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A STRONG CONVERGENCE THEOREM BY A RELAXED EXTRAGRADIENT METHOD FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES AND STRICT PSEUDO-CONTRACTIONS

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Abstract. In this paper, we introduce a new iterative scheme based on the relaxed extragradient method for finding a common element of the set of solutions of a general system of variational inequalities and the set of fixed points of N strict pseudo-contractions in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above sets under some controlling conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is said to be the metric projection of H onto

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C. It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{1.1}$$

for every $x, y \in H$. Moreover, we know that $P_C x$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{1.2}$$

for all $x \in H$, $y \in C$.

Recall that a mapping $S : C \to C$ is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x-y\|^2 + \kappa\|(I-S)x - (I-S)y\|^2$$

for all $x, y \in C$. We use $F(S)$ to denote the set of fixed points of S, i.e., $F(S) = \{x \in C : Sx = x\}$. Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $f : C \to C$ is called contraction if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$.

For two given nonlinear operators $A, B : C \to H$, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\lambda > 0$ and $\mu > 0$ are two constants.This is so-called a general system of Variational inequalities, which is defined by Verma [9]. If $A = B$, then problem (1.3) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which is said to be the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.4) reduces to the classical variational inequality, denoted by $VI(A, C)$, which is to find an $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0$$

for all $x \in C$. The variational problem is one of the important branches of sciences, and the variational inequality has been extensively studied. See, e.g. [4, 5, 6, 9, 11].

Let C be a closed convex subset of real Hilbert space H. It is known that A is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$$

for all $u, v \in C$. Recently, Ceng et al. [4] introduced and studied a relaxed extragradient method for finding solutions of problem (1.3). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let S be a nonexpansive mapping and suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$. Then, they proved that the iterative sequence $\{x_n\}$ strongly converges to a common element under some parameters controlling conditions. Very recently, for approximating a common element of the set of fixed points of a strict pseudo-contraction and the set of solutions of problem (1.3), Yao et al. [11] introduced a new iterative scheme:

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n S y_n, \end{cases}$$

where Q is a contraction and S is a strict pseudo-contraction. Furthermore, they also obtained a strong convergence theorem in a real Hilbert space.

Motivated and inspired by the above works, in this paper, we consider a new iterative scheme based on the extragradient method for finding a common element of the set of solutions of (1.3) and the set of fixed points of N strict pseudo-contractions. We also prove that the iterative scheme strongly converges to a common element of the above sets.

2. Preliminaries

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 ([4]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.3) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu B x) - \lambda A P_C(x - \mu B x)], \ \forall x \in C,$$

where $y^* = P_C(x^* - \mu B x^*)$.

Remark 2.1 ([4]). *If the mappings $A, B : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone respectively, then $G : C \to C$ is a nonexpansive mapping provided $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$.*

Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

Lemma 2.2 ([7]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.3 ([8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)\, y_n+\beta_n\, x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty} (\parallel y_{n+1} - y_n \parallel - \parallel x_{n+1} - x_n \parallel) \le 0$. Then $\lim_{n\to\infty} \parallel y_n - x_n \parallel = 0$.*

Lemma 2.4 ([10]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \;\; n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([2]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \ne \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and if $\{(I-T)x_n\}$ converges strongly to y (for short, $(I-T)x_n \to y$), then $(I-T)x = y$.*

Proposition 2.6 ([1]). *Assume C is a closed convex subset of a Hilbert space H.*

(i) *If $T : C \to C$ is a κ-strict pseudo-contraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \le \frac{1+\kappa}{1-\kappa}\|x-y\|, \forall x, y \in C.$$

(ii) *If $T : C \to C$ is a κ-strict pseudo-contraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \widetilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\widetilde{x} = 0$.*
(iii) *If $T : C \to C$ is a κ-strict pseudo-contraction, then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*
(iv) *Given an integer $N \ge 1$, assume, for each $1 \le i \le N$, $T_i : C \to C$ is a κ_i-strict pseudo-contraction for some $0 \le \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\Sigma_{i=1}^N \lambda_i = 1$. Then $\Sigma_{i=1}^N \lambda_i T_i$ is a κ-strict pseudo-contraction with $\kappa = \max\{\kappa_i : 1 \le i \le N\}$.*

(v) *Let* $\{T_i\}_{i=1}^N$ *and* $\{\lambda_i\}_{i=1}^N$ *be given as in* (iv) *above. Suppose that* $\{T_i\}_{i=1}^N$ *has a common fixed point. Then*

$$F\Big(\sum_{i=1}^{N}\lambda_i T_i\Big)=\bigcap_{i=1}^{N}F(T_i).$$

Lemma 2.7 ([3])**.** *Let* $S: C\to H$ *be a* $\kappa-$ *strict pseudo-contraction. Define* $T: C\to H$ *by* $Tx=\lambda x+(1-\lambda)Sx$ *for each* $x\in C$*. Then, as* $\lambda\in[\kappa,1)$*,* T *is a nonexpansive mapping such that* $F(T)=F(S)$*.*

Lemma 2.8 ([4])**.** *In a real Hilbert space* H*, there holds the inequality*

$$\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle,\ \forall x,y\in H.$$

3. Main Results

Now we state and prove our main result of this paper.

Theorem 3.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $A,B: C\to H$ *be* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively. Let* $S_i: C\to C$ *be a* κ_i*-strict pseudo-contraction for some* $0\le\kappa_i<1$*. Let* $\kappa=\max\{\kappa_i: 1\le i\le N\}$*. Assume the set* $\bigcap_{i=1}^N F(S_i)\bigcap\Omega\ne\emptyset$*. Assume also* $\{\eta_i^{(n)}\}_{i=1}^N$ *are sequences of positive numbers such that* $\sum_{i=1}^N\eta_i^{(n)}=1$ *for all* $n\ge1$ *and* $\inf_{n\ge1}\eta_i^{(n)}>0$ *for all* $1\le i\le N$*. Let the mapping* V_n *be defined by* $V_n=\sum_{i=1}^N\eta_i^{(n)}S_i$*. Let* $f: C\to C$ *be a contraction with coefficient* $\rho\in[0,\frac{1}{2})$*. Suppose* $x_1\in C$ *and* $\{x_n\}$ *is generated by the following algorithm:*

$$\begin{cases} y_n=P_C(x_n-\mu Bx_n),\\ V_n^{\delta_n}=\delta_n I+(1-\delta_n)V_n,\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n V_n^{\delta_n}P_C(y_n-\lambda Ay_n),\end{cases}$$

where $\lambda\in(0,2\alpha)$*,* $\mu\in(0,2\beta)$*,* $\{\delta_n\}\subset[\kappa,b]$ *for some* $b\in[\kappa,1)$*, and* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\eta_i^{(n)}\}$ *are sequences in* $[0,1]$ *such that*

(i) $\alpha_n+\beta_n+\gamma_n=1,\forall n\ge1$*;*
(ii) $\lim_{n\to\infty}\alpha_n=0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n=\infty$*;*
(iii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$*;*
(iv) $\lim_{n\to\infty}\mid\delta_{n+1}-\delta_n\mid=0$*;*
(v) $\lim_{n\to\infty}\mid\eta_i^{(n+1)}-\eta_i^{(n)}\mid=0$*, for* $1\le i\le N$*.*

Then $\{x_n\}$ *converges strongly to* $\bar{x}=P_{\cap_{i=1}^N F(S_i)\cap\Omega}f(\bar{x})$ *and* $(\bar{x},\bar{y})$ *is a solution of problem* (1.3)*, where* $\bar{y}=P_C(\bar{x}-\mu B\bar{x})$*.*

Proof. Let $Q = P_{\cap_{i=1}^N F(S_i)\cap\Omega}$. Then Qf is a contraction of C into C. In fact, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \rho\|x - y\|$$

for all $x, y \in C$. This implies that Qf is a contraction on C. Since H is complete, there exists a unique element $x^* \in C$ such that $x^* = Qf(x^*)$.

The following proof is divided into several steps.

Step 1: Show that $\{x_n\}$ is bounded firstly.

Let $x^* \in \bigcap_{i=1}^N F(S_i) \bigcap \Omega$. By Proposition 2.6 ($v$), Lemma 2.7 and Lemma 2.1, we know that $V_n x^* = x^*$, $V_n^{\delta_n} x^* = x^*$ and

$$x^* = P_C[P_C(x^* - \mu Bx^*) - \lambda A P_C(x^* - \mu Bx^*)].$$

Put $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$. Then $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n V_n^{\delta_n} t_n.$$

Observe that

$$\begin{aligned} &\|P_C(I - \lambda A)x - P_C(I - \lambda A)y\|^2 \leq \|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ =&\|x - y - \lambda(Ax - Ay)\|^2 \\ =&\|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ \leq&\|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \leq \|x - y\|^2 \end{aligned} \tag{3.1}$$

and similarly,

$$\begin{aligned} &\|P_C(I - \mu B)x - P_C(I - \mu B)y\|^2 \leq \|(I - \mu B)x - (I - \mu B)y\|^2 \\ \leq&\|x - y\|^2 + \mu(\mu - 2\beta)\|Bx - By\|^2 \leq \|x - y\|^2 \end{aligned} \tag{3.2}$$

for all $x, y \in H$. Thus from (3.1) and (3.2), we have

$$\begin{aligned} &\|t_n - x^*\| = \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\| \\ \leq&\|y_n - y^*\| = \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\| \leq \|x_n - x^*\|. \end{aligned} \tag{3.3}$$

Hence, it follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\| = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n V_n^{\delta_n} t_n - x^*\| \\
\le&\alpha_n\|f(x_n)-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|V_n^{\delta_n}t_n - x^*\| \\
\le&\alpha_n\|f(x_n)-f(x^*)\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| + \gamma_n\|V_n^{\delta_n}t_n - x^*\| \\
\le&(\alpha_n\rho+\beta_n)\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| + \gamma_n\|t_n-x^*\| \\
\le&(\alpha_n\rho+\beta_n+\gamma_n)\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| \\
=&(1-\alpha_n(1-\rho))\|x_n-x^*\| + \alpha_n(1-\rho)\frac{\|f(x^*)-x^*\|}{1-\rho} \\
\le&\max\{\|x_n-x^*\|, \frac{1}{1-\rho}\|f(x^*)-x^*\|\} \\
\le&\max\{\|x_1-x^*\|, \frac{1}{1-\rho}\|f(x^*)-x^*\|\}.
\end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{t_n\}$, $\{y_n\}$, $\{Ay_n\}$, $\{Bx_n\}$, $\{f(x_n)\}$, $\{S_i t_n\}$, $\{V_n t_n\}$ and $\{V_n^{\delta_n} t_n\}$ are also bounded.

Step 2: Show $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From (3.1) and (3.2), we also observe that

$$\begin{aligned}
&\|t_{n+1}-t_n\| = \|P_C(y_{n+1}-\lambda Ay_{n+1}) - P_C(y_n-\lambda Ay_n)\| \\
\le&\|y_{n+1}-y_n\| = \|P_C(x_{n+1}-\mu Bx_{n+1}) - P_C(x_n-\mu Bx_n)\| \\
\le&\|x_{n+1}-x_n\|.
\end{aligned} \tag{3.4}$$

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, where $z_n = \frac{\alpha_n f(x_n)+\gamma_n V_n^{\delta_n} t_n}{1-\beta_n}$. Then we get

$$\begin{aligned}
&\|z_{n+1}-z_n\| = \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}V_{n+1}^{\delta_{n+1}}t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)+\gamma_n V_n^{\delta_n}t_n}{1-\beta_n}\| \\
\le& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)\| \\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|V_{n+1}^{\delta_{n+1}}t_{n+1} - V_n^{\delta_n}t_n\| + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right|\|V_n^{\delta_n}t_n\| \\
\le& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\rho\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\Big(\|f(x_n)\| + \|V_n^{\delta_n}t_n\|\Big) \\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|V_{n+1}^{\delta_{n+1}}t_{n+1} - V_n^{\delta_n}t_n\|.
\end{aligned} \tag{3.5}$$

Using the convexity of $\|\cdot\|$, we have

$$\begin{aligned}
&\|V_{n+1}^{\delta_{n+1}}t_{n+1} - V_n^{\delta_n}t_n\| \leq \|V_{n+1}^{\delta_{n+1}}t_{n+1} - V_{n+1}^{\delta_{n+1}}t_n\| + \|V_{n+1}^{\delta_{n+1}}t_n - V_n^{\delta_n}t_n\| \\
\leq&\|t_{n+1} - t_n\| + \|\delta_{n+1}t_n + (1-\delta_{n+1})V_{n+1}t_n - \delta_n t_n - (1-\delta_n)V_n t_n\| \\
\leq&\|x_{n+1} - x_n\| + \|(\delta_{n+1} - \delta_n)(t_n - V_n t_n) + (1-\delta_{n+1})(V_{n+1}t_n - V_n t_n)\| \\
\leq&\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|t_n - V_n t_n\| \\
&+ (1-\delta_{n+1})\|\sum_{i=1}^N \eta_i^{(n+1)} S_i t_n - \sum_{i=1}^N \eta_i^{(n)} S_i t_n\| \\
\leq&\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|t_n - V_n t_n\| \\
&+ (1-\delta_{n+1})\sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|\|S_i t_n\| \\
\leq&\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1 + (1-\delta_{n+1})M_2\sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|,
\end{aligned} \tag{3.6}$$

where $M_1 = \sup_{n\geq 1}\{\|t_n - V_n t_n\|\}$ and $M_2 = \sup_{n\geq 1, 1\leq i\leq N}\{\|S_i t_n\|\}$. Combining (3.5) and (3.6), we have

$$\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\leq&\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big|\Big(\|f(x_n)\| + \|V_n^{\delta_n}t_n\|\Big) \\
&+ \Big(\frac{\alpha_{n+1}\rho}{1-\beta_{n+1}} + \frac{\gamma_{n+1}}{1-\beta_{n+1}} - 1\Big)\|x_{n+1} - x_n\| \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big[|\delta_{n+1} - \delta_n|M_1 + (1-\delta_{n+1})M_2\sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|\Big] \\
\leq&\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big|\Big(\|f(x_n)\| + \|V_n^{\delta_n}t_n\|\Big) + \frac{\alpha_{n+1}(\rho-1)}{1-\beta_{n+1}}\|x_{n+1} - x_n\| \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big[|\delta_{n+1} - \delta_n|M_1 + (1-\delta_{n+1})M_2\sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|\Big].
\end{aligned}$$

This implies that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Hence by Lemma 2.3 we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.7}$$

From (3.4) and (3.7), it follows that $\lim_{n\to\infty}\|t_{n+1} - t_n\| = \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

Step 3: Show $\lim_{n\to\infty}\|V_n^{\delta_n}t_n - x_n\| = 0$.

Note that

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(V_n^{\delta_n} t_n - x_n).$$

This together with (ii) and (3.7) implies that $\|V_n^{\delta_n} t_n - x_n\| \to 0$ as $n \to \infty$.

Step 4: $\lim_{n\to\infty} \|Ay_n - Ay^*\| = 0$ and $\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0$.

Since $x^* \in \bigcap_{i=1}^N F(S_i) \bigcap \Omega$, from (3.1), (3.2), (3.3) and Lemma 2.2, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n V_n^{\delta_n} t_n - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
=&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\Big(\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2\Big) \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - y^*\|^2 \\
=&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n\mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\gamma_n\lambda(2\alpha - \lambda)\|Ay_n - Ay^*\|^2 \le \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|
\end{aligned} \tag{3.8}$$

and similarly

$$\begin{aligned}
&\gamma_n\mu(2\beta - \mu)\|Bx_n - Bx^*\|^2 \\
\le&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|.
\end{aligned} \tag{3.9}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, from (3.8) and (3.9) we derive $\lim_{n\to\infty} \|Ay_n - Ay^*\| = 0$ and $\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0$.

Step 5: Show $\lim_{n\to\infty} \|(x_n - y_n) - (x^* - y^*)\| = 0$.

From (1.1) and (3.1) we get

$$
\begin{aligned}
&\|y_n - y^*\|^2 = \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\
\leq&\langle (x_n - \mu Bx_n) - (x^* - \mu Bx^*), y_n - y^*\rangle \\
=&\frac{1}{2}[\|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\|^2 + \|y_n - y^*\|^2 \\
&- \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2] \\
\leq&\frac{1}{2}[\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2] \\
=&\frac{1}{2}[\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(x_n - y_n) - (x^* - y^*)\|^2 - \mu^2\|Bx_n - Bx^*\|^2 \\
&+ 2\mu\langle (x_n - y_n) - (x^* - y^*), Bx_n - Bx^*\rangle].
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\|y_n - y^*\|^2 \leq \|x_n - x^*\|^2 &- \|(x_n - y_n) - (x^* - y^*)\|^2 \\
&+ 2\mu\langle (x_n - y_n) - (x^* - y^*), Bx_n - Bx^*\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - y^*\|^2 \\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\
&- \gamma_n\|(x_n - y_n) - (x^* - y^*)\|^2 + 2\gamma_n\mu\langle (x_n - y_n) - (x^* - y^*), Bx_n - Bx^*\rangle \\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|(x_n - y_n) - (x^* - y^*)\|^2 \\
&+ 2\gamma_n\mu\langle (x_n - y_n) - (x^* - y^*), Bx_n - Bx^*\rangle
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\gamma_n\|(x_n - y_n) - (x^* - y^*)\|^2 \\
\leq&\alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x_{n+1}\|) \\
&+ 2\gamma_n\mu\|(x_n - y_n) - (x^* - y^*)\|\|Bx_n - Bx^*\|.
\end{aligned}
$$

Note that $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Bx_n - Bx^*\| \to 0$ as $n \to \infty$, then we immediately deduce $\lim_{n\to\infty} \|(x_n - y_n) - (x^* - y^*)\| = 0$.

Step 6: Show $\lim_{n\to\infty} \|(y_n - t_n) + (x^* - y^*)\| = 0$.

Now, from Lemma 2.8 and (1.1)

$$\begin{aligned}
&\|(y_n - t_n) + (x^* - y^*)\|^2 \\
=&\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)] \\
&+ \lambda(Ay_n - Ay^*)\|^2 \\
\le&\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)]\|^2 \\
&+ 2\lambda\langle Ay_n - Ay^*, (y_n - t_n) + (x^* - y^*)\rangle \\
\le&\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
&+ 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\|.
\end{aligned}$$

Since

$$\|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\| = \|t_n - x^*\| \ge \|V_n^{\delta_n} t_n - V_n^{\delta_n} x^*\|,$$

it follows that

$$\begin{aligned}
&\|(y_n - t_n) + (x^* - y^*)\|^2 \\
\le&\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|V_n^{\delta_n} t_n - V_n^{\delta_n} x^*\|^2 \\
&+ 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\| \\
\le&\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - (V_n^{\delta_n} t_n - x^*)\| \\
&\times \Big(\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|V_n^{\delta_n} t_n - x^*\|\Big) + 2\lambda\|Ay_n - Ay^*\| \\
&\times \|(y_n - t_n) + (x^* - y^*)\| \\
=&\|(x^* - y^*) - (x_n - y_n) + (x_n - V_n^{\delta_n} t_n) - \lambda(Ay_n - Ay^*)\| \\
&\times \Big(\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|V_n^{\delta_n} t_n - x^*\|\Big) + 2\lambda\|Ay_n - Ay^*\| \\
&\times \|(y_n - t_n) + (x^* - y^*)\| \\
\le&\Big(\|(x^* - y^*) - (x_n - y_n)\| + \|x_n - V_n^{\delta_n} t_n\| + \lambda\|Ay_n - Ay^*\|\Big) \\
&\times \Big(\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|V_n^{\delta_n} t_n - x^*\|\Big) + 2\lambda\|Ay_n - Ay^*\| \\
&\times \|(y_n - t_n) + (x^* - y^*)\|.
\end{aligned}$$

Since $\|(x_n - y_n) - (x^* - y^*)\| \to 0$, $\|V_n^{\delta_n} t_n - x_n\| \to 0$ and $\|Ay_n - Ay^*\| \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \|(y_n - t_n) + (x^* - y^*)\| = 0$.

Step 7: Show $\lim_{n\to\infty} \|V_n^{\delta_n} t_n - t_n\| = 0$.

We observe that

$$\|V_n^{\delta_n} t_n - t_n\| \le \|V_n^{\delta_n} t_n - x_n\| + \|(x_n - y_n) - (x^* - y^*)\| + \|(y_n - t_n) + (x^* - y^*)\|.$$

Combining the above results, we get $\|V_n^{\delta_n} t_n - t_n\| \to 0$ as $n \to \infty$.

Step 8: $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \le 0$ where $\bar{x} = P_{\cap_{i=1}^N F(S_i)\cap\Omega} f(\bar{x})$.

Indeed, since $\{V_n^{\delta_n}t_n\}$ is a bounded sequence in C, we can choose a subsequence $\{V_{n_j}^{\delta_{n_j}}t_{n_j}\}$ of $\{V_n^{\delta_n}t_n\}$ such that $V_{n_j}^{\delta_{n_j}}t_{n_j} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, V_n^{\delta_n}t_n - \bar{x}\rangle = \lim_{j\to\infty}\langle f(\bar{x}) - \bar{x}, V_{n_j}^{\delta_{n_j}}t_{n_j} - \bar{x}\rangle$. Since for all $1 \le i \le N$, $\{\eta_i^{(n_j)}\}$ is bounded, there exists a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ such that $\eta_i^{(n_{j_k})} \to \eta_i$ (as $k \to \infty$) for all $1 \le i \le N$. Without loss of generality, we can assume that

$$\eta_i^{(n_j)} \to \eta_i \ (as\ j \to \infty), 1 \le i \le N.$$

Since $\|V_n^{\delta_n}t_n - t_n\| \to 0$, we obtain $t_{n_j} \rightharpoonup z$ as $j \to \infty$. Now we claim that $z \in \bigcap_{i=1}^N F(S_i) \bigcap \Omega$. First, it is easy to get each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. We also have

$$V_{n_j}x \to Vx \ (as\ j \to \infty)$$

for all $x \in C$, where $V = \sum_{i=1}^N \eta_i S_i$. Using Proposition 2.6 (iv) and (v), V is κ-strict pseudo-contraction and $F(V) = \bigcap_{i=1}^N F(S_i)$. Observe that

$$\begin{aligned}&\|Vt_{n_j} - t_{n_j}\| \le \|Vt_{n_j} - V_{n_j}t_{n_j}\| + \|V_{n_j}t_{n_j} - t_{n_j}\| \\ \le &\sum_{i=1}^N |\eta_i - \eta_i^{(n_j)}|\|S_i t_{n_j}\| + \frac{1}{1-\delta_{n_j}}\|V_{n_j}^{\delta_{n_j}}t_{n_j} - t_{n_j}\|.\end{aligned}$$

Thus by $\eta_i^{(n_j)} \to \eta_i$ and $\|V_{n_j}^{\delta_{n_j}}t_{n_j} - t_{n_j}\| \to 0$, we obtain $\|Vt_{n_j} - t_{n_j}\| \to 0$. So by the demiclosedness principle (proposition 2.6 (ii)), it follows that $z \in F(V) = \bigcap_{i=1}^N F(S_i)$. Next, we prove that $z \in \Omega$. From Lemma 2.1 and Remark 2.1 we note that

$$\begin{aligned}&\|t_n - G(t_n)\| = \|P_C[P_C(x_n - \mu Bx_n) - \lambda AP_C(x_n - \mu Bx_n)] - G(t_n)\| \\ =&\|G(x_n) - G(t_n)\| \le \|x_n - t_n\| \le \|x_n - V_n^{\delta_n}t_n\| + \|V_n^{\delta_n}t_n - t_n\|.\end{aligned}$$

Since $\|V_n^{\delta_n}t_n - t_n\| \to 0$ and $\|x_n - V_n^{\delta_n}t_n\| \to 0$ as $n \to \infty$, we get $\|t_n - G(t_n)\| \to 0$. According to Lemma 2.5 we obtain $z \in \Omega$. Therefore there holds $z \in \bigcap_{i=1}^N F(S_i) \bigcap \Omega$. Hence it follows from (1.2) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, V_n^{\delta_n}t_n - \bar{x}\rangle \\ =&\lim_{j\to\infty}\langle f(\bar{x}) - \bar{x}, V_{n_j}^{\delta_{n_j}}t_{n_j} - \bar{x}\rangle = \langle f(\bar{x}) - \bar{x}, z - \bar{x}\rangle \le 0.\end{aligned} \tag{3.10}$$

Step 9: Show $\lim_{n\to\infty} x_n = \bar{x}$.

Note that

$$
\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
=&\langle \alpha_n f(x_n)+\beta_n x_n+\gamma_n V_n^{\delta_n} t_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\
=&\alpha_n\langle f(x_n)-\bar{x}, x_{n+1}-\bar{x}\rangle+\beta_n\langle x_n-\bar{x}, x_{n+1}-\bar{x}\rangle+\gamma_n\langle V_n^{\delta_n} t_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\
\leq&\alpha_n\langle f(x_n)-\bar{x}, x_{n+1}-x_n\rangle+\alpha_n\langle f(x_n)-f(\bar{x}), x_n-\bar{x}\rangle\\
&+\alpha_n\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle+\frac{\beta_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)\\
&+\frac{\gamma_n}{2}(\|t_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)\\
\leq&\alpha_n\|f(x_n)-\bar{x}\|\|x_{n+1}-x_n\|+\alpha_n\rho\|x_n-\bar{x}\|^2+\alpha_n\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle\\
&+\frac{\beta_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)+\frac{\gamma_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)\\
\leq&[\frac{1}{2}(1-\alpha_n)+\alpha_n\rho]\|x_n-\bar{x}\|^2+\frac{1}{2}(1-\alpha_n)\|x_{n+1}-\bar{x}\|^2\\
&+\alpha_n\|f(x_n)-\bar{x}\|\|x_{n+1}-x_n\|+\alpha_n\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq [1-\alpha_n(1-2\rho)]\|x_n-\bar{x}\|^2+\alpha_n(1-2\rho)\\
&\times\Big(\frac{2}{1-2\rho}\|f(x_n)-\bar{x}\|\|x_{n+1}-x_n\|+\frac{2}{1-2\rho}\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle\Big).
\end{aligned}
$$

Consequently, according to (3.10) and Lemma 2.4, we deduce that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

As direct consequences of Theorem 3.1, we obtain two corollaries.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Let $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the set $\bigcap_{i=1}^N F(S_i) \bigcap \Omega \neq \emptyset$. Assume also $\{\eta_i^{(n)}\}_{i=1}^N$ are sequences of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \geq 1$ and $\inf_{n\geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping V_n be defined by $V_n = \sum_{i=1}^N \eta_i^{(n)} S_i$. Let $f : C \to C$ be a contraction with coefficient $\rho \in [0, \frac{1}{2})$. Suppose $x_1 \in C$ and $\{x_n\}$ is generated by the following algorithm:*

$$
\begin{cases}
y_n = P_C(x_n - \mu A x_n),\\
V_n^{\delta_n} = \delta_n I + (1-\delta_n)V_n,\\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n V_n^{\delta_n} P_C(y_n - \lambda A y_n),
\end{cases}
$$

where $\lambda, \mu \in (0, 2\alpha)$, $\{\delta_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\eta_i^{(n)}\}$ are sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\lim_{n\to\infty} \mid \delta_{n+1} - \delta_n \mid = 0$;
(v) $\lim_{n\to\infty} \mid \eta_i^{(n+1)} - \eta_i^{(n)} \mid = 0$, *for* $1 \leq i \leq N$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{\cap_{i=1}^N F(S_i)\cap\Omega} f(\bar{x})$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (1.4), *where* $\bar{y} = P_C(\bar{x} - \mu A\bar{x})$.

Proof. Set $B = A$ in Theorem 3.1. Then from Theorem 3.1 we obtain the desired result. □

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the set $\bigcap_{i=1}^N F(S_i) \bigcap \Omega \neq \emptyset$. Assume $\{\eta_i^{(n)}\}_{i=1}^N$ are sequences of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \geq 1$ and $\inf_{n\geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping V_n be defined by $V_n = \sum_{i=1}^N \eta_i^{(n)} S_i$. Suppose $u, x_1 \in C$ and $\{x_n\}$ is generated by the following algorithm:*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ V_n^{\delta_n} = \delta_n I + (1 - \delta_n) V_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n V_n^{\delta_n} P_C(y_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, $\{\delta_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\eta_i^{(n)}\}$ are sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\lim_{n\to\infty} \mid \delta_{n+1} - \delta_n \mid = 0$;
(v) $\lim_{n\to\infty} \mid \eta_i^{(n+1)} - \eta_i^{(n)} \mid = 0$, *for* $1 \leq i \leq N$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{\cap_{i=1}^N F(S_i)\cap\Omega} u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (1.3), *where* $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof. Set $f(x_n) = u$ for all $n \geq 1$ in Theorem 3.1. Then by Theorem 3.1 we obtain the desired result. □

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