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SHRINKING PROJECTION METHODS FOR A FAMILY OF MAXIMAL MONOTONE OPERATORS

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Abstract. We deal with a common zero problem for a countable family of maximal monotone operators. Using the shrinking projection method, we obtain strong convergence of an iterative sequence. This result can be applied to a system of equilibrium problems and the iterative sequence converges strongly to their common solution.

1. Introduction

Let E be a real Banach space and A a set-valued mapping of E into E^* such that

$$\langle x - y, x^* - y^* \rangle \geq 0$$

whenever $x, y \in E$ and $x^*, y^* \in E^*$ satisfy that $x^* \in Ax$ and $y^* \in Ay$. This mapping is called a monotone operator of E into E^*. Finding a zero of a monotone operator is a significant problem in nonlinear analysis because it contains various important problems such as convex optimization problems, saddle point problems, equilibrium problems, and others. We call it a zero point problem for A. It is also closely related with fixed point problems for nonexpansive mappings because the resolvent operator for A is a nonexpansive mapping in the case where E is a Hilbert space.

In 2008, Takahashi, Takeuchi, and Kubota established a strong convergence theorem by a new type of projection method.

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Theorem 1.1 (Takahashi-Takeuchi-Kubota [12]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0,a]$, where $0 < a < 1$. For a point $x \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} &= \{z \in H : \|z - y_n\| \leq \|z - x_n\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

This iterative scheme is known as the shrinking projection method. We remark that their original result is a strongly convergent iterative scheme to a common fixed point of a family of nonexpansive mappings under certain conditions.

Inspired by this result, Kimura and Takahashi [8] obtained the following convergence theorem for the zero point problem for a maximal monotone operator defined on a Banach space.

Theorem 1.2 (Kimura-Takahashi [8]). *Let E be a strictly convex reflexive Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let A be a maximal monotone operator of E into E^* satisfying that $A^{-1}0 \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ and $\{\rho_n\} \subset]0,\infty[$ be real sequences such that $\liminf_{n\to\infty} \alpha_n < 1$ and that $\inf_{n\in\mathbb{N}} \rho_n > 0$. For a point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in E$, $C_1 = E$, and*

$$\begin{aligned} y_n &= J^*(\alpha_n J x_n + (1-\alpha_n)J(J+\rho_n A)^{-1} J x_n), \\ C_{n+1} &= \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x \in C$, where P_K is the metric projection of E onto a nonempty closed convex subset K of E.

On the other hand, using another type of resolvent, Kimura, Nakajo, and Takahashi proved a strong convergence theorem for a countable family of monotone operators as follows:

Theorem 1.3 (Kimura-Nakajo-Takahashi [7]). *Let E be a strictly convex, smooth, and reflexive Banach space having the Kadec-Klee property. Let $\{A_j : j \in I\}$ be a countable family of maximal monotone operators of E into E^* and suppose that $Z = \bigcap_{j\in I} A_j^{-1}0 \neq \emptyset$. Let $\{\rho_n\} \subset]0,\infty[$ be a real sequence. Let $i : \mathbb{N} \to I$ and suppose that for each $j \in I$, there exists a subsequence $\{n_k\}$ of*

$\mathbb{N}$ *such that* $i(n_k) = j$ *for every* $k \in \mathbb{N}$ *and that* $\inf_{k\in\mathbb{N}} \rho_{n_k} > 0$. *Let* $x \in E$ *and generate a sequence* $\{x_n\}$ *by the following iterative scheme:* $x_1 \in E$, $C_1 = E$, *and*

$$\begin{aligned} y_n &= (I + \rho_n J^* A_{i(n)})^{-1} x_n, \\ C_{n+1} &= \{u \in E : \langle y_n - u, J(x_n - y_n)\rangle \geq 0\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_Z x \in E$.

Motivated by these results, we prove strong convergence of an iterative scheme by the shrinking projection method for a countable family for maximal monotone operators, which generalizes Theorem 1.2. This result can be applied to a system of equilibrium problems and the iterative sequence converges to their common solution.

2. Preliminaries

In what follows, E is a real Banach space with a norm $\|\cdot\|$. The dual space of E is denoted by E^* and its norm is also denoted by $\|\cdot\|$.

The normalized duality mapping on E is denoted by J. That is,

$$Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^*\rangle = \|x^*\|^2\}$$

for $x \in E$, where $\langle x, x^*\rangle = x^*(x) \in \mathbb{R}$. Suppose that E is strictly convex, reflexive, and smooth. Then, J is a single-valued one-to-one mapping of E onto E^*. In this case, J^{-1} coincides with the duality mapping J^* on E^*. If the norm of E is Fréchet differentiable, then J is norm-to-norm continuous.

We say E has the Kadec-Klee property if a weakly convergent sequence $\{x_n\}$ of E with a limit x_0 converges strongly to x_0 whenever $\{\|x_n\|\}$ converges to $\|x_0\|$. It is known that E^* has a Fréchet differential norm if and only if E is reflexive, is strictly convex, and has the Kadec-Klee property. For more details, see [11].

Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. Define $\text{s-Li}_n K_n$ and $\text{w-Ls}_n K_n$ as follows: $x \in \text{s-Li}_n K_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in K_n$ for all $n \in \mathbb{N}$; $y \in \text{w-Ls}_n K_n$ if and only if there exist a subsequence $\{K_{n_i}\}$ of $\{K_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in K_{n_i}$ for all $i \in \mathbb{N}$. If K_0 satisfies that

$$K_0 = \operatorname*{s-Li}_n K_n = \operatorname*{w-Ls}_n K_n,$$

then we say that $\{K_n\}$ converges to K_0 in the sense of Mosco [10] and we write $K_0 = \text{M-lim}_{n\to\infty} K_n$. It is easy to show that if $\{K_n\}$ is decreasing with respect to inclusion, then $\{K_n\}$ converges to $\bigcap_{n=1}^{\infty} K_n$ in the sense of Mosco. For more details, see [3].

Let K be a nonempty closed convex subset of a strictly convex reflexive Banach space E. Then, for arbitrarily fixed $x \in E$, there exists a unique point $y_x \in K$ such that

$$\|x - y_x\| = \inf_{y \in K} \|x - y\| .$$

Using this point, we define the metric projection $P_K : E \to K$ by $P_K x = y_x$ for $x \in E$.

The following theorem proved by Tsukada [13] plays an important role in the main result.

Theorem 2.1 (Tsukada [13]). *Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of a strictly convex reflexive Banach space E having the Kadec-Klee property. If $\{K_n\}$ converges to a nonempty closed convex subset K_0 of E in the sense of Mosco, then $\{P_{K_n} x\}$ converges strongly to $P_{K_0} x$ for each $x \in E$.*

A set-valued mapping $A : E \rightrightarrows E^*$ is said to be monotone if the inequality

$$\langle x - y, x^* - y^* \rangle \geq 0$$

holds for any $x, y \in E$ and $x^*, y^* \in E^*$ satisfying $x^* \in Ax$ and $y^* \in Ay$. A monotone operator A is said to be maximal if the graph of A is not a proper subset of the graph of any other monotone operator. A point $z \in E$ satisfying that $0 \in Az$ is called a zero of A and the set of such points is denoted by $A^{-1}0$. We know that if A is maximal monotone, then $A^{-1}0$ is a closed convex subset of E.

Suppose that a Banach space E is strictly convex, reflexive, and smooth. Then, for a maximal monotone operator $A : E \rightrightarrows E^*$ and a positive real number ρ, a mapping $J + \rho A : E \rightrightarrows E^*$ has a single-valued inverse. Further, its range is the whole space E^*. Therefore, we may define a single-valued mapping $(J + \rho A)^{-1} J : E \to E$, which is called a resolvent of A for $\rho > 0$. For more details, see [2, 11] and others.

Let E be a smooth Banach space. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$. We know several fundamental properties of ϕ as follows: $\phi(x, y) \geq 0$ for all $x, y \in E$. For a sequence $\{y_n\}$ in E and $x \in E$, $\{y_n\}$ is bounded if and only if $\{\phi(x, y_n)\}$ is bounded. Suppose that E be a strictly convex reflexive smooth Banach space. Let $A : E \rightrightarrows E^*$ be maximal monotone and $\rho > 0$. Then, we know from [9] that

$$\phi(z, (J + \rho A)^{-1} Jx) \leq \phi(z, (J + \rho A)^{-1} Jx) + \phi((J + \rho A)^{-1} Jx, x) \leq \phi(z, x)$$

for any $z \in A^{-1}0$ and $x \in E$. For more details of ϕ, see, for example, [5].

3. The main result

We consider a common zero problem for a countable family $\{A_j : j \in I\}$ of maximal monotone operators defined on a real Banach space E. The set of solutions of this problem is

$$\bigcap_{j\in I} A_j^{-1}0 = \{z \in E : A_j z = 0 \text{ for all } j \in I\}.$$

Using the shrinking projection method, we obtain strong convergence of an iterative scheme.

Theorem 3.1. *Let E be a strictly convex reflexive Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $\{A_j : j \in I\}$ be a countable family of maximal monotone operators of E into E^* and suppose that $Z = \bigcap_{j\in I} A_j^{-1}0 \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ and $\{\rho_n\} \subset]0,\infty[$ be real sequences. Let $i : \mathbb{N} \to I$ and suppose that for each $j \in I$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $i(n_k) = j$ for every $k \in \mathbb{N}$, $\lim_{k\to\infty} \alpha_{n_k} < 1$, and $\inf_{k\in\mathbb{N}} \rho_{n_k} > 0$. Let $x \in E$ and generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in E$, $C_1 = E$, and*

$$\begin{aligned} y_n &= J^*(\alpha_n Jx_n + (1-\alpha_n)J(J+\rho_n A_{i(n)})^{-1}Jx_n), \\ C_{n+1} &= \{u \in E : \phi(u,y_n) \leq \phi(u,x_n)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}}x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_Z x \in E$, where P_K is the metric projection of E onto a nonempty closed convex subset K of E.

Proof. For the well-definedness of the iterative sequence $\{x_n\}$, we suppose that $x_1, x_2, \ldots, x_n$ are defined and $C_1, C_2, \ldots, C_n$ are nonempty closed convex subsets of E which include Z. Then, since

$$\begin{aligned} C_{n+1} &= \{u \in E : \phi(u,y_n) \leq \phi(u,x_n)\} \cap C_n \\ &= \{u \in E : \langle u, Jx_n - Jy_n\rangle + (\|y_n\|^2 - \|x_n\|^2)/2 \leq 0\} \cap C_n, \end{aligned}$$

C_{n+1} is a closed convex subset of E. Let $w_n = (J+\rho_n A_{i(n)})^{-1}Jx_n$ for $n \in \mathbb{N}$ and $z \in Z = \bigcap_{j\in I} A_j^{-1}0$. Since $\phi(z,w_n) \leq \phi(z,x_n)$ for $n \in \mathbb{N}$, we have that

$$\begin{aligned} &\phi(z,y_n) \\ &= \|z\|^2 - 2\langle z, \alpha_n Jx_n + (1-\alpha_n)Jw_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)Jw_n\|^2 \\ &\leq \|z\|^2 - 2\alpha_n\langle z, Jx_n\rangle - 2(1-\alpha_n)\langle z, Jw_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|w_n\|^2 \\ &= \alpha_n\left(\|z\|^2 - 2\langle z, Jx_n\rangle + \|x_n\|^2\right) + (1-\alpha_n)\left(\|z\|^2 - 2\langle z, Jw_n\rangle + \|w_n\|^2\right) \\ &= \alpha_n\phi(z,x_n) + (1-\alpha_n)\phi(z,w_n) \\ &\leq \phi(z,x_n). \end{aligned}$$

It follows that $z \in C_{n+1}$ and hence $Z \subset C_{n+1}$. Since Z is nonempty, so is C_{n+1}. Therefore $x_{n+1} = P_{C_{n+1}} x_n$ is well defined. Since $x_1 \in E$ is given and $C_1 = E$ is obviously nonempty, closed, convex, and contains Z, we obtain that $\{x_n\}$ is well defined by induction.

By definition, a sequence $\{C_n\}$ is decreasing with respect to inclusion. Thus we have that

$$\operatorname{M-lim}_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n \supset Z \neq \emptyset.$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.1 we have that $\{x_n\} = \{P_{C_n} x\}$ converges strongly to $x_0 = P_{C_0} x \in E$. This also implies that $\lim_{n\to\infty} \phi(x_0, x_n) = 0$. Since x_0 belongs to C_n for every $n \in \mathbb{N}$, we get that $0 \leq \phi(x_0, y_n) \leq \phi(x_0, x_n)$ for $n \in \mathbb{N}$ and, as $n \to \infty$, we have that

$$\lim_{n\to\infty} \phi(x_0, y_n) = 0.$$

Since

$$0 \leq \lim_{n\to\infty} (\|x_0\| - \|y_n\|)^2 \leq \lim_{n\to\infty} \phi(x_0, y_n) = 0,$$

we have that $\lim_{n\to\infty} \|y_n\| = \|x_0\|$. We also have that $\{Jy_n\} \subset E^*$ is bounded. For fixed $j \in I$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $i(n_k) = j$ for every $k \in \mathbb{N}$, $\{\alpha_{n_k}\}$ converges to $\alpha_0 \in [0, 1[$, $\inf_{k\in\mathbb{N}} \rho_{n_k} > 0$, and $\{Jy_{n_k}\}$ converges weakly to $y_0^* \in E^*$. Then we have that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \phi(x_0, y_{n_k}) \\
&= \lim_{k\to\infty} \left(\|x_0\|^2 - 2\langle x_0, Jy_{n_k}\rangle + \|y_{n_k}\|^2 \right) \\
&= 2\|x_0\|^2 - 2 \lim_{k\to\infty} \langle x_0, Jy_{n_k}\rangle \\
&= 2(\|x_0\|^2 - \langle x_0, y_0^*\rangle)
\end{aligned}$$

and thus

$$\|x_0\|^2 = \langle x_0, y_0^*\rangle \leq \|x_0\| \|y_0^*\| \leq \|x_0\| \lim_{k\to\infty} \|Jy_{n_k}\| = \|x_0\| \lim_{k\to\infty} \|y_{n_k}\| = \|x_0\|^2.$$

It follows that $\|x_0\|^2 = \langle x_0, y_0^*\rangle = \|y_0^*\|^2$ and hence $Jx_0 = y_0^*$. We also have that

$$\|Jx_0\| = \|x_0\| = \lim_{k\to\infty} \|y_{n_k}\| = \lim_{k\to\infty} \|Jy_{n_k}\|.$$

Since E is reflexive and its norm is Fréchet differentiable, E^* has the Kadec-Klee property. Therefore $\{Jy_{n_k}\}$ converges strongly to Jx_0. We also have that

$$\begin{aligned} Jy_{n_k} - Jx_{n_k} &= JJ^*(\alpha_{n_k} Jx_{n_k} + (1-\alpha_{n_k})Jw_{n_k}) - Jx_{n_k} \\ &= \alpha_{n_k} Jx_{n_k} + (1-\alpha_{n_k})Jw_{n_k} - Jx_{n_k} \\ &= (1-\alpha_{n_k})(Jw_{n_k} - Jx_{n_k}) \end{aligned}$$

for every $k \in \mathbb{N}$, and it follows that

$$0 = \lim_{k\to\infty} \|Jy_{n_k} - Jx_0\| = (1-\alpha_0) \lim_{k\to\infty} \|Jw_{n_k} - Jx_{n_k}\|.$$

Since $\alpha_0 < 1$, we have that $\lim_{k\to\infty} \|Jx_0 - Jw_{n_k}\| = 0$. From the Fréchet differentiability of the norm on E^*, J^* is norm-to-norm continuous and thus we have that $\{w_{n_k}\}$ converges strongly to x_0. Let $v \in E$ and $v^* \in E^*$ be such that $v^* \in A_j v$. Since $i(n_k) = j$ for any $k \in \mathbb{N}$, we have that

$$w_{n_k} = (J + \rho_{n_k} A_{i(n_k)})^{-1} Jx_{n_k} = (J + \rho_{n_k} A_j)^{-1} Jx_{n_k}$$

and thus

$$\frac{1}{\rho_{n_k}}(Jx_{n_k} - Jw_{n_k}) \in A_j w_{n_k}$$

for every $k \in \mathbb{N}$. Using the monotonicity of A_j, we have that

$$\left\langle w_{n_k} - v, \frac{1}{\rho_{n_k}}(Jx_{n_k} - Jw_{n_k}) - v^* \right\rangle \geq 0$$

for $k \in \mathbb{N}$. As $k \to \infty$ we have that $\langle x_0 - v, 0 - v^* \rangle \geq 0$. Since A_j is maximal monotone, it follows that $0 \in A_j x_0$. Therefore we obtain that $x_0 \in \bigcap_{j\in I} A_j^{-1}0 = Z$ and hence $x_0 = P_Z x$, which is the desired result. □

In the case where the number of the operators is finite, that is, the index set is $I = \{0, 1, 2, \ldots, N-1\}$, we may use a mapping $i : \mathbb{N} \to I$ defined by $i(n) = n \bmod N$ for $n \in \mathbb{N}$. Thus we obtain the following result.

Theorem 3.2 (Kimura [6]). *Let E be a strictly convex reflexive Banach space having the Kadec-Klee property and a Fréchet differentiable norm. Let $\{A_i : i \in I\}$ be a finite family of maximal monotone operators of E into E^* with an index set $I = \{0, 1, 2, \ldots, N-1\}$ and suppose that $Z = \bigcap_{i\in I} A_i^{-1}0 \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ and $\{\rho_n\} \subset]0, \infty[$ be sequences such that $\liminf_{k\to\infty} \alpha_{Nk+i} < 1$ for every $i \in I$ and that $\inf_{n\in\mathbb{N}} \rho_n > 0$. For a point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in E$, $C_1 = E$, and*

$$\begin{aligned} y_n &= J^*(\alpha_n Jx_n + (1-\alpha_n)J(J + \rho_n A_{(n \bmod N)})^{-1} Jx_n), \\ C_{n+1} &= \{u \in E : \phi(u, y_n) \leq \phi(u, x_n)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_Z x \in E$.

4. Application to an infinite system of equilibrium problems

Let C be a nonempty convex subset of a Banach space E. For a function $f : C \times C \to \mathbb{R}$, we consider the following problem which is called an equilibrium problem for f: Find $x \in C$ such that $f(x, y) \geq 0$ for all $y \in C$. The set of solutions to this problem is denoted by $EP(f)$. We assume the following conditions:

(i) $f(x, x) = 0$ for every $x \in C$;
(ii) $f(x, y) + f(y, x) \leq 0$ for every $x, y \in C$;
(iii) $f(x, \cdot)$ is convex and lower semicontinuous for every $x \in C$;
(iv) $\limsup_{t \downarrow 0} f(ty + (1 - t)x, y) \leq f(x, y)$ for every $x, y \in C$.

Equilibrium problems are closely related to the zero point problems for maximal monotone operators. Indeed, suppose that C is a closed convex subset of a strictly convex reflexive smooth Banach space E. For $f : C \times C \to \mathbb{R}$ satisfying four conditions above, define $A_f : E \rightrightarrows E^*$ by

$$A_f x = \begin{cases} \{x^* \in E^* : f(x, y) \geq \langle y - x, x^* \rangle \text{ for all } y \in C\} & (x \in C) \\ \emptyset & (x \notin C). \end{cases}$$

Then, A_f is a maximal monotone operator satisfying $A_f^{-1} 0 = EP(f)$. In this case, the resolvent $z = (J + \rho A_f)^{-1} Jx$ for $\rho > 0$ and $x \in E$ is the unique element which satisfies

$$f(z, y) + \frac{1}{\rho} \langle y - z, Jz - Jx \rangle \geq 0$$

for all $y \in C$. For more details, see [4, 1].

Let us consider an infinite system of equilibrium problems for $\{f_n\}$. Using the fact mentioned above, we may apply Theorem 3.1 to approximate a common solution $x_0 \in \bigcap_{n=1}^{\infty} EP(f_n)$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space E having the Kadec-Klee property and a Fréchet differentiable norm. Let $\{f_n\}$ be a countable family of functions of $C \times C$ into $\mathbb{R}$ satisfying the conditions* (i)–(iv) *and suppose that the set of common solutions $Z = \bigcap_{n=1}^{\infty} EP(f_n)$ to the equilibrium problems for $\{f_n\}$ is nonempty. Let $\{\alpha_n\} \subset [0, 1]$ and $\{\rho_n\} \subset]0, \infty[$ be real sequences. Let $i : \mathbb{N} \to I$ and suppose that for each $j \in I$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $i(n_k) = j$ for every $k \in \mathbb{N}$, $\lim_{k\to\infty} \alpha_{n_k} < 1$, and $\inf_{k \in \mathbb{N}} \rho_{n_k} > 0$. Let $x \in E$ and generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in E$, $C_1 = E$, and*

$$\begin{aligned} y_n &= J^*(\alpha_n J x_n + (1 - \alpha_n) J F_{\rho_n} x_n), \\ C_{n+1} &= \{u \in E : \phi(u, y_n) \leq \phi(u, x_n)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$, where $F_{\rho_n} x_n$ is the unique element in C satisfying that

$$f_n(F_{\rho_n} x_n, y) + \frac{1}{\rho_n} \langle y - F_{\rho_n} x_n, JF_{\rho_n} x_n - Jx_n \rangle \geq 0$$

for all $y \in C$. Then, $\{x_n\}$ converges strongly to $x_0 = P_Z x \in \bigcap_{n=1}^{\infty} EP(f_n)$.

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