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STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce a new iterative scheme for finding the common element of the set of fixed points of an asymptotically nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a general system of variational inequalities for inverse-strongly monotone mappings in Hilbert spaces. We prove that the sequence converges strongly to a common element of the above three sets under some parameters controlling conditions. This main result improve and extend the corresponding results announced by many others. Using this theorem, we obtain three corollaries.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergence, respectively. Let C be a nonempty closed convex subset of H. Recall that a self-mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be asymptotically nonexpansive if there exists a sequence $\{h_n\}$ in $[1, \infty)$ with $lim_{n\to\infty} h_n = 1$ such that $\|T^n x - T^n y\| \leq h_n\|x - y\|$ for all $x, y \in C$ and $n \in N$. We use $F(T)$ to denote the set of fixed points of T, that is,

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$F(T) = \{x \in C : Tx = x\}$. A mapping f of C into itself is called contraction if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \le \rho\|x - y\|$ for all $x, y \in C$.

It is clear that every contraction is nonexpansive, and every nonexpansive mapping is asymptotically nonexpansive. The converses do not hold. The asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings. For details, we refer the reader to [6].

Let $F : C \times C \to R$ be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \ge 0 \tag{1.1}$$

for all $y \in C$. The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \ge 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. For solving the equilibrium problem for a bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \le F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2 \tag{1.2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{1.3}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{1.4}$$

for all $x \in H$, $y \in C$.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \ge 0 \tag{1.5}$$

for all $v \in C$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0. \tag{1.6}$$

A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \ge \alpha\|Au - Av\|^2$ for all

$u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.

For finding an element of the set of fixed points of a nonexpansive mapping T and the set of solutions of a variational inequality problem, Takahashi and Toyoda [11] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C(x_n - \lambda_n A x_n), \tag{1.7}$$

where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. Motivated by the idea of Korpelevich [7], Nadezhkina and Takahashi [9], Zeng and Yao [17] and Yao and Yao [16] proposed some so-called extragradient methods for finding a common element of $F(T) \cap VI(A, C)$.

Let $A, B : C \to H$ be two mappings. Now we concern the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.8}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.8) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.9}$$

which is defined by Verma [12] (see also [13]) and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.9) reduces to the classical variational inequality problem (1.5). For solving problem (1.8), recently, Ceng et al. [3] introduced and studied a relaxed extragradient method. Based on the relaxed extragradient method and the viscosity approximation method, W. Kumam and P. Kumam [8] constructed a new viscosity relaxed extragradient approximation method for finding an element of the set of fixed points of a nonexpansive mapping T, the set of solutions of the equilibrium problem (1.1) and the set of solutions of a general system of variational inequalities (1.8). Very recently, based on the extragradient method, Yao et al. [15] proposed an iterative method for finding a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space.

Motivated and inspired by the above works, in this paper, we introduce an iterative method based on the extragradient method and viscosity method for finding the common element of the set of fixed points of an asymptotically nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a general system of variational inequalities for inverse-strongly monotone mappings in real Hilbert spaces. We establish some strong convergence theorems for our iterative scheme.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([10]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)\, y_n + \beta_n\, x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \le 0$. Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.*

Lemma 1.2 ([4]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I-T)x_n\}$ converges strongly to 0, then $x = Tx$.*

Lemma 1.3 ([14]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \ \ n \ge 0,$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} | \delta_n | < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4 ([2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into R satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0$ *for all* $y \in C$.

Lemma 1.5 ([5]). *Assume that $F : C \times C \to R$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \ \forall y \in C\}$

for all $x \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*
   $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$;
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Lemma 1.6 ([3]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.8) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x-\mu Bx) - \lambda A P_C(x-\mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu B x^*)$.

Note that the mapping G is nonexpansive provided $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Throughout this paper, the set of fixed points of the mapping G is denoted by Γ.

Lemma 1.7. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 1.8 ([1]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to 0, then $x = Tx$.*

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying $(A1) - (A4)$, the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively. Let $T : C \to C$ be an asymptotically non-expansive mapping with $\Omega := F(T) \bigcap EP(F) \bigcap \Gamma \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction. Suppose $x_1 \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n P_C(y_n - \lambda A y_n), \end{cases} \tag{2.1}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ is a real sequence such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{h_n - 1}{\alpha_n} = 0$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \mid r_{n+1} - r_n \mid = 0$,

(v) T satisfies the asymptotically regularity on C: for any bounded subset K of C,

$$\limsup_{n\to\infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0,$$

then $\{x_n\}$ given by (2.1) converges strongly to $x^* = P_\Omega f(x^*)$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.8), where $y^* = P_C(x^* - \mu B x^*)$.

Proof. Let $Q = P_\Omega$. Then Qf is a contraction of C into itself. Since C is a closed set of H, there exists a unique element of $x^* \in C$ such that $x^* = Qf(x^*)$.

For any $x, y \in C$ and $\lambda \in (0, 2\alpha)$, we note that

$$\begin{aligned} &\|(I-\lambda A)x-(I-\lambda A)y\|^2 = \|x-y-\lambda(Ax-Ay)\|^2 \\ =&\|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ \leq&\|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \leq \|x-y\|^2, \end{aligned} \tag{2.2}$$

which implies that $I - \lambda A$ is nonexpansive. In the same way we can obtain that $I - \mu B$ is also nonexpansive and

$$\|(I-\mu B)x-(I-\mu B)y\|^2 \leq \|x-y\|^2 + \mu(\mu-2\beta)\|Bx-By\|^2 \tag{2.3}$$

for all $x, y \in C$ and $\mu \in (0, 2\beta)$.

Let $\{T_{r_n}\}$ be a sequence of mapping defined as in Lemma 1.5 and let $x^* \in \Omega$. Then $x^* = Tx^* = T_{r_n}x^*$ and $x^* = P_C[P_C(x^*-\mu Bx^*) - \lambda AP_C(x^*-\mu Bx^*)]$. Putting $y^* = P_C(x^*-\mu Bx^*)$ and $v_n = P_C(y_n - \lambda Ay_n)$, we have $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n v_n.$$

From (2.1) we have that

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|, \tag{2.4}$$

$$\|y_n-y^*\| = \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\| \leq \|u_n-x^*\| \leq \|x_n-x^*\| \tag{2.5}$$

and

$$\|v_n-x^*\| = \|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\| \leq \|y_n-y^*\| \leq \|x_n-x^*\|. \tag{2.6}$$

By condition (ii), for any x^*, $0 < \varepsilon < 1-\rho$ and sufficient large $n \geq 1$, we have $\gamma_n(h_n - 1) < \varepsilon\alpha_n$ and hence

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ \leq&\alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|T^n v_n - x^*\| \\ \leq&\alpha_n(\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + \beta_n\|x_n - x^*\| + \gamma_n h_n\|v_n - x^*\| \\ \leq&\alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n h_n\|x_n - x^*\| \\ =&[1 - \alpha_n + \alpha_n\rho + \gamma_n(h_n - 1)]\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ \leq&\max\{\|x_n - x^*\|, (1-\rho-\varepsilon)^{-1}\|f(x^*) - x^*\|\}. \end{aligned}$$

By induction, we have that

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, (1-\rho-\varepsilon)^{-1}\|f(x^*) - x^*\|\}, \forall n \in N. \tag{2.7}$$

Thus the sequence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}, \{y_n\}, \{v_n\}$, $\{T^n y_n\}$,$\{Bu_n\}$ and $\{Ay_n\}$ are also bounded.

Next, we claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Indeed, we define a sequence $\{s_n\}$ by $x_{n+1}=\beta_n x_n+(1-\beta_n)s_n,\ \forall n\ge 1$. Thus, we have

$$\begin{aligned} s_{n+1}-s_n =&\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\ =&\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}T^{n+1}v_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n T^n v_n}{1-\beta_n}\\ =&\Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n)\Big]\\ &+\frac{\gamma_{n+1}}{1-\beta_{n+1}}[T^{n+1}v_{n+1}-T^{n+1}v_n]\\ &+T^{n+1}v_n-T^n v_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}T^{n+1}v_n+\frac{\alpha_n}{1-\beta_n}T^n v_n. \end{aligned}\tag{2.8}$$

We note that

$$\begin{aligned} \|v_{n+1}-v_n\| =&\|P_C(I-\lambda A)y_{n+1}-P_C(I-\lambda A)y_n\|\le\|y_{n+1}-y_n\|\\ =&\|P_C(I-\mu B)u_{n+1}-P_C(I-\mu B)u_n\|\le\|u_{n+1}-u_n\|. \end{aligned}\tag{2.9}$$

From $u_n=T_{r_n}x_n$ and $u_{n+1}=T_{r_{n+1}}x_{n+1}$, we have that

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C\tag{2.10}$$

and

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,\quad \forall y\in C.\tag{2.11}$$

Putting $y=u_{n+1}$ in (2.10) and $y=u_n$ in (2.11) respectively, we obtain

$$F(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0$$

and

$$F(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.$$

So, from $(A2)$ we have

$$\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle\ge 0.$$

Hence

$$\langle u_{n+1}-u_n,u_n-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\ge 0$$

and

$$\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\ge 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number c such that $r_n > c > 0$ for all $n \geq 1$. Then we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\leq \langle u_{n+1}-u_n, x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle \\ &\leq \|u_{n+1}-u_n\|\big\{\|x_{n+1}-x_n\|+ \mid 1-\frac{r_n}{r_{n+1}} \mid \|u_{n+1}-x_{n+1}\|\big\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}} \mid r_{n+1}-r_n \mid \|u_{n+1}-x_{n+1}\| \\ &\leq \|x_{n+1}-x_n\| + \frac{L_1}{c} \mid r_{n+1}-r_n \mid,\end{aligned} \tag{2.12}$$

where $L_1 = \sup\{\|u_n - x_n\| : n \geq 1\}$. Substituting (2.12) into (2.9), we have

$$\|v_{n+1}-v_n\| \leq \|x_{n+1}-x_n\| + \frac{L_1}{c} \mid r_{n+1}-r_n \mid . \tag{2.13}$$

Combining (2.8) and (2.13), we have

$$\begin{aligned}&\|s_{n+1}-s_n\| - \|x_{n+1}-x_n\| \\ \leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|T^{n+1}v_n\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|T^n v_n\|) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|T^{n+1}v_{n+1} - T^{n+1}v_n\| + \|T^{n+1}v_n - T^n v_n\| - \|x_{n+1}-x_n\| \\ \leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|T^{n+1}v_n\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|T^n v_n\|) \\ &+ \frac{\gamma_{n+1}h_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\| + \|T^{n+1}v_n - T^n v_n\| - \|x_{n+1}-x_n\| \\ \leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|T^{n+1}v_n\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|T^n v_n\|) \\ &+ (\frac{\gamma_{n+1}h_{n+1}}{1-\beta_{n+1}} - 1)\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}h_{n+1}}{1-\beta_{n+1}}\frac{L_1}{c} \mid r_{n+1}-r_n \mid \\ &+ \|T^{n+1}v_n - T^n v_n\| \\ =&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|T^{n+1}v_n\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|T^n v_n\|) \\ &+ [\frac{\gamma_{n+1}(h_{n+1}-1)}{1-\beta_{n+1}} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}]\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}h_{n+1}}{1-\beta_{n+1}}\frac{L_1}{c} \mid r_{n+1}-r_n \mid \\ &+ \|T^{n+1}v_n - T^n v_n\|.\end{aligned}$$

Thus it follows from conditions $(ii), (iv)$ (v) and $h_n \to 1$ that

$$\limsup_{n\to\infty}(\|s_{n+1}-s_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

By Lemma 1.1 we get $\lim_{n\to\infty}\|s_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|s_n - x_n\| = 0. \tag{2.14}$$

From (iv), (2.9), (2.12) and (2.13), we also have $v_{n+1} - v_n \to 0$, $u_{n+1} - u_n \to 0$ and $y_{n+1} - y_n \to 0$ as $n \to \infty$.

Further, we can obtain that $\lim_{n\to\infty}\|Ay_n - Ay^*\| = 0$ and $\lim_{n\to\infty}\|Bu_n - Bx^*\| = 0$. Indeed, from (2.2) – (2.6) we get that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|v_n - x^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|(I-\lambda A)y_n - (I-\lambda A)y^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\{\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2\}\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2\\
=&\alpha_n\|f(x_n) - x^*\|^2 + [1 - \alpha_n + \gamma_n(h_n^2 - 1)]\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|v_n - x^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|y_n - y^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|(I-\mu B)u_n - (I-\mu B)x^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\{\|u_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2\}\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2\\
=&\alpha_n\|f(x_n) - x^*\|^2 + [1 - \alpha_n + \gamma_n(h_n^2 - 1)]\|x_n - x^*\|^2\\
&+ \gamma_n h_n^2\mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\gamma_n\lambda(2\alpha - \lambda)\|Ay_n - Ay^*\|^2\\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2\\
&+ \gamma_n(h_n^2 - 1)\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
=&\alpha_n\|f(x_n)-x^*\|^2+[\gamma_n(h_n^2-1)-\alpha_n]\|x_n-x^*\|^2\\
&+\gamma_n(h_n^2-1)\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\\
&+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\\
\leq&\alpha_n\|f(x_n)-x^*\|^2+[\gamma_n(h_n^2-1)-\alpha_n]\|x_n-x^*\|^2\\
&+\gamma_n(h_n^2-1)\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\\
&+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\gamma_n\mu(2\beta-\mu)\|Bu_n-Bx^*\|^2\\
\leq&\alpha_n\|f(x_n)-x^*\|^2+[\gamma_n(h_n^2-1)-\alpha_n]\|x_n-x^*\|^2\\
&+\gamma_n(h_n^2-1)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
=&\alpha_n\|f(x_n)-x^*\|^2+[\gamma_n(h_n^2-1)-\alpha_n]\|x_n-x^*\|^2\\
&+\gamma_n(h_n^2-1)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\\
&+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\\
\leq&\alpha_n\|f(x_n)-x^*\|^2+[\gamma_n(h_n^2-1)-\alpha_n]\|x_n-x^*\|^2\\
&+\gamma_n(h_n^2-1)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\\
&+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
$$

Since $\alpha_n\to 0$, $h_n\to 1$, $\|x_n-x_{n+1}\|\to 0$ and $\liminf_{n\to\infty}\gamma_n>0$, we obtain $\lim_{n\to\infty}\|Ay_n-Ay^*\|=0$ and $\lim_{n\to\infty}\|Bu_n-Bx^*\|=0$.

Now we show that $\|Tv_n-v_n\|\to 0$ as $n\to\infty$. Noting that P_C is firmly nonexpansive, from $\|u_n-x^*\|\leq\|x_n-x^*\|$ we have

$$
\begin{aligned}
&\|y_n-y^*\|^2\\
=&\|P_C(I-\mu B)u_n-P_C(I-\mu B)x^*\|^2\\
\leq&\langle(I-\mu B)u_n-(I-\mu B)x^*,y_n-y^*\rangle\\
=&\frac{1}{2}[\|(I-\mu B)u_n-(I-\mu B)x^*\|^2+\|y_n-y^*\|^2\\
&-\|(I-\mu B)u_n-(I-\mu B)x^*-(y_n-y^*)\|^2]\\
\leq&\frac{1}{2}[\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|u_n-y_n-\mu(Bu_n-Bx^*)-(x^*-y^*)\|^2]\\
\leq&\frac{1}{2}[\|x_n-x^*\|^2+\|y_n-y^*\|^2-\|u_n-y_n-(x^*-y^*)\|^2\\
&+2\mu\langle u_n-y_n-(x^*-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2]
\end{aligned}
$$

and from $\|y_n - y^*\| \le \|x_n - x^*\|$ we also have

$$\begin{aligned}
&\|v_n - x^*\|^2 \\
=&\|P_C(I-\lambda A)y_n - P_C(I-\lambda A)y^*\|^2 \\
\le&\langle (I-\lambda A)y_n - (I-\lambda A)y^*, v_n - x^*\rangle \\
=&\frac{1}{2}[\|(I-\lambda A)y_n - (I-\lambda A)y^*\|^2 + \|v_n - x^*\|^2 \\
&- \|(I-\lambda A)y_n - (I-\lambda A)y^* - (v_n - x^*)\|^2] \\
\le&\frac{1}{2}[\|y_n - y^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n - \lambda(Ay_n - Ay^*) - (y^* - x^*)\|^2] \\
\le&\frac{1}{2}[\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n + (x^* - y^*)\|^2 \\
&+ 2\lambda\langle y_n - v_n + (x^* - y^*), Ay_n - Ay^*\rangle - \lambda^2\|Ay_n - Ay^*\|^2].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|y_n - y^*\|^2 \le&\|x_n - x^*\|^2 - \|u_n - y_n - (x^* - y^*)\|^2 \\
&+ 2\mu\langle u_n - y_n - (x^* - y^*), Bu_n - Bx^*\rangle
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\|v_n - x^*\|^2 \le&\|x_n - x^*\|^2 - \|y_n - v_n + (x^* - y^*)\|^2 \\
&+ 2\lambda\langle y_n - v_n + (x^* - y^*), Ay_n - Ay^*\rangle.
\end{aligned} \tag{2.17}$$

By (2.6) and (2.16) we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|v_n - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|y_n - y^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\{\|x_n - x^*\|^2 \\
&- \|u_n - y_n - (x^* - y^*)\|^2 + 2\mu\langle u_n - y_n - (x^* - y^*), Bu_n - Bx^*\rangle\} \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + [1 - \alpha_n + \gamma_n(h_n^2 - 1)]\|x_n - x^*\|^2 \\
&- \gamma_n h_n^2\|u_n - y_n - (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\mu\|u_n - y_n - (x^* - y^*)\| \cdot \|Bu_n - Bx^*\|
\end{aligned}$$

which implies that

$$
\begin{aligned}
&\gamma_n\|u_n - y_n - (x^* - y^*)\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|u_n - y_n - (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\mu\|u_n - y_n - (x^* - y^*)\| \cdot \|Bu_n - Bx^*\| \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\
&+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|u_n - y_n - (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\mu\|u_n - y_n - (x^* - y^*)\| \cdot \|Bu_n - Bx^*\|.
\end{aligned}
\tag{2.18}
$$

It follows from (2.17) that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|v_n - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\{\|x_n - x^*\|^2 \\
&- \|y_n - v_n + (x^* - y^*)\|^2 + 2\lambda\langle y_n - v_n + (x^* - y^*), Ay_n - Ay^*\rangle\} \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + [1 - \alpha_n + \gamma_n(h_n^2 - 1)]\|x_n - x^*\|^2 \\
&- \gamma_n h_n^2\|y_n - v_n + (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\lambda\|y_n - v_n + (x^* - y^*)\| \cdot \|Ay_n - Ay^*\|
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\gamma_n\|y_n - v_n + (x^* - y^*)\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|y_n - v_n + (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\lambda\|y_n - v_n + (x^* - y^*)\| \cdot \|Ay_n - Ay^*\| \qquad (2.19)\\
\le&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\
&+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|y_n - v_n + (x^* - y^*)\|^2 \\
&+ 2\gamma_n h_n^2\lambda\|y_n - v_n + (x^* - y^*)\| \cdot \|Ay_n - Ay^*\|.
\end{aligned}
$$

Note that $\|x_n - x_{n+1}\| \to 0$, $\alpha_n \to 0$, $h_n \to 1$, $\|Bu_n - Bx^*\| \to 0$, $\|Ay_n - Ay^*\| \to 0$ and $\liminf_{n\to\infty} \gamma_n > 0$. From (2.18) and (2.19) we deduce

$$\lim_{n\to\infty} \|u_n - y_n - (x^* - y^*)\| = 0 \tag{2.20}$$

and

$$\lim_{n\to\infty} \|y_n - v_n + (x^* - y^*)\| = 0. \tag{2.21}$$

Since T_{r_n} is firmly nonexpansive for each $n \geq 1$, we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \leq \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\ &= \langle u_n - x^*, x_n - x^*\rangle = \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2)\end{aligned}$$

and hence $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. It follows from (2.5) and (2.6) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ \leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|v_n - x^*\|^2 \\ \leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n h_n^2\|u_n - x^*\|^2 \\ \leq&\alpha_n\|f(x_n) - x^*\|^2 + [1 - \alpha_n + \gamma_n(h_n^2 - 1)]\|x_n - x^*\|^2 - \gamma_n h_n^2\|x_n - u_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\gamma_n\|x_n - u_n\|^2 \\ \leq&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|x_n - u_n\|^2 \\ \leq&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &+ [\gamma_n(h_n^2 - 1) - \alpha_n]\|x_n - x^*\|^2 - \gamma_n(h_n^2 - 1)\|x_n - u_n\|^2.\end{aligned}$$

So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.22}$$

Moreover, from (2.20) and (2.21) we obtain

$$\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{2.23}$$

From $x_{n+1} - x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n v_n - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(T^n v_n - x_n)$, we get $\gamma_n(T^n v_n - x_n) = x_{n+1} - x_n - \alpha_n(f(x_n) - x_n)$. It follows from $x_{n+1} - x_n \to 0$ and $\alpha_n \to 0$ that

$$\lim_{n\to\infty} \|T^n v_n - x_n\| = 0. \tag{2.24}$$

Since $\|T^n v_n - v_n\| \leq \|T^n v_n - x_n\| + \|x_n - v_n\|$, from (2.23) and (2.24) we have $\|T^n v_n - v_n\| \to 0$. It follows from

$$\begin{aligned}\|v_n - Tv_n\| &\leq \|v_n - T^n v_n\| + \|T^n v_n - T^{n+1}v_n\| + \|T^{n+1}v_n - Tv_n\| \\ &\leq \|T^n v_n - T^{n+1}v_n\| + (1 + h_1)\|v_n - T^n v_n\|\end{aligned}$$

that

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{2.25}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0,$$

where $x^* = P_\Omega f(x^*)$. As $\{v_n\}$ is bounded, we can choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, v_n - x^*\rangle = \lim_{i\to\infty}\langle f(x^*) - x^*, v_{n_i} - x^*\rangle.$$

From $\|Tv_n - v_n\| \to 0$ and Lemma 1.2, we obtain $z \in F(T)$. Let us show $z \in EP(F)$. Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From $(A2)$, we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

From $\|u_n - x_n\| \to 0$ and $\|x_n - v_n\| \to 0$ we get $u_{n_i} \rightharpoonup z$. Since $\|u_{n_i} - x_{n_i}\| \to 0$ and $\liminf_{n\to\infty} r_n > 0$, it follows from $(A4)$ that

$$0 \geq F(y, z), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So from $(A1)$ and $(A4)$ we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From $(A3)$, we get $0 \leq F(z, y)$ for all $y \in C$ and $z \in EP(F)$. We shall show $z \in \Omega$. Since G is nonexpansive, we have that

$$\begin{aligned}
&\|v_n - G(v_n)\| \\
=&\|P_C[P_C(u_n - \mu Bu_n) - \lambda AP_C(u_n - \mu Bu_n)] - G(v_n)\| \\
=&\|G(u_n) - G(v_n)\| \\
\leq&\|u_n - v_n\| \\
\to&0.
\end{aligned}$$

From Lemma 1.8 we have $z \in F(G)$ and hence $z \in \Gamma$. Hence $z \in \Omega$. It follows from $\|x_n - v_n\| \to 0$, $x^* = P_\Omega f(x^*)$ and (1.3) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle =&\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - v_n + v_n - x^*\rangle \\
\leq&\limsup_{n\to\infty}\langle f(x^*) - x^*, v_n - x^*\rangle \\
=&\lim_{i\to\infty}\langle f(x^*) - x^*, v_{n_i} - x^*\rangle \\
=&\langle f(x^*) - x^*, z - x^*\rangle \\
\leq&0.
\end{aligned} \tag{2.26}$$

At last, we show that $\lim_{n\to\infty} x_n = x^*$. From Lemma 1.7 we get that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le&\|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(T^n v_n - x^*)\|^2 \\
\le&\|\beta_n(x_n - x^*) + \gamma_n(T^n v_n - x^*)\|^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
\le&(\beta_n\|x_n - x^*\| + \gamma_n h_n\|x_n - x^*\|)^2 + 2\alpha_n\rho\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\
&+ 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
\le&[1 - \alpha_n + \gamma_n(h_n - 1)]^2\|x_n - x^*\|^2 + \alpha_n\rho(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&+ 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
\le&[1 - (2 - \rho)\alpha_n + \alpha_n^2 + 3(h_n - 1)]\|x_n - x^*\|^2 + \alpha_n\rho\|x_{n+1} - x^*\|^2 \\
&+ 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le&\frac{1 - (2 - \rho)\alpha_n}{1 - \rho\alpha_n}\|x_n - x^*\|^2 + \frac{\alpha_n^2 + 3(h_n - 1)}{1 - \rho\alpha_n}\|x_n - x^*\|^2 \\
&+ \frac{2\alpha_n}{1 - \rho\alpha_n}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
:=&(1 - \sigma_n)\|x_n - x^*\|^2 + \delta_n,
\end{aligned}
$$

where $\sigma_n = \frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}$ and $\delta_n = \frac{\alpha_n^2+3(h_n-1)}{1-\rho\alpha_n}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$. It follows from (ii), (2.26) and Lemma 1.3 that $x_n \to x^*$. This completes the proof. □

As direct consequences of Theorem 2.1, we obtain three corollaries.

Corollary 2.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying $(A1) - (A4)$, the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse strongly monotone, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping with $\Omega := F(T) \bigcap EP(F) \bigcap \Gamma \neq \emptyset$. For fixed $u \in C$ and given $x_1 \in C$ arbitrarily, $\{x_n\}$ is generated by*

$$
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \ \forall y \in C, \\
y_n = P_C(u_n - \mu B u_n), \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T^n P_C(y_n - \lambda A y_n),
\end{cases}
\tag{2.27}
$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ is a real sequence such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{h_n-1}{\alpha_n} = 0$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} | r_{n+1} - r_n | = 0$,

(v) T *satisfies the asymptotically regularity on* C*: for any bounded subset* K *of* C,

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0,$$

then $\{x_n\}$ *given by* (2.27) *converges strongly to* $x^* = P_\Omega f(x^*)$ *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (1.8)*, where* $y^* = P_C(x^* - \mu B x^*)$.

Corollary 2.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let the mappings* $A, B : C \to H$ *be* α*-inverse-strongly monotone and* β*-inverse strongly monotone, respectively. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $\Omega := F(T) \bigcap \Gamma \neq \emptyset$*. Let* $f : C \to C$ *be a* ρ*-contraction. Suppose* $x_1 \in C$ *and* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n P_C(y_n - \lambda A y_n), \end{cases} \tag{2.28}$$

where $\lambda \in (0, 2\alpha)$*,* $\mu \in (0, 2\beta)$*,* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{h_n - 1}{\alpha_n} = 0$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(iv) T *satisfies the asymptotically regularity on* C*: for any bounded subset* K *of* C,

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0,$$

then $\{x_n\}$ *given by* (2.28) *converges strongly to* $x^* = P_\Omega f(x^*)$ *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (1.8)*, where* $y^* = P_C(x^* - \mu B x^*)$.

Corollary 2.3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let the mappings* $A, B : C \to H$ *be* α*-inverse-strongly monotone and* β*-inverse strongly monotone, respectively. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $\Omega := F(T) \bigcap \Gamma \neq \emptyset$*. For fixed* $u \in C$ *and given* $x_1 \in C$ *arbitrarily,* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T^n P_C(y_n - \lambda A y_n), \end{cases} \tag{2.29}$$

where $\lambda \in (0, 2\alpha)$*,* $\mu \in (0, 2\beta)$*,* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{h_n-1}{\alpha_n} = 0$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) T *satisfies the asymptotically regularity on* C*: for any bounded subset* K *of* C,

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0,$$

then $\{x_n\}$ *given by* (2.29) *converges strongly to* $x^* = P_\Omega f(x^*)$ *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (1.8)*, where* $y^* = P_C(x^* - \mu B x^*)$.

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