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STRONG CONVERGENCE THEOREMS FOR GENERALIZED EQUILIBRIUM PROBLEMS AND STRICT PSEUDO-CONTRACTIONS

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Abstract. Let $\{S_i\}_{i=1}^N$ be N strict pseudo-contractions defined on a closed convex subset C of a real Hilbert space H. Consider the problem of finding a common element of the fixed point set of these mappings and the solution set of generalized equilibrium problems by parallel and cyclic algorithms. In this paper, we propose new iterative schemes for solving this problem and prove these schemes converge strongly by monotone hybrid methods.

1. Introduction

Let H be a real Hilbert Space with the inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

Recall that a mapping $S : C \to C$ is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \ \forall x, y \in C.$$

Clearly, the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mapping S on C such that

$$\|Sx - Sy\| \leq \|x - y\|, \ \forall x, y \in C.$$

That is, S is nonexpansive if and only if S is a $0-$strict pseudo-contraction.

In this paper, we use $F(S)$ to denote the fixed point set of S (i.e., $F(S) = \{x \in C : Sx = x\}$).

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A nonlinear mapping $A : C \to H$ is said to be *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C.$$

Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \ \geq 0, \forall v \in C. \tag{1.1}$$

Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the relation:

$$\langle x - y, y - z\rangle \geq 0 \ \text{ for all } \ z \in C,$$

where P_C denotes the metric projection from H onto C. From the above we see that $u \in C$ is a solution to problem (1.1) if and only if u satisfies the following equation:

$$u = P_C(u - \rho Au), \tag{1.2}$$

where $\rho > 0$ is a constant. This implies that problem (1.1) and (1.2) are equivalent. This alternative formula is very important from the numerical analysis point of view.

Let $A : C \to H$ be an α-inverse strongly monotone mapping, F a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } \ F(x, y) + \langle Ax, y - x\rangle \ \geq 0, \ \forall y \in C. \tag{1.3}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$.

Next, we give two special cases of problem (1.3).

(i) if $A \equiv 0$, then problem (1.3) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } \ F(x, y) \geq 0, \ \forall y \in C. \tag{1.4}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$.

(ii) if $F \equiv 0$, then problem (1.3) is reduced to the variational inequality problem (1.1).

Problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini-max problems, Nash equilibrium problem in noncooperative games and others; see, for instance, [2, 4, 5, 6].

For solving the equilibrium problem, let us assume that the bi-function F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e.$F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) $F(x, \cdot)$ is convex and lower semicontionuous for each $x \in C$.

Recently, Takahashi and Takahashi [9] considered the problem (1.3) by an iterative method. To be more precise, they proved the following theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bi-function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap EP(F, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)z_n],\ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$, and $\lambda_n \in [0, 2\alpha]$, satisfy

$$0 < c \leq \beta_n \leq d < 1,\ 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0,\ \lim_{n\to\infty}\alpha_n = 0,\ \textit{and} \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to $z = P_{F(S)\cap EP(F,A)}u$.

Very recently, Qin, Kang and Cho [8] further considered the generalized equilibrium problem (1.3). They obtained the following result in a real Hilbert space.

Theorem 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and f a contraction with the $\tau \in [0, 1)$ of C into itself. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), respectively. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1 - k)Tx$, $\forall x \in C$. Assume that $F = EP(F_1, A) \cap EP(F_2, B) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \forall v \in C, \\ y_n = \gamma_n u_n + (1 - \gamma_n)v_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)y_n], \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$, $r \in (0, 2\alpha)$ and $s \in (0, 2\beta)$. If the above control sequences satisfy the following restrictions

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
(c) $\lim_{n\to\infty}\gamma_n = \gamma \in (0, 1)$,

then $\{x_n\}$ will converge strongly to $z \in F$, where $z = P_F u$.

In 2008, Takahashi et al. ([10] Theorem 4.1) proved the following theorem by a new hybrid method.

Theorem 1.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in N, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in N$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In this paper, motivated by [1, 8, 9, 10], applying parallel and cyclic algorithms, we obtain strong convergence theorems for finding a common element of the fixed point set of a finite family of strict pseudo-contractions and the solution set of the problem (1.3) by the monotone hybrid methods.

We will use the notations:

1. $\rightharpoonup$ for the weak convergence and $\to$ for the strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega-$limit set of $\{x_n\}$.

2. Preliminaries

We need some facts and tools in a real Hilbert space H which are listed below.

Lemma 2.1. *Let H be a real Hilbert space. Then the following identities hold.*
(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \forall x, y \in H$.
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2, \forall t \in [0,1], \forall x, y \in H$.

Lemma 2.2. [7] *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ is a sequence in H and $u \in H$. Let $q = P_C u$. Suppose $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \text{ for all } n.$$

Then $x_n \to q$.

Lemma 2.3. [3] *Let $S : C \to H$ be a κ-strict pseudo-contraction. Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.*

Proposition 2.4. [1] *Assume C is a nonempty closed convex subset of a real Hilbert space H.*

(i) *If $T : C \to C$ is a κ-strict pseudo-contraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \forall x, y \in C.$$

(ii) *If $T : C \to C$ is a κ-strict pseudo-contraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

(iii) *If $T : C \to C$ is a $\kappa-$strict pseudo-contraction, then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

(iv) *Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a κ-strict pseudo-contraction, with $\kappa = max\{\kappa_i : 1 \leq i \leq N\}$.*

(v) *Let $\{T_i\}_{i=1}^N$ and $\{\lambda_i\}$ be given as in (iv) above. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F\Big(\sum_{i=1}^N \lambda_i T_i\Big) = \bigcap_{i=1}^N F(T_i).$$

Lemma 2.5. [2] *Let C be a nonempty closed convex subset of H, let F be bi-function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in C.$$

Lemma 2.6. [4] *For $r > 0$, $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C | F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Then, the following statements hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) *$F(T_r) = EP(F)$;*

(iv) *$EP(F)$ is closed and convex.*

3. Parallel Algorithm

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and*

$B: C \to H$ a β-inverse-strongly monotone mapping, respectively. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a κ_i−strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Let $\kappa = max\{\kappa_i : 1 \leq i \leq N\}$. Assume that

$$\Omega = \cap_{i=1}^{N} F(S_i) \cap EP(F_1, A) \cap EP(F_2, B) \neq \emptyset.$$

Assume also that $\{\eta_i^{(n)}\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \eta_i^{(n)} = 1$ for all $n \in \mathbb{N}$ and $\inf_{n\geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping V_n be defined by

$$V_n = \sum_{i=1}^{N} \eta_i^{(n)} S_i.$$

Given $x_1 \in C = C_1$, let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \forall v \in C, \\ z_n = \gamma_n u_n + (1 - \gamma_n)v_n, \\ V_n^{\lambda_n} = \lambda_n I + (1 - \lambda_n)V_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)V_n^{\lambda_n} z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$, $r \in (0, 2\alpha)$ and $s \in (0, 2\beta)$. If the above control sequences satisfy the following restrictions:

(i) *$\alpha_n \subset [0, a]$ with $a < 1$;*
(ii) $\lim_{n\to\infty} \gamma_n = \gamma \in (0, 1)$;
(iii) $\lambda_n \in [\kappa, b]$, $\kappa < b < 1$.

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega x_1$.

Proof. First, we claim that the mappings $I - rA$ and $I - sB$ are nonexpansive. Indeed, for each $x, y \in C$, we have

$$\begin{aligned} \|(I - rA)x - (I - rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r\langle x - y, Ax - Ay\rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r\alpha\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 - r(2\alpha - r)\|Ax - Ay\|^2. \end{aligned}$$

It follows from the condition $r \in (0, 2\alpha)$ that the mapping $I - rA$ is nonexpansive, so is $I - sB$. Note that u_n can be rewritten as $u_n = T_r(I - rA)x_n$ and v_n can be rewritten as $v_n = T_s(I - sB)x_n$ for each $n \geq 1$. Let $p \in \Omega$, it follows from Lemma 2.3 and Proposition 2.4 that

$$p = T_r(I - rA)p = T_s(I - sB)p = V_n p = V_n^{\lambda_n} p.$$

Thus we have

$$\|u_n - p\| \le \|x_n - p\| \text{ and } \|v_n - p\| \le \|x_n - p\|.$$

The proof is divided into seven steps.

Step 1. Show that $\{x_n\}$ is well defined.

Indeed, C_{n+1} is the intersection of C_n with the half space $\{z \in C : 2\langle x_n - y_n, z\rangle \le \|x_n\|^2 - \|y_n\|^2\}$. Since $C_1 = C$ is closed and convex, it is obvious that C_n is closed and convex for each $n \in N$.

Take $p \in \Omega$, since $V_n^{\lambda_n}$ is nonexpansive, we have

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)V_n^{\lambda_n} z_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|V_n^{\lambda_n} z_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\| \\
&= \alpha_n\|x_n - p\| + (1-\alpha_n)\|\gamma_n u_n + (1-\gamma_n)v_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)[\gamma_n\|u_n - p\| + (1-\gamma_n)\|v_n - p\|] \\
&\le \|x_n - p\|
\end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. So $p \in C_n$ for all n. Hence $\Omega \subset C_n$ holds for all $n \ge 1$.

Step 2. Show that

$$\|x_n - x_1\| \le \|x_1 - x^*\|, \text{ where } x^* = P_\Omega x_1. \tag{3.3}$$

Notice the facts $\Omega \subset C_n$ and $x_n = P_{C_n}x_1$ imply

$$\|x_n - x_1\| \le \|x_1 - p\| \text{ for all } p \in \Omega.$$

Then $\{x_n\}$ is bounded and (3.3) holds. From (3.2) and Proposition 2.4, we also obtain $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, $\{y_n\}$, $\{S_i x_n\}$ and $\{V_n x_n\}$ are bounded. From the nonexpansivity of $V_n^{\lambda_n}$, it follows that $\{V_n^{\lambda_n} x_n\}$ is also bounded.

Step 3. Show that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

Since $x_n = P_{C_n}x_1, x_{n+1} = P_{C_{n+1}}x_1$ and $C_{n+1} \subset C_n$, by the property of the projection, we have

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - x_{n+1}\|,
\end{aligned}$$

that is, $\|x_n - x_1\| \le \|x_1 - x_{n+1}\|$. The sequence $\{\|x_n - x_1\|\}$ is nondecreasing. Since $\{\|x_n - x_1\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Moreover,

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 - (x_n - x_1)\|^2 \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1\rangle \\
&\le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\end{aligned}$$

Then (3.4) holds.

Step 4. Show that

$$\|Ax_n - Ap\| \to 0 \text{ and } \|Bx_n - Bp\| \to 0.$$

By $x_{n+1} = P_{C_{n+1}} x_1$, it follows that

$$\begin{aligned} &\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|, \\ &\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \le 2\|x_n - x_{n+1}\| \to 0. \end{aligned} \tag{3.5}$$

For each $p \in \Omega$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n) V_n^{\lambda_n} z_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|\gamma_n u_n + (1-\gamma_n) v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\gamma_n \|u_n - p\|^2 + (1-\gamma_n)\|v_n - p\|^2] \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\gamma_n \|T_r(I - rA)x_n - p\|^2 \\ &\quad + (1-\gamma_n)\|T_s(I - sB)x_n - p\|^2] \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[\gamma_n \|x_n - p - r(Ax_n - Ap)\|^2 \\ &\quad + (1-\gamma_n)\|x_n - p - s(Bx_n - Bp)\|^2] \\ &= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\gamma_n(\|x_n - p\|^2 - 2r\langle x_n - p, Ax_n - Ap\rangle \\ &\quad + r^2\|Ax_n - Ap\|^2) \\ &\quad + (1-\alpha_n)(1-\gamma_n)(\|x_n - p\|^2 - 2s\langle x_n - p, Bx_n - Bp\rangle \\ &\quad + s^2\|Bx_n - Bp\|^2) \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\gamma_n[\|x_n - p\|^2 - r(2\alpha - r)\|Ax_n - Ap\|^2] \\ &\quad + (1-\alpha_n)(1-\gamma_n)[\|x_n - p\|^2 - s(2\beta - s)\|Bx_n - Bp\|^2] \\ &= \|x_n - p\|^2 - (1-\alpha_n)\gamma_n r(2\alpha - r)\|Ax_n - Ap\|^2 \\ &\quad - (1-\alpha_n)(1-\gamma_n) s(2\beta - s)\|Bx_n - Bp\|^2. \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} (1-\alpha_n)\gamma_n r(2\alpha - r)\|Ax_n - Ap\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \|y_n - x_n\|(\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

From the conditions (i), (ii) and (3.5), we see that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.7}$$

Similarly, from the conditions (i), (ii), (3.5) and (3.6), we obtain that

$$\lim_{n\to\infty} \|Bx_n - Bp\| = 0. \tag{3.8}$$

Step 5. Show that

$$\|V_n x_n - x_n\| \to 0. \tag{3.9}$$

By $u_n = T_r(I - rA)x_n$, it follows that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_r(I - rA)x_n - T_r(I - rA)p\|^2 \\ &\le \langle (I - rA)x_n - (I - rA)p, u_n - p\rangle \\ &= \frac{1}{2}[\|(I - rA)x_n - (I - rA)p\|^2 + \|u_n - p\|^2 \\ &\quad - \|(I - rA)x_n - (I - rA)p - (u_n - p)\|^2] \\ &\le \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r(Ax_n - Ap)\|^2] \\ &= \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - (\|x_n - u_n\|^2 - 2r\langle x_n - u_n, Ax_n - Ap\rangle \\ &\quad + r^2\|Ax_n - Ap\|^2)].\end{aligned}$$

Hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Ap\|. \tag{3.10}$$

Similarly, we can obtain that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s\|x_n - v_n\|\|Bx_n - Bp\|. \tag{3.11}$$

From (3.6), we get

$$\|y_n - p\|^2 \le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\gamma_n\|u_n - p\|^2 + (1 - \gamma_n)\|v_n - p\|^2]. \tag{3.12}$$

Substituting (3.10) and (3.11) into (3.12), we see that

$$\begin{aligned}\|y_n - p\|^2 \le &\|x_n - p\|^2 - (1 - \alpha_n)\gamma_n\|x_n - u_n\|^2 - (1 - \alpha_n)(1 - \gamma_n)\|x_n - v_n)\|^2 \\ &+ 2r\|x_n - u_n\|\|Ax_n - Ap\| + 2s\|x_n - v_n\|\|Bx_n - Bp\|.\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}&(1 - \alpha_n)\gamma_n\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2r\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s\|x_n - v_n\|\|Bx_n - Bp\| \\ &\le \|y_n - x_n\|(\|x_n - p\| + \|y_n - p\|) + 2r\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s\|x_n - v_n\|\|Bx_n - Bp\|.\end{aligned}$$

From the conditions (i), (ii), (3.5), (3.7) and (3.8), we obtain that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

Similarly, from (3.13), the conditions (i), (ii), (3.5), (3.7) and (3.8), we have that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\|z_n - x_n\| &= \|\gamma_n u_n + (1-\gamma_n)v_n - x_n\| \\ &\le \gamma_n\|u_n - x_n\| + (1-\gamma_n)\|v_n - x_n\|.\end{aligned}$$

In view of the condition (ii), (3.14) and (3.15), we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.16}$$

Since

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)V_n^{\lambda_n} z_n - x_n\| \\ &= (1-\alpha_n)\|V_n^{\lambda_n} z_n - x_n\|.\end{aligned}$$

From condition (i) and (3.6), we obtain

$$\|V_n^{\lambda_n} z_n - x_n\| \to 0. \tag{3.17}$$

It follows that

$$\begin{aligned}\|x_n - V_n^{\lambda_n} x_n\| &\le \|x_n - V_n^{\lambda_n} z_n\| + \|V_n^{\lambda_n} z_n - V_n^{\lambda_n} x_n\| \\ &\le \|x_n - V_n^{\lambda_n} z_n\| + \|z_n - x_n\|.\end{aligned}$$

From (3.16) and (3.17), we see that

$$\lim_{n\to\infty} \|x_n - V_n^{\lambda_n} x_n\| = 0.$$

Since

$$\begin{aligned}\|V_n^{\lambda_n} x_n - x_n\| &= \|\lambda_n x_n + (1-\lambda_n)V_n x_n - x_n\| \\ &= (1-\lambda_n)\|V_n x_n - x_n\| \\ &\ge (1-b)\|V_n x_n - x_n\|.\end{aligned}$$

Condition (iii) implies

$$\lim_{n\to\infty} \|V_n x_n - x_n\| = 0. \tag{3.18}$$

Step 6. Show that

$$\omega_w(x_n) \subset \Omega. \tag{3.19}$$

We first show $\omega_w(x_n) \subset \cap_{i=1}^N F(S_i)$. To see this, we take $\omega \in \omega_w(x_n)$ and assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$ for some subsequence $\{x_{n_j}\}$ of x_n.

Without loss of generality, we may assume that

$$\eta_i^{(n_j)} \to \eta_i \text{ (as } j \to \infty),\ 1 \le i \le N. \tag{3.20}$$

It is easily seen that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. We also have

$$V_{n_j} x \to Vx \text{ (as } j \to \infty) \text{ for all } x \in C,$$

where $V = \sum_{i=1}^{N} \eta_i S_i$. Note that by Proposition 2.4, V is $\kappa-$strict pseudo-contraction and $F(V) = \cap_{i=1}^{N} F(S_i)$. Since

$$\begin{aligned}\|Vx_{n_j} - x_{n_j}\| &\le \|V_{n_j}x_{n_j} - Vx_{n_j}\| + \|V_{n_j}x_{n_j} - x_{n_j}\| \\ &\le \sum_{i=1}^{N} |\eta_i^{(n_j)} - \eta_i|\|S_i x_{n_j}\| + \|V_{n_j}x_{n_j} - x_{n_j}\|,\end{aligned}$$

we obtain by virtue of (3.9) and (3.20)

$$\|Vx_{n_j} - x_{n_j}\| \to 0.$$

So by the demiclosedness principle (Proposition 2.4 (ii)), it follows that $\omega \in F(V) = \cap_{i=1}^{N} F(S_i)$ and hence the fact that $\omega_w(x_n) \subset \cap_{i=1}^{N} F(S_i)$ holds.

Next, we define a mapping $R : C \to C$ by

$$Rx = \gamma T_r(I - rA)x + (1-\gamma)T_s(I - sB)x, \ \forall x \in C,$$

where $(0,1) \ni \gamma = \lim_{n\to\infty} \gamma_n$. From Proposition 2.4 (iv), we see that R is a nonexpansive mapping with

$$F(R) = F(T_r(I - rA)) \cap F(T_s(I - sB)) = EP(F_1, A) \cap EP(F_2, B).$$

Note that

$$\begin{aligned}\|x_n - Rx_n\| &\le \|z_n - x_n\| + \|z_n - Rx_n\| \\ &= \|z_n - x_n\| + \|\gamma_n u_n + (1-\gamma_n)v_n - [\gamma u_n + (1-\gamma)v_n]\| \\ &\le \|z_n - x_n\| + |\gamma_n - \gamma|M,\end{aligned}$$

where M is an appropriate constant such that $M \ge \sup_{n\ge1}\{\|u_n\| + \|v_n\|\}$. This implies that

$$\lim_{j\to\infty} \|x_{n_j} - Rx_{n_j}\| = 0.$$

In view of Proposition 2.4 (ii), we obtain that $\omega \in F(R)$. That is,

$$\omega \in EP(F_1, A) \cap EP(F_2, B) \cap \cap_{i=1}^{N} F(S_i).$$

Hence (3.19) holds.

Step 7. Show that $x_n \to x^* = P_\Omega x_1$.

From (3.3), (3.19) and Lemma 2.2, we conclude that $x_n \to x^*$, where $x^* = P_\Omega x_1$. □

4. Cyclic Algorithm

Let C be a closed and convex subset of a Hilbert space H and let $\{S_i\}_{i=0}^{N-1}$ be N κ_i-strict pseudo-contractions on C such that the common fixed point set

$$\bigcap_{i=0}^{N-1} F(S_i) \ne \emptyset.$$

Let $x_0 \in C$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0, 1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned}
&x_1 = \alpha_0 x_0 + (1-\alpha_0)S_0 x_0,\\
&x_2 = \alpha_1 x_1 + (1-\alpha_1)S_1 x_1,\\
&\cdots\\
&x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})S_{N-1} x_{N-1},\\
&x_{N+1} = \alpha_N x_N + (1-\alpha_N)S_0 x_N,\\
&\cdots .
\end{aligned}$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_{[n]} x_n,$$

where $S_{[n]} = S_i$, with $i = n\ (mod)\ N,\ 0 \le i \le N-1$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping, respectively. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \le \kappa_i < 1$. Let $\kappa = max\{\kappa_i : 0 \le i \le N-1\}$. Assume that $\Omega = \cap_{i=0}^{N-1} F(S_i) \cap EP(F_1, A) \cap EP(F_2, B) \ne \emptyset$. Given $x_0 \in C = C_0$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases}
F_1(u_n, u) + \langle Ax_n, u-u_n\rangle + \frac{1}{r}\langle u-u_n, u_n-x_n\rangle \ge 0, \forall u \in C,\\
F_2(v_n, v) + \langle Bx_n, v-v_n\rangle + \frac{1}{s}\langle v-v_n, v_n-x_n\rangle \ge 0, \forall v \in C,\\
z_n = \gamma_n u_n + (1-\gamma_n)v_n,\\
S_{[n]}^{\lambda_n} = \lambda_n I + (1-\lambda_n)S_{[n]},\\
y_n = \alpha_n x_n + (1-\alpha_n)S_{[n]}^{\lambda_n} z_n,\\
C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\
x_{n+1} = P_{C_{n+1}} x_0,
\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$, $r \in (0, 2\alpha)$ and $s \in (0, 2\beta)$. If the above control sequences satisfy the following restrictions:

(i) $\alpha_n \subset [0, a]$ *with* $a < 1$*;*
(ii) $\lim_{n\to\infty} \gamma_n = \gamma \in (0, 1)$;
(iii) $\lambda_n \in [\kappa, b],\ \kappa < b < 1$.

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega x_0$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. The main points are:

Step 1. The sequence $\{x_n\}$ is well defined.

Step 2. $\|x_n - x_0\| \le \|x^* - x_0\|$ for all n, where $x^* = P_\Omega x_1$.

Step 3. $\|x_{n+1} - x_n\| \to 0$.

Step 4. $\|Ax_n - Ap\| \to 0$ and $\|Bx_n - Bp\| \to 0$.

Step 5. $\|S_{[n]}x_n - x_n\| \to 0$.

To prove the above steps, one simply replaces V_n with $S_{[n]}$ in the corresponding step of Theorem 3.1.

Step 6. $\omega_w(x_n) \subset \Omega$.

Indeed, let $\omega \in \omega_w(x_n)$ and $x_{n_m} \rightharpoonup \omega$ for some subsequence $\{x_{n_m}\}$ of $\{x_n\}$. We may assume that $l = n_m$ $(mod\ N)$ for all m. Since by $\|x_{n+1} - x_n\| \to 0$, we also have $x_{n_m+j} \rightharpoonup \omega$ for all $j \geq 0$, we deduce that

$$\|x_{n_m+j} - S_{[l+j]}x_{n_m+j}\| = \|x_{n_m+j} - S_{[n_m+j]}x_{n_m+j}\| \to 0.$$

Then the demiclosedness principle implies that $\omega \in F(S_{[l+j]})$ for all j. This ensures that $\omega \in \cap_{i=1}^{N} F(S_i)$.

The proof of $\omega \in EP(F_1, A) \cap EP(F_2, B)$ is similar to that of Theorem 3.1.

Step 7. The sequence x_n converges strongly to x^*.

The strong convergence to x^* of $\{x_n\}$ is the consequence of Step 2, Step 6 and Lemma 2.2. □

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