = 0$ is a common fixed point of S and T.

Corollary 3.6. *Let $(X, \sqsubseteq, D)$ be an ordered complete generalized and normal cone metric space with a cone P with non-empty interior. Let S be a self-mappings on X such that there exists a point $x_0 \in X$ with $x_0 \sqsubseteq Sx_0$. Suppose that the mapping S satisfies the following:*

$$D(Sx, Sy, Sy) \preceq kD(x, y, y)$$

for all $x, y \in X$ satisfying $x \sqsubseteq y$, where $k \in (0,1)$. Assume that the following conditions hold:

(a) *S is nondecreasing;*

(b) *if $\{x_n\} \subset X$ is a nondecreasing sequence and converges to a point $z \in X$, then $x_n \sqsubseteq z$.*

Then S has a fixed point in X.

Proof. The desired result is obtained directly from Theorem 3.4 by setting $T = I$, where I is the identity mapping on X. □

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. anal. Appl. **341** (2008), 416–420.

[2] M. Abbas, B. E. Rhoades and T. Nazir, *Common fixed points for four maps in cone metric spaces*, Appl. Math. Comp. **216** (2010), 80–86.

[3] I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo. **58** (2009), 319–325.

[4] I. Altun, B. Damnjanović, D. Djorić, *Fixed point and common fixed theorems on ordered cone metric space*, Appl. Math. Lett. (2009) doi:10.1016/j.aml.2009.09.16.

[5] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[6] Z. Kadelburg, M. Pavlović and S. Radenović,*Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Comp. Math. Appl. (2010). doi:10.1016/j.camwa.2010.02.039.

[7] D. llić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.

[8] Z. Mustafa, H. Obiedat and F. Awawdeh,*Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory and Applications, vol. 2008, Article ID 189870, 12 pages, 2008.

[9] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces*, Fixed Point Theory and Applications, vol. 2009, Article ID 283028, 10 pages, 2009.

[10] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces* , J. Nonlinear and Convex Anal. **7** (2006), 289–297.

[11] Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory and Applications, vol. 2009, Article ID 917175, 10 pages, 2009.

[12] S. Radenović, Common fixed points under contractive conditions in cone metric spaces, Comput. Math. Appl. 58 (2009) 1273–1278.

[13] S. Rezapour and R. Hamlbarani, *Some note on the paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **345** (2008), 719–724.

[14] S. Radenović and B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. **57** (2009), 1701–1707.