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A NEWTON–TYPE METHOD IN $\mathcal{K}$–NORMED SPACES

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Abstract. We use our new idea of recurrent functions to provide a new semilocal convergence result for a Newton–type method (NTM) for solving a nonlinear operator equation in a $\mathcal{K}$–normed space setting. Using more precise majorizing sequences than before [3], [8], we show how to expand the convergence domain of (NTM) under the same computational cost as before [3], [8]. A numerical examples shows how to solve an equation in cases not covered before.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

where, F, G are defined on a closed ball $\overline{U}(x_0, R)$ centered at some point x_0 of a Banach space $\mathcal{X}$ with $R > 0$, and with values in $\mathcal{X}$. Operator F is differentiable, whereas the differentiability of G is not assumed.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations,

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and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We propose the Newton–type method (NTM)

$$x_{n+1} = x_n - F'(x_n)^{-1}\ (F(x_n) + G(x_n)) \quad (n \geq 0) \tag{1.2}$$

to generate a sequence approximating $x^\star$. A survey of local as well as semilocal convergence results for (NTM), under Lipschitz or Hölder type continuity conditions can be found in [3], [4], [8] (see also [1]–[21]).

This study is motivated by the elegant works in [8], [20], and optimization considerations, where $\mathcal{X}$ is a real Banach space ordered by a closed convex cone $\mathcal{K}$. Note that passing from scalar majorants to vector majorants enlarges the range of applications, since the latter uses the spectral radius which is usually smaller than its norm used by the former.

In [3], Argyros used tighter vector majorants than before [8] and provided under the same hypotheses:

(a) Sufficient convergence conditions which are always weaker than before.
(b) Tighter error bounds on the distances involved, and an at least as precise information on the location of the solution $x^\star$ are provided.

Some applications are also provided in [3]. In particular Argyros showed as a special case that the famous Newton–Kantorovich hypothesis is weakened.

Here, we extend the convergence domain of (NTM) even further than [3], [8] using our new idea of recurrent functions. Numerical examples are also provided to show that our results apply to solve equation but not earlier ones.

2. Semilocal convergence analysis for (NTM)

In order to make the study as self–contained as possible we need to reintroduce some concepts involving $\mathcal{K}$–normed spaces [3], [8], [20].

Let $\mathcal{X}$ be a real Banach space ordered by a closed convex cone $\mathcal{K}$. We say that cone $\mathcal{K}$ is regular if every increasing sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_n \leq \cdots$$

which is bounded above, converges in norm. Moreover, If

$$\lambda_n^0 \leq \lambda_n \leq \lambda_n^1 \quad \text{and} \quad \lim_{n\to\infty} \lambda_n^0 = \lim_{n\to\infty} \lambda_n^1 = \lambda^\star$$

then the regularity of $\mathcal{K}$ implies $\lim\limits_{n\to\infty} \lambda_n = \lambda^\star$.

Let $\alpha, \beta \in \mathcal{X}$, then we define the conic segment $\langle \alpha, \beta \rangle = \{\lambda : \alpha \leq \lambda \leq \beta\}$. An operator Q in $\mathcal{X}$ is called positive if $Q(\lambda) \in \mathcal{K}$ for all $\lambda \in \mathcal{K}$. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{X})$ the space of all bounded linear operators in $\mathcal{X}$, and $\mathcal{L}_{\text{sym}}(\mathcal{X}^2, \mathcal{X})$ the space of bilinear, symmetric, bounded operators from $\mathcal{X}^2$ to $\mathcal{X}$. Using the standard linear isometry between $\mathcal{L}(\mathcal{X}^2, \mathcal{X})$, and $\mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{X}))$, we consider the former embedded into the latter.

Let $\mathcal{D}$ be a linearly connected subset of $\mathcal{K}$, and φ be a continuous operator from $\mathcal{D}$ into $\mathcal{L}(\mathcal{X}, \mathcal{X})$ or $\mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{X}))$. We say that the line integral of φ is independent of the path if for every polygonal line L in $\mathcal{D}$, the line integral depends only on the initial and final point of L. We define

$$\int_{r_0}^{r} \varphi(t)\; dt = \int_0^1 \varphi((1-s)\; r_0 + s\; r)\,(r - r_0)\; ds. \tag{2.1}$$

We need the definition of $\mathcal{K}$–normed space:

Definition 2.1. *Let $\mathcal{X}$ be a real linear space. Then $\mathcal{X}$ is said to be $\mathcal{K}$–normed if operator $\,]\cdot[\,: \mathcal{X} \to \mathcal{X}$ satisfies:*

$$\begin{aligned} &]x[\; \geq 0 \quad (x \in \mathcal{X});\\ &]x[\; = 0 \Leftrightarrow x = 0;\\ &]\mu\, x[\; = |\mu|\;]x[\quad (x \in \mathcal{X}, \mu \in \mathbb{R});\\ &]x + y[\; \leq]x[\; + \;]y[\quad (x, y \in \mathcal{X}). \end{aligned} \tag{2.2}$$

Definition 2.2. *Let $x_0 \in \mathcal{X}$ and $r \in \mathcal{K}$. Then we denote*

$$\overline{U}(x_0, r) = \{x \in \mathcal{X} \;:]x - x_0[\; \leq r\}. \tag{2.3}$$

Using $\mathcal{K}$–norm we can define convergence on $\mathcal{X}$. A sequence $\{y_n\}$ $(n \geq 0)$ in $\mathcal{X}$ is said to be

(a) *convergent to a limit $y \in \mathcal{X}$ if*

$$\lim_{n\to\infty}]y_n - y[\; = 0 \quad \text{in } \mathcal{X} \tag{2.4}$$

and we write

$$(\mathcal{X}) - \lim_{n\to\infty} y_n = y;$$

(b) *a Cauchy sequence if*

$$\lim_{m,n\to\infty} \rbrack y_m - y_n \lbrack = 0.$$

The space $\mathcal{X}$ is complete if every Cauchy sequence is convergent.

We use the following conditions:

F is differentiable on the $\mathcal{K}$–ball $U(x_0, R)$, and for every $r \in \mathcal{S} = \langle 0, R\rangle$, there exist positive operators $\omega_0(r)$, $\overline{\omega}(r) \in \mathcal{L}_{\text{sym}}(\mathcal{X}^2, \mathcal{X})$ such that for all $z \in \mathcal{X}$ and for all $x, y \in \overline{U}(x_0, r)$:

$$\rbrack (F'(x) - F'(x_0))\ (z) \lbrack \leq \omega_0(r)\ (\rbrack x - x_0 \lbrack, \rbrack z \lbrack) \tag{2.5}$$

and

$$\rbrack (F'(x) - F'(y))\ (z) \lbrack \leq \overline{\omega}(r)\ (\rbrack x - y \lbrack, \rbrack z \lbrack) \tag{2.6}$$

where operators ω_0, $\overline{\omega} : \mathcal{S} \to \mathcal{L}_{\text{sym}}(\mathcal{X}^2, \mathcal{X})$ are increasing, with $\omega_0(0) = \overline{\omega}(0) = 0$. Moreover, the line integral of $\overline{\omega}$ (similarly for ω_0) is independent of the path, and the same is true for the operator $\omega : \mathcal{S} \to \mathcal{L}(\mathcal{X}, \mathcal{X})$ given by

$$\omega(r) = \int_0^r \overline{\omega}(t)\ dt. \tag{2.7}$$

Note that in general

$$\omega_0(r) \leq \overline{\omega}(r) \quad \text{for all } r \in \mathcal{S}, \tag{2.8}$$

and $\dfrac{\overline{\omega}}{\omega_0}$ can be arbitrarily large [4].

The Newton–Leibniz formula holds for F on $\overline{U}(x_0, R)$:

$$F(x) - F(y) = \int_x^y F'(z)\ dz, \tag{2.9}$$

for all segments $[x, y] \in \overline{U}(x_0, R)$; for every $r \in \mathcal{S}$ there exists a positive operator $\omega_1(r) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ such that:

$$\rbrack G(x) - G(y) \lbrack \leq \omega_1(r)\ (\rbrack x - y \lbrack) \quad \text{for all } x, y \in \overline{U}(x_0, r), \tag{2.10}$$

where, $\omega_1 : \mathcal{S} \to \mathcal{L}(\mathcal{X}, \mathcal{X})$, is increasing, $\omega_1(0) = 0$, and the line integral of ω_1 is independent of the path;

Operator $F'(x_0)$ is invertible and satisfies:

$$\rbrack F'(x_0)(y) \lbrack \leq b \rbrack y \lbrack \quad \text{for all} \quad y \in \mathcal{X} \tag{2.11}$$

for some positive operator $b \in \mathcal{L}(\mathcal{X}, \mathcal{X})$.

Let

$$\eta = \rbrack F'(x_0)^{-1}\ (F(x_0) + G(x_0)) \lbrack. \tag{2.12}$$

Define operator $f : \mathcal{S} \to \mathcal{X}$ by:

$$f(r) = \eta + b \int_0^r \omega(t)\ dt + b\ \int_0^r \omega_1(t)\ dt. \tag{2.13}$$

Using the monotonicity of operators ω, ω_1, we see that f is order convex, i.e., for all r, $\overline{r} \in \mathcal{S}$, with $r \leq \overline{r}$,

$$f((1-s)r + s\ \overline{r}) \leq (1-s)\ f(r) + s\ f(\overline{r}) \quad \text{for all} \quad s \in [0,1]. \tag{2.14}$$

We will use the following results whose proofs can be found in [8]:

Lemma 2.3. (a) *If Lipschitz condition (2.6) holds then*

$$\big](F'(x+y) - F'(x))\ (z)\big[\leq (\omega(r+\big]y\big[) - \omega(r))(\big]z\big[) \tag{2.15}$$

for all r, $r+\ ]y[\ \in \mathcal{S}$, $x \in \overline{U}(x_0, r)$, $z \in \mathcal{X}$.

(b) *If Lipschitz condition (2.10) holds then*

$$]G(x+y) - G(x)[\ \leq \int_r^{r+]y[} \omega_1(t)\ dt \tag{2.16}$$

for all r, $r+\ ]y[\ \in \mathcal{S},\ x \in \overline{U}(x_0, r)$.

Lemma 2.4. *Denote by* $\mathrm{Fix}(f)$ *the set of all fixed points of the operator* f*, and assume:*

$$\mathrm{Fix}(f) \neq \emptyset. \tag{2.17}$$

Then there is a minimal element $r^\star$ *in* $\mathrm{Fix}(f)$*, which can be found by applying the method of successive approximations*

$$r = f(r) \tag{2.18}$$

with 0 *as the starting point.*

The set

$$B(f, r^\star) = \{r \in \mathcal{S}\ :\ \lim_{n\to\infty} f^n(r) = r^\star\} \tag{2.19}$$

is the attracting zone of $r^\star$.

Remark 2.5. ([8]) Let $r \in \mathcal{S}$. If

$$f(r) \leq r \tag{2.20}$$

and

$$\langle 0, r\rangle \cap \mathrm{Fix}(f) = \{r^\star\} \tag{2.21}$$

then

$$\langle 0, r\rangle \subseteq B(f, r^\star). \tag{2.22}$$

Note that the successive approximations

$$\varepsilon_{n+r} = \delta(\varepsilon_n) \quad (\varepsilon_0 = r) \quad (n \in \mathcal{N}) \tag{2.23}$$

converges to a fixed point $\varepsilon^\star$ of f, satisfying $0 \leq s^\star \leq r$. Hence, we conclude $s^\star = r^\star$, which implies $r \in B(f, r^\star)$.

In particular, we have:

$$\langle 0, (1-s)\ r^\star + s\ r\rangle \subseteq B(f, r^\star) \tag{2.24}$$

for every $r \in \mathrm{Fix}(f)$, with $\langle 0, r\rangle \cap \mathrm{Fix}(f) = \{r^\star, r\}$, and for all $\lambda \in [0,1)$.

In the scalar case $\mathcal{X} = \mathbb{R}$, we have

$$B(f, r^\star) = [0, r^\star] \cup \{r \in \mathcal{S}\ :\ r^\star < r, f(q) < q,\ (r^\star < q \leq r)\}. \tag{2.25}$$

We will also use the notation

$$E(r^\star) = \bigcup_{r \in B(f, r^\star)} \overline{U}(x_0, r). \tag{2.26}$$

Returning back to method (1.2), we consider the sequences of approximations

$$r_{n+1} = r_n - (b\ \omega_0(r_n) - \mathcal{I})^{-1}\ (f(r_n) - r_n) \quad (r_0 = 0,\ n \geq 0) \tag{2.27}$$

and

$$\overline{r}_{n+1} = \overline{r}_n - (b\ \omega(\overline{r}_n) - \mathcal{I})^{-1}\ (f(\overline{r}_n) - \overline{r}_n) \quad (\overline{r}_0 = 0,\ n \geq 0) \tag{2.28}$$

for the majorant equation (2.18).

Lemma 2.6. ([3]) *If operators*

$$\mathcal{I} - b\ \omega_0(r), \quad r \in [0, r^\star) \tag{2.29}$$

are invertible with positive inverses, then sequence $\{r_n\}$ $(n \geq 0)$ given by (2.27) is well defined for all $n \geq 0$, monotonically increasing and convergent to $r^\star$.

Remark 2.7. If equality holds in (2.8), then sequence $\{\overline{r}_n\}$ becomes $\{r_n\}$ $(n \geq 0)$ and Lemma 2.6 reduces to [8, Lemma 3, p. 555].

Moreover as it can easily be seen using induction on n

$$r_{n+1} - r_n \leq \overline{r}_{n+1} - \overline{r}_n \tag{2.30}$$

and

$$r_n \leq \overline{r}_n \tag{2.31}$$

for all $n \geq 0$. Furthermore if strict inequality holds in (2.8) so does in (2.30) and (2.31). If $\{r_n\}$ $(n \geq 0)$ is a majorizing sequence for method (1.2), then (2.30) shows that the error bounds on the distances $\parallel x_{n+1} - x_n \parallel$ are tighter. It turns out that this is indeed the case.

We can show the semilocal convergence theorem for method (1.2).

Theorem 2.8. *Assume hypotheses (2.6), (2.7), (2.9)–(2.11), (2.17) hold, and operators (2.29) are invertible with positive inverses.*

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by (NTM) is well defined, remains in the $\mathcal{K}$–ball $U(x_0, r^\star)$ for all $n \geq 0$, and converges to a solution $x^\star$ of equation (1.1) in $E(r^\star)$, where $E(r^\star)$ is given by (2.26).

Moreover the following error bounds hold for all $n \geq 0$:

$$]x_{n+1} - x_n[\; \leq r_{n+1} - r_n, \tag{2.32}$$

and

$$]x^\star - x_n[\; \leq r^\star - r_n, \tag{2.33}$$

where sequence $\{r_n\}$ is given by (2.27).

Proposition 2.9. *We first show (2.32) using induction on $n \geq 0$ (by (2.12)). For $n = 0$;*

$$]x_1 - x_0[\; = \;]F'(x_0)^{-1}(F(x_0) + G(x_0))[\; = \eta = r_1 - r_0. \tag{2.34}$$

Assume:

$$]x_k - x_{k-1}[\; \leq r_k - r_{k-1}, \qquad k = 1, 2, \cdots, n. \tag{2.35}$$

Using (2.35) we get

$$]x_n - x_0[\; \leq \sum_{k=1}^{n}]x_k - x_{k-1}[\; \leq \sum_{k=1}^{n} (r_k - r_{k-1}) = r_n. \tag{2.36}$$

Define operators $Q_n : \mathcal{X} \to \mathcal{X}$ by

$$Q_n = -F'(x_0)^{-1}(F'(x_n) - F'(x_0)). \tag{2.37}$$

By (2.5) and (2.11) we get

$$\begin{aligned}]Q_n(z)[\; &= \;]F'(x_0)^{-1}(F'(x_n) - F'(x_0))(z)[\\ &\leq \; b](F'(x_n) - F'(x_0))(z)[\leq b\ \omega_0(r_n)(]z[), \end{aligned} \tag{2.38}$$

and

$$]Q_n^i(z)[\; \leq (b\ \omega_0(r_n))^i(]z[) \quad (i \geq 1). \tag{2.39}$$

Hence

$$\sum_{i=0}^{\infty}]Q_n^i(z)[\; \leq \sum_{j=0}^{\infty} (b\ \omega_0(r_n))^i\ (]z[). \tag{2.40}$$

That is, series $\displaystyle\sum_{i=0}^{\infty} Q_n^i(z)$ is convergent in $\mathcal{X}$. Hence operator $\mathcal{I} - Q_n$ is invertible, and

$$](\mathcal{I} - Q_n)^{-1}(z)[\; \leq (\mathcal{I} - b\ \omega_0(r_n))^{-1}\ (]z[). \tag{2.41}$$

Operator $F'(x_n)$ is invertible for all $n \geq 0$, since $F'(x_n) = F'(x_0)\ (\mathcal{I} - Q_n)$, and for all $x \in \mathcal{X}$ we have:

$$\begin{aligned}]F'(x_n)^{-1}(x)[\; &= \;](\mathcal{I} - Q_n)^{-1}\ F'(x_0)^{-1}(x)[\\ &\leq \; (\mathcal{I} - b\ \omega_0(r_n))^{-1}(]F'(x_0)^{-1}(x)[) \\ &\leq \; (\mathcal{I} - b\ \omega_0(r_n))^{-1}(b\ ]x[). \end{aligned} \tag{2.42}$$

Using (2.3) we obtain the approximation

$$\begin{aligned} &]x_{n+1} - x_n[\\ &= \ ]F'(x_n)^{-1} \ (F(x_n) + G(x_n)) \\ &\qquad - F'(x_n)^{-1} \ (F'(x_{n-1}) \ (x_n - x_{n-1}) + F(x_{n-1}) + G(x_{n-1}))[. \end{aligned} \tag{2.43}$$

It now follows from (2.5)–(2.11), (2.13), (2.27) and (2.43)

$$\begin{aligned} &]x_{n+1} - x_n[\\ &\leq]F'(x_n)^{-1}(F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})[\\ &\quad +]F'(x_n)^{-1}(G(x_n) - G(x_{n-1})) \\ &\leq (\mathcal{I} - b\omega_0(r_n))^{-1} \\ &\quad \times \left\{ b \left] \int_0^1 (F'((1-\lambda)x_{n-1} + \lambda x_n) - F'(x_{n-1})) \ (x_n - x_{n-1}) d_\lambda \right[\right\} \\ &\quad + (\mathcal{I} - b \ \omega_0(r_n))^{-1} \ (b \ ]G(x_n) - G(x_{n-1})[) \\ &\leq (\mathcal{I} - b \, \omega_0(r_n))^{-1} \left\{ b \int_0^1 (\omega((1-\lambda) r_{n-1} + \lambda r_n) \right. \\ &\quad \left. - \omega(r_{n-1}))(r_n - r_{n-1}) d_\lambda \right\} + (\mathcal{I} - b \ \omega_0(r_n))^{-1} \left(b \ \int_{r_{n-1}}^{r_n} \omega_1(t) \ dt \right) \\ &= (\mathcal{I} - b \ \omega_0(r_n))^{-1} \left\{ b \ \int_{r_{n-1}}^{r_n} \omega(t) \ dt - b \ \omega(r_{n-1})(r_n - r_{n-1}) \right. \\ &\quad \left. + \ b \ \int_{r_{n-1}}^{r_n} \omega_1(t) \ dt \right\} \\ &= (\mathcal{I} - b \ \omega_0(r_n))^{-1}(f(r_n) - f(r_{n-1}) - b \ \omega(r_{n-1})(r_n - r_{n-1})) \\ &= (\mathcal{I} - b \ \omega_0(r_n))^{-1}((f(r_n) - r_n) - (f(r_{n-1}) - r_{n-1}) \\ &\quad - (b \ \omega(r_{n-1}) - \mathcal{I}) \ (r_n - r_{n-1})) \\ &= (\mathcal{I} - b \ \omega_0(r_n))^{-1} \ ((f(r_n) - r_n) - (f(r_{n-1}) - r_{n-1}) \\ &\quad - (b \ \omega(r_{n-1}) - \mathcal{I})(r_n - r_{n-1})) \\ &\leq (\mathcal{I} - b \ \omega_0(r_n))^{-1}((f(r_n) - r_n) - (f(r_{n-1}) - r_{n-1}) \\ &\quad - (b \ \omega_0(r_{n-1}) - \mathcal{I})(r_n - r_{n-1})) \\ &= (\mathcal{I} - b \ \omega_0(r_n))^{-1}(f(r_n) - r_n) \\ &= r_{n+1} - r_n. \end{aligned} \tag{2.44}$$

By Lemma 2.5, sequence $\{r_n\}$ $(n \geq 0)$ converges to $r^\star$. Hence $\{x_n\}$ is a convergent sequence, and its limit is a solution of equation (1.1). Therefore x_n converges to $x^\star$.

Finally (2.33) follows from (2.32) by using standard majorization techniques ([4], [10]). The uniqueness part is omitted since it follows exactly as in [8, Theorem 2].

This completes the proof of Theorem 2.8.

Remark 2.10. It follows immediately from (2.44) that sequence

$$\begin{aligned} & t_0 = t_0, \quad t_1 = \eta, \\ & t_{n+1} - t_n = (\mathcal{I} - b\,\omega_0(t_n))^{-1} \Big\{ b \int_{t_{n-1}}^{t_n} \omega(t)\,dt - b\,\omega(t_{n-1})(t_n - t_{n-1}) \\ & \qquad + b \int_{t_{n-1}}^{t_n} \omega_1(t) dt \Big\} \quad (n \geq 1) \end{aligned} \tag{2.45}$$

is also a tighter majorizing sequence of $\{x_n\}$ $(n \geq 0)$ and converges to some $t^\star$ in $\langle 0, r^\star \rangle$.

The proof of Theorem 2.8 was also essentially given in [3], but the iteration (2.45) uses $\omega_1(t_n)\,(t_n - t_{n-1})$ instead of $\int_{t_{n-1}}^{t_n} \omega_1(t)\ dt$.

Moreover the following hold for all $n \geq 0$

$$]x_1 - x_0[\leq t_1 - t_0 = r_1 - r_0, \tag{2.46}$$

$$]x_{n+1} - x_n[\leq t_{n+1} - t_n \leq r_{n+1} - r_n, \tag{2.47}$$

$$]x^\star - x_n[\leq t^\star - t_n \leq r^\star - r_n, \tag{2.48}$$

$$t_n \leq r_n, \tag{2.49}$$

and

$$t^\star \leq r^\star. \tag{2.50}$$

That is, $\{t_n\}$ is a tighter majorizing sequence than $\{r_n\}$ and the information on the location of the solution $x^\star$ is more precise. Therefore, Argyros [3] remarks that if studying the convergence of $\{t_n\}$ without assuming (2.17) can lead to weaker sufficient convergence conditions for (NTM). In Theorem 2.8, Argyros responds to this question.

We need the following definition of some operators.

Definition 2.11. Define operators:

$$f_n,\, h_n,\, p_n \ :\ [0,1) \longrightarrow \mathcal{X}$$

and

$$q\ :\ I_q = \left[1, \frac{1}{1-\gamma}\right] \times [0,1)^4 \longrightarrow \mathcal{X}, \quad \gamma \in [0,1)$$

by

$$\begin{aligned} f_n(\gamma) = b\, \Big\{ & \int_0^1 \left(\omega\Big((\frac{1-\gamma^{n-1}}{1-\gamma} + t\, \gamma^{n-1})\, \eta \Big) - \omega\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\ \eta \Big) \right) dt \\ & + \omega_1\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\ \eta \Big) + \gamma\, \omega_0\Big(\frac{1-\gamma^{n}}{1-\gamma}\ \eta \Big) \Big\} - \gamma, \end{aligned} \tag{2.51}$$

$$
\begin{aligned}
& h_n(\gamma) \\
& = b\Big\{ \int_0^1 \Big(\omega\Big((\frac{1-\gamma^n}{1-\gamma}+t\,\gamma^n)\,\eta\Big)-\omega\Big((\frac{1-\gamma^{n-1}}{1-\gamma}+t\,\gamma^{n-1})\,\eta\Big)\Big)\,dt \\
& \quad +\Big(\omega\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\eta\Big)-\omega\Big(\frac{1-\gamma^n}{1-\gamma}\eta\Big)\Big) \\
& \quad +\Big(\omega_1\Big(\frac{1-\gamma^n}{1-\gamma}\,\eta\Big)-\omega_1\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\eta\Big)\Big) \\
& \quad +\gamma\,\Big(\omega_0\Big(\frac{1-\gamma^{n+1}}{1-\gamma}\;\eta\Big)-\omega_0\Big(\frac{1-\gamma^n}{1-\gamma}\;\eta\Big)\Big)\Big\},
\end{aligned} \tag{2.52}
$$

$$
\begin{aligned}
\overline{p_n}(\gamma) \;=\; & \int_0^1 \Big(\omega\Big((\frac{1-\gamma^{n+1}}{1-\gamma}+t\;\gamma^{n+1})\;\eta\Big)+\omega\Big((\frac{1-\gamma^{n-1}}{1-\gamma}+t\;\gamma^{n-1})\;\eta\Big) \\
& -2\;\omega\Big((\frac{1-\gamma^n}{1-\gamma}+t\;\gamma^n)\;\eta\Big)\Big)\;dt \\
& +\Big(2\;\omega\Big(\frac{1-\gamma^n}{1-\gamma}\;\eta\Big)-\omega\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\;\eta\Big)-\omega\Big(\frac{1-\gamma^{n+1}}{1-\gamma}\;\eta\Big)\Big) \\
& +\Big(\omega_1\Big(\frac{1-\gamma^{n+1}}{1-\gamma}\;\eta\Big)+\omega_1\Big(\frac{1-\gamma^{n-1}}{1-\gamma}\;\eta\Big)-2\;\omega_1\Big(\frac{1-\gamma^n}{1-\gamma}\;\eta\Big)\Big) \\
& +\gamma\;\Big(\omega_0\Big(\frac{1-\gamma^{n+2}}{1-\gamma}\;\eta\Big)+\omega_0\Big(\frac{1-\gamma^n}{1-\gamma}\;\eta\Big)-2\;\omega_0\Big(\frac{1-\gamma^{n+1}}{1-\gamma}\;\eta\Big)\Big),
\end{aligned}
$$

$$
p_n(\gamma)=b\;\overline{p_n}(\gamma), \tag{2.53}
$$

$$
\begin{aligned}
& \overline{q}(v_1,v_2,v_3,v_4,\gamma) \\
= \; & \int_0^1 \Big(\omega((v_1+v_2+v_3+t\,v_4)\;\eta)+\omega((v_1+t\;v_2)\;\eta) \\
& -2\;\omega((v_1+v_2+t\;v_3)\;\eta)\Big)\;dt \\
& +\Big(2\;\omega((v_1+v_2)\;\eta)-\omega(v_1\;\eta)-\omega((v_1+v_2+v_3)\;\eta)\Big) \\
& +\Big(\omega_1((v_1+v_2+v_3)\;\eta)+\omega_1(v_1\;\eta)-2\;\omega_1((v_1+v_2)\;\eta)\Big) \\
& +\gamma\;\Big(\omega_0((v_1+v_2+v_3+v_4)\,\eta)+\omega_0((v_1+v_2)\,\eta) \\
& -2\;\omega_0((v_1+v_2+v_3)\;\eta)\Big),
\end{aligned}
$$

$$
q(v_1,v_2,v_3,v_4,\gamma)=b\;\overline{q}(v_1,v_2,v_3,v_4,\gamma), \tag{2.54}
$$

where, η is given by (2.12). Moreover, define function $f_\infty : [0,1) \longrightarrow \mathcal{X}$ by

$$f_\infty(\gamma) = \lim_{n \longrightarrow \infty} f_n(\gamma). \tag{2.55}$$

It then follows from (2.51), and (2.55) that

$$f_\infty(\gamma) = b\left(\omega(\frac{\eta}{1-\gamma}) + \gamma\ \omega_0(\frac{\eta}{1-\gamma})\right) - \gamma. \tag{2.56}$$

It can also easily be seen from (2.51)–(2.54) that the following identities hold:

$$f_{n+1}(\gamma) = f_n(\gamma) + h_n(\gamma), \tag{2.57}$$

$$h_{n+1}(\gamma) = h_n(\gamma) + p_n(\gamma), \tag{2.58}$$

and for

$$v_1 = \sum_{i=0}^{n-2} \gamma^i, \quad v_2 = \gamma^{n-1}, \quad v_3 = \gamma^n, \quad v_4 = \gamma^{n+1}, \tag{2.59}$$

we have

$$q(v_1, v_2, v_3, v_4, \gamma) = p_n(\gamma). \tag{2.60}$$

We need the following result on majorizing sequences for (NTM).

Lemma 2.12. *Assume:*

Operator $\mathcal{I} - b\ \omega_0(\eta)$ is positive, invertible, and with a positive inverse; there exists $\alpha \in (0,1)$, such that:

$$\frac{\eta}{1-\alpha} \le R; \tag{2.61}$$

$$0 \le \left(\mathcal{I} - b\ \omega_0(\eta)\right)^{-1} b\ \left(\int_0^1 \omega(t\ \eta)\ dt - \omega(0) + \omega_1(0)\right) \le \alpha\ \mathcal{I}; \tag{2.62}$$

$$q(v_1, v_2, v_3, v_4, \gamma) \ge 0 \quad \text{on} \quad I_q, \tag{2.63}$$

$$h_1(\alpha) \ge 0, \tag{2.64}$$

and

$$f_\infty(\alpha) \le 0, \tag{2.65}$$

where, 0 and $\mathcal{I}$ is the zero endomorphism and the identity opertor on $\mathcal{X}$, respectively. Then iteration $\{t_n\}$ $(n \ge 0)$ given by (2.45) is non–decreasing, bounded from above by

$$t^{\star\star} = \frac{\eta}{1-\alpha}, \tag{2.66}$$

and converges to its unique least upper bound $t^\star$ satisfying

$$t^\star \in \langle 0, t^{\star\star} \rangle. \tag{2.67}$$

Moreover the following error bounds hold for all $n \ge 0$:

$$0 \le t_{n+1} - t_n \le \alpha\ (t_n - t_{n-1}) \le \alpha^n\ \eta, \tag{2.68}$$

and

$$t^\star - t_n \leq \frac{\eta}{1-\alpha}\ \alpha^n. \tag{2.69}$$

Proposition 2.13. *Estimate (2.68) is true, if*

$$\begin{aligned} & 0 \\ & \leq \Big(\mathcal{I} - b\omega_0(\eta)\Big)^{-1} b\Big(\int_0^1 \omega(t_{n-1} + t(t_n - t_{n-1}))dt - \omega(t_{n-1}) + \omega_1(t_{n-1})\Big) \\ & \leq \alpha\mathcal{I} \end{aligned} \tag{2.70}$$

hold for all $n \geq 1$.

In view of (2.62), and (2.66), estimate (2.70) holds for $n = 1$. We also have by (2.45), and (2.70) that

$$0 \leq t_2 - t_1 \leq \alpha\ (t_1 - t_0).$$

Let us assume that (2.68), and (2.70) hold for all $k \leq n$. Then, we have

$$t_n \leq \frac{1-\alpha^n}{1-\alpha}\ \eta. \tag{2.71}$$

Using the induction hypotheses, and (2.70), we have by Lemma 2.4 that $(\mathcal{I} - b\ \omega_0(t_n))^{-1}$ exists, and is positive. Moreover, (2.68) and (2.70) shall hold if

$$\begin{aligned} & b\left\{\int_0^1 \left(\omega\Big(\big(\frac{1-\alpha^{n-1}}{1-\alpha} + t\ \alpha^{n-1}\big)\ \eta\Big) - \omega\Big(\frac{1-\alpha^{n-1}}{1-\alpha}\ \eta\Big)\right) dt \right. \\ & \left. + \omega_1\Big(\frac{1-\alpha^{n-1}}{1-\alpha}\ \eta\Big) + \alpha\ \omega_0\Big(\frac{1-\alpha^n}{1-\alpha}\ \eta\Big)\right\} - \alpha \leq 0. \end{aligned} \tag{2.72}$$

Estimate (2.72) motivates us to define functions f_n (for $\gamma = \alpha$), and show instead

$$f_n(\alpha) \leq 0. \tag{2.73}$$

We have by (2.57)–(2.60), (2.63) and (2.64) that

$$f_{n+1}(\alpha) \geq f_n(\alpha). \tag{2.74}$$

In view of (2.55) and (2.74), estimate (2.73) shall hold, if (2.65) is true. The induction is completed. It follows that iteration $\{t_n\}$ is non–decreasing, bounded from above by $t^{\star\star}$ (given by (2.66)), and as such it converges to $t^\star$ satisfying (2.67).

Finally, estimate (2.69) follows from (2.68) by using standard majorizing techniques ([4], [10]). That completes the proof of Lemma 2.12.

We also state a result from [3], so we can compare with Lemma 2.12.

Lemma 2.14. ([3]) *Assume there exist parameters $\eta \geq 0$, $\delta \in [0, 2)$ such that*

(I) *Operators*

$$\mathcal{I} - b\,\omega_0\left(2\,(2\,\mathcal{I} - \delta\,\mathcal{I})^{-1}\left(\mathcal{I} - \left(\frac{\delta\mathcal{I}}{2}\right)^{n+1}\right)\eta\right) \tag{2.75}$$

be positive, invertible, and with positive inverses for all $n \geq 0$;

(II)

$$2(\mathcal{I} - b\,\omega_0(\eta))^{-1}\left(b\,\omega_1(\eta) + b\int_0^1 \omega(s\eta)\,ds - b\,\omega(0)\right) \leq \delta\,\mathcal{I}; \tag{2.76}$$

(III)

$$\begin{aligned}
&2\,b\int_0^1 \omega\left(2\,(2\mathcal{I} - \delta\,\mathcal{I})^{-1}\left(\mathcal{I} - \left(\frac{\delta\,\mathcal{I}}{2}\right)^{n+1}\right)\eta + s\,\left(\frac{\delta\,\mathcal{I}}{2}\right)^{n+1}\eta\right)\,ds\\
&-2\,b\,\omega\left(2\,(2\mathcal{I} - \delta\,\mathcal{I})^{-1}\left(\mathcal{I} - \left(\frac{\delta\,\mathcal{I}}{2}\right)^{n+1}\right)\eta\right)\\
&+2\,b\,\omega_1\left(2\,(2\mathcal{I} - \delta\,\mathcal{I})^{-1}\left(\mathcal{I} - \left(\frac{\delta\,\mathcal{I}}{2}\right)^{n+1}\eta\right)\right)\\
&+\delta\,b\,\omega_0\left(2\,(2\mathcal{I} - \delta\mathcal{I})^{-1}\left(\mathcal{I} - \left(\frac{\delta\,\mathcal{I}}{2}\right)^{n+1}\right)\eta\right)\\
&\leq 2\,b\int_0^1 \omega(s\,\eta)\,ds - 2\,b\,\omega(0) + 2\,b\,\omega_1(\eta) + \delta\,b\,\omega_0(\eta),
\end{aligned} \tag{2.77}$$

for all $n \geq 0$. Then iteration $\{t_n\}$ $(n \geq 0)$ given by (2.45) is non–decreasing, bounded above by

$$t^{\star\star} = 2\,(2\mathcal{I} - \delta\,\mathcal{I})^{-1}\,\eta, \tag{2.78}$$

converges to some $t^\star$, such that

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.79}$$

Moreover the following error bounds hold for all $n \geq 0$:

$$0 \leq t_{n+2} - t_{n+1} \leq \frac{\delta\,\mathcal{I}}{2}\,(t_{n+1} - t_n) \leq \left(\frac{\delta\mathcal{I}}{2}\right)^{n+1}\eta. \tag{2.80}$$

We can show the main semilocal convergence theorem for (NTM).

Theorem 2.15. *Assume:*

hypotheses (2.5)–(2.7), (2.9)–(2.11), hypotheses of Lemma 2.12, (2.75)–(2.77) hold, and

$$t^{\star\star} \leq R, \tag{2.81}$$

where $t^{\star\star}$ is given by (2.66).

Then sequence $\{x_n\}$ $(n \geq)$ *generated by (NTM) is well defined, remains in the* $\mathcal{K}$*–ball* $U(x_0, t^\star)$ *for all* $n \geq 0$ *and converges to a solution* $x^\star$ *of equation (1.1), which is unique in* $E(t^\star)$*.*

Moreover the following error bounds hold for all $n \geq 0$*:*

$$]x_{n+1} - x_n[\ \leq t_{n+1} - t_n \tag{2.82}$$

and

$$]x^\star - x_n[\ \leq t^\star - t_n, \tag{2.83}$$

where sequence $\{t_n\}$ $(n \geq 0)$ *and* $t^\star$ *are given by (2.45) and (2.79) respectively.*

Proposition 2.16. *The proof is identical to Theorem 2.8 with sequence* t_n *replacing* r_n *until the derivation of (2.45). But then the right hand side of (54) with these changes becomes* $t_{n+1} - t_n$*. By Lemma 2.14,* $\{t_n\}$ *converges to* $t^\star$*. Hence* $\{x_n\}$ *is a convergent sequence, its limit converges to a solution of equation (1.1). Therefore,* $\{x_n\}$ *converges to* $x^\star$*. Estimate (2.83) follows from (2.82) by using standard majorization techniques (*[4], [10]*). The uniqueness part is omitted since it follows exactly as in* [8, Theorem 2]. *That completes the proof of Theorem 2.15.*

3. Special cases and applications

Remark 3.1. The hypotheses of Lemma 2.12 are easier to verify than Lemma 2.14.

Application 3.2. *Assume operator* $]\ [$ *is given by a norm* $\| \cdot \|$*, and set* $G(x) = 0$ *for all* $x \in \overline{U}(x_0, R)$*. Choose for all* $r \in \mathcal{S}$*:*

$$\overline{\omega}(r) = \ell\ r, \tag{3.1}$$

$$\omega_0(r) = \ell_0\ r \tag{3.2}$$

and

$$\omega_1(r) = 0. \tag{3.3}$$

That is, we are consider Lipschitz and center–Lipschitz conditions of the form:

$$\| F'(x) - F'(y) \| \leq \ell\ \ \| x - y \|, \tag{3.4}$$

and

$$\| F'(x) - F'(x_0) \| \leq \ell_0\ \ \| x - x_0 \|, \tag{3.5}$$

for all $x,\ y \in \overline{U}(x_0, R)$*.*

Remark 3.3. Let $\mathcal{X} = \mathbb{R}$, $x_0 = 0$, and define function F on $\mathcal{X}$ by

$$F(x) = c_0 x + c_1 + c_2\ \sin e^{c_3 x}, \tag{3.6}$$

where c_i, $i = 0, 1, 2, 3$ are given parameters. It can easily be seen by (3.4), (3.5) and (3.6) that for c_3 large and c_2 sufficiently small $\frac{\ell}{\ell_0}$ can be arbitrarily large.

Remark 3.4. The sufficient convergence condition in [8] using (2.13) resuces to the famous for its simplicity and clarity Newton–Kantorovich hypotheses for solving nonlinear equation [10]:

$$h_K = \beta\ \ell\ \eta \le \frac{1}{2}. \tag{3.7}$$

Moreover, the conditions of Lemma 2.14 becomes for $\delta = 1$ [3]:

$$h_A = \beta\ \frac{\ell + \ell_0}{2}\ \eta \le \frac{1}{2}. \tag{3.8}$$

Furthemore, the conditions of Lemma 2.12 give for

$$\alpha = \frac{4\ \ell}{\ell + \sqrt{\ell^2 + 8\ \ell_0\ \ell}}, \tag{3.9}$$

$$h_{AH} = \beta\ \overline{\ell}\ \eta \le \frac{1}{2}, \tag{3.10}$$

where,

$$\overline{\ell} = \frac{1}{8}\ (\ell + 4\ \ell_0 + \sqrt{\ell^2 + 8\ \ell_0\ \ell}). \tag{3.11}$$

It follows that

$$h_K \le \frac{1}{2} \Longrightarrow h_A \le \frac{1}{2} \Longrightarrow h_{AH} \le \frac{1}{2} \tag{3.12}$$

but not vice versa unless if $\ell_0 = \ell$.

Hence, we have expanded the applicability of (NM) under the same computational cost as in [3], [8].

Note that in practice the computation of ℓ requires that of ℓ_0. Hence, (3.5) is not an additional hypothesis.

Example 3.5. Let $\mathcal{X} = \mathbb{R}^2$, be equipped with the max–norm, and

$$x_0 = (1,1)^T, \quad \mathcal{D} = \{x : \parallel x - x_0 \parallel \le 1 - a\}, \quad a \in \left[0, \frac{1}{2}\right).$$

Define function F on $\mathcal{D}$ by

$$F(x) = (\xi_1^3 - a,\ \xi_2^3 - a), \qquad x = (\xi_1, \xi_2)^T.$$

The Fréchet–derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\ \xi_1^2 & 0 \\ 0 & 3\ \xi_2^2 \end{bmatrix}.$$

Using (3.7), we obtain:

$$h_K = \frac{2}{3}\ (1-a)\ (2-a) > \frac{1}{2} \quad \text{for all} \quad a \in \left[0, \frac{1}{2}\right).$$

That is no guarantee that (NM) converges to the solution $x^\star = (\sqrt[3]{a}, \sqrt[3]{a})^T$ of equation $F(x) = 0$, starting at x_0.

However from (3.8), we get:

$$h_A = \frac{1}{6}\ (1-a)\ (3-a+2\ (2-a)) \le \frac{1}{2} \quad \text{for all} \quad a \in I_A = \left[\frac{5-\sqrt{13}}{2}, \frac{1}{2}\right)$$

which improves (3.7).

Finally, by (3.10), we get that

$$h_A \le \frac{1}{2} \quad \text{for all} \quad a \in I_{AH} = \left[.4500339002, \frac{1}{2}\right) \supseteq I_A.$$

Remark 3.6. The results obtained here hold under even weaker conditions. Indeed, since (2.6) is not "directly" used in the proofs above, it can be replaced by the weaker condition (2.15) throughout this study. As we showed in Lemma 2.3

$$(2.6) \Longrightarrow (2.15)$$

but not necessarily vice versa unless if operator ω is convex [21, p. 674].

Conclusion

Using our new idea of recurrent functions, and a combination of Lipschitz/center–Lipschitz conditions, we provided a semilocal convergence analysis for (NTM) to approximate a locally unique solution of nonlinear equations in $\mathcal{K}$–normed space. In the particular case of Newton's method, our analysis has the following advantages over the work in [8]: weaker sufficient convergence conditions, and larger convergence domain. Numerical examples further validating the results are also provided in this study.

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