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SOME UNIQUE FIXED POINT THEOREMS FOR RATIONAL EXPRESSIONS IN CONE b-METRIC SPACES

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Abstract. In this paper, we establish some unique fixed point theorems for rational expressions of Dass and Gupta and an almost Dass and Gupta type in the setting of cone b-metric spaces. Our results extend and generalize the corresponding results of [7, 15] and many others from the existing literature.

1. Introduction and preliminaries

Fixed point theory plays a very important and significant role in the development of nonlinear analysis. In this direction, the first important result was proved by Banach in 1922 for contraction mapping in complete metric space, known as the Banach contraction principle [3]. The Banach contraction principle with rational expressions have been expanded and some fixed and common fixed point theorems have been obtained in [9], [10] and [15].

In 1989, Bakhtin [4] introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces (see, [5, 6]). In 2007, Huang and Zhang [12] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 14, 18, 21] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

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In [13], Hussain and Shah introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces, in 2011. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space.

In this paper, we establish some unique fixed theorems for rational expressions in the framework of cone b-metric spaces. Our results extend and generalize several results from the existing literature (see, e.g., [7, 15, 19]).

Definition 1.1. ([12]) Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(c_1) P is closed, nonempty and $P \neq \{0\}$;
(c_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [20]).

There exist two kinds of cones-normal (with the normal constant M) and non-normal ones (see [8]).

Let E be a real Banach space, $P \subset E$ a cone and '$\leq$' partial ordering defined by P. Then P is called normal if there is a number $M > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq M\|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number M satisfying (1.1) is called the normal constant of P.

Example 1.2. ([20]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$ and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$. Hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3. ([12, 22]) Let X be a nonempty set. Suppose that the mapping $d \colon X \times X \to E$ satisfies:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ $x, y, z \in X$.

Then d is called a cone metric [12] on X and (X, d) is called a cone metric space [12].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.4. ([12]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d \colon X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $M = 1$.

Example 1.5. ([17]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0 \text{ for all } n\}$, (X, ρ) be a metric space and $d \colon X \times X \to E$ defined by $d(x, y) = \{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.6. ([13]) Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d \colon X \times X \to E$ is said to be cone b-metric if and only if, for all x, y, $z \in X$, the following conditions are satisfies:

(b_1) $0 \leq d(x, y)$ with $x \neq y$ and $d(x, y) = 0 \iff x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Remark 1.7. The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric space which are not cone metric space.

Example 1.8. ([11]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d \colon X \times X \to E$ defined by $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.9. ([11]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}_+$ defined by $d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}$, where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X,d) is a cone b-metric space with the coefficient $s = 2^{1/p} > 1$, but not a cone metric space.

Example 1.10. ([11]) Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d\colon X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

Definition 1.11. ([13]) Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

- $\{x_n\}$ is a Cauchy sequence whenever, if for every $c \in E$ with $0 \ll c$, then there is a natural number N such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$;
- $\{x_n\}$ converges to x whenever, for every $c \in E$ with $0 \ll c$, then there is a natural number N such that for all $n \geq N$, $d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
- (X,d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Let us recall ([12]) that if P is a normal solid cone, then $x_n \in X$ is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$. Further, $x_n \in X$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

In the following (X,d) will stands for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and '$\leq$' is partial ordering in E with respect to P.

Definition 1.12. ([2]) Let (X,d) be a complete partially ordered metric space. A self mapping $T\colon X \to X$ is called an almost Dass and Gupta contraction (Arshad et al. contraction) if it satisfies the following condition:

$$\begin{aligned} d(Tx,Ty) \;\leq\; & \frac{\alpha\, d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)} + \beta\, d(x,y) \\ & + L\, \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}, \end{aligned} \tag{1.3}$$

for all $x, y \in X$, where $L \geq 0$ and $\alpha,\ \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Definition 1.13. ([7]) Let (X, d) be a metric space. A self mapping $T\colon X \to X$ is called Dass and Gupta contraction if it satisfies the following condition:

$$d(Tx, Ty) \quad \leq \quad \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y), \tag{1.4}$$

for all $x, y \in X$ and $\alpha,\ \beta \in [0, 1)$ with $\alpha + \beta < 1$.

2. Main results

In this section, we shall prove some fixed point theorems for rational expressions in the framework of cone b-metric spaces.

Theorem 2.1. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P be a normal cone with normal constant M. Suppose that the mapping $T\colon X \to X$ satisfies the rational contraction* (1.3) *for all $x, y \in X$, where $L \geq 0$ and $\alpha,\ \beta \in [0, 1)$ with $s\alpha + s\beta < 1$. Then T has a unique fixed point in X.*

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (1.3), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \frac{\alpha\, d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, x_n) \\ &\quad + L \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \frac{\alpha\, d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, x_n) \\ &\quad + L \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &\leq \alpha\, d(x_n, x_{n+1}) + \beta\, d(x_{n-1}, x_n). \end{aligned} \tag{2.1}$$

This implies that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \left(\frac{\beta}{1-\alpha}\right) d(x_{n-1}, x_n) \\ &= k\, d(x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

where $k = \frac{\beta}{1-\alpha}$, since $s\alpha + s\beta < 1$, it is clear that $0 < k < 1/s$. By induction, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq k\, d(x_{n-1}, x_n) \leq k^2\, d(x_{n-2}, x_{n-1}) \leq \ldots \\ &\leq k^n\, d(x_0, x_1). \end{aligned} \tag{2.3}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq sk^n d(x_1, x_0) + s^2 k^{n+1} d(x_1, x_0) + s^3 k^{n+2} d(x_1, x_0) \\ &\quad + \cdots + s^m k^{n+m-1} d(x_1, x_0) \\ &= sk^n [1 + sk + s^2k^2 + s^3k^3 + \cdots + (sk)^{m-1}] d(x_1, x_0) \\ &\leq \Big[\frac{sk^n}{1 - sk}\Big] d(x_1, x_0). \end{aligned}$$

Since P is a normal cone with normal constant M, so we get $\|d(x_n, x_m)\| \leq M \frac{sk^n}{1-sk} \|d(x_1, x_0)\|$. This implies $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$, since $0 < sk < 1$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone b-metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now, since

$$\begin{aligned} d(Tp, p) &\leq s[d(Tp, Tx_n) + d(Tx_n, p)] \\ &= sd(Tp, Tx_n) + sd(Tx_n, p) \\ &\leq s\Big[\frac{\alpha\, d(x_n, Tx_n)[1 + d(p, Tp)]}{1 + d(p, x_n)} + \beta\, d(p, x_n) \\ &\quad + L \min\{d(p, Tp), d(p, Tx_n), d(x_n, Tp)\}\Big] + sd(Tx_n, p) \\ &= s\Big[\frac{\alpha\, d(x_n, x_{n+1})[1 + d(p, Tp)]}{1 + d(p, x_n)} + \beta\, d(p, x_n) \\ &\quad + L \min\{d(p, Tp), d(p, x_{n+1}), d(x_n, Tp)\}\Big] + sd(x_{n+1}, p). \end{aligned}$$

As $x_n \to p$ and $x_{n+1} \to p$ as $n \to \infty$, we get

$$\|d(Tp, p)\| \leq M\Big[s\beta\|d(p, x_n)\| + s\|d(x_{n+1}, p)\|\Big] \to 0 \text{ as } n \to \infty.$$

Hence $\|d(Tp, p)\| = 0$. Thus we get $Tp = p$, that is, p is a fixed point of T.

Uniqueness. Let p' be another fixed point of T, that is, $Tp' = p'$ such that $p \neq p'$. Then from (1.3), we have

$$d(p, p') = d(Tp, Tp')$$

$$\begin{aligned} &\leq \frac{\alpha\, d(p', Tp')[1 + d(p, Tp)]}{1 + d(p, p')} + \beta\, d(p, p') \\ &\quad + L \min\{d(p, Tp), d(p, Tp'), d(p', Tp)\} \\ &\leq \frac{\alpha\, d(p', p')[1 + d(p, p)]}{1 + d(p, p')} + \beta\, d(p, p') \\ &\quad + L \min\{d(p, p), d(p, p'), d(p', p)\} \\ &\leq \beta\, d(p, p') \\ &< d(p, p'), \end{aligned}$$

since $0 < \beta < 1$, which is a contradiction. Hence $\|d(p, p')\| = 0$ and so $p = p'$. Thus p is a unique fixed point of T. This completes the proof. □

From Theorem 2.1, we obtain the following result as corollary.

Corollary 2.2. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P be a normal cone with normal constant M. Suppose that the mapping $T\colon X \to X$ satisfies the Dass and Gupta rational contraction* (1.4) *for all $x, y \in X$ and α, $\beta \in [0, 1)$ with $s\alpha + s\beta < 1$. Then T has a unique fixed point in X.*

Proof. The proof of Corollary 2.2 immediately follows from Theorem 2.1 by taking $L = 0$. This completes the proof. □

Theorem 2.3. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P be a normal cone with normal constant M. Suppose that the mapping $T\colon X \to X$ satisfies the rational contraction:*

$$\begin{aligned} d(Tx, Ty) &\leq \alpha\, d(x, y) + \frac{\beta\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \\ &\quad + \frac{\gamma\,[d(y, Ty) + d(y, Tx)]}{1 + d(y, Ty) d(y, Tx)}, \end{aligned} \tag{2.4}$$

for all $x, y \in X$ and α, β, $\gamma \in [0, 1)$ with $s\alpha + \beta + \gamma < 1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.4), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha\, d(x_{n-1}, x_n) + \frac{\beta\, d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} \\ &\quad + \frac{\gamma\,[d(x_n, Tx_n) + d(x_n, Tx_{n-1})]}{1 + d(x_n, Tx_n) d(x_n, Tx_{n-1})} \\ &= \alpha\, d(x_{n-1}, x_n) + \frac{\beta\, d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} \\ &\quad + \frac{\gamma\,[d(x_n, x_{n+1}) + d(x_n, x_n)]}{1 + d(x_n, x_{n+1}) d(x_n, x_n)} \\ &\leq \alpha\, d(x_{n-1}, x_n) + (\beta + \gamma) d(x_n, x_{n+1}). \end{aligned} \tag{2.5}$$

This implies that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \Big(\frac{\alpha}{1 - \beta - \gamma}\Big) d(x_{n-1}, x_n) \\ &= f\, d(x_{n-1}, x_n), \end{aligned} \tag{2.6}$$

where $f = \Big(\frac{\alpha}{1-\beta-\gamma}\Big)$, since $s\alpha + \beta + \gamma < 1$, it is clear that $0 < f < 1/s$. By induction, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq f\, d(x_{n-1}, x_n) \leq f^2\, d(x_{n-2}, x_{n-1}) \leq \dots \\ &\leq f^n\, d(x_0, x_1). \end{aligned} \tag{2.7}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \dots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq sf^n d(x_1, x_0) + s^2 f^{n+1} d(x_1, x_0) + s^3 f^{n+2} d(x_1, x_0) \\ &\quad + \dots + s^m f^{n+m-1} d(x_1, x_0) \\ &= sf^n[1 + sf + s^2 f^2 + s^3 f^3 + \dots + (sf)^{m-1}] d(x_1, x_0) \\ &\leq \Big[\frac{sf^n}{1 - sf}\Big] d(x_1, x_0). \end{aligned}$$

Since P is a normal cone with normal constant M, so we get $\|d(x_n, x_m)\| \leq M \frac{sf^n}{1-sf} \|d(x_1, x_0)\|$. This implies $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$, since $0 <$

$sf < 1$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now, since

$$\begin{aligned}
d(Tu, u) &\leq s[d(Tu, Tx_n) + d(Tx_n, u)] \\
&= sd(Tu, Tx_n) + sd(Tx_n, u) \\
&\leq s\Big[\alpha\, d(u, x_n) + \frac{\beta\, d(x_n, Tx_n)[1 + d(u, Tu)]}{1 + d(u, x_n)} \\
&\quad + \frac{\gamma\, [d(x_n, Tx_n) + d(x_n, Tu)]}{1 + d(x_n, Tx_n)d(x_n, Tu)}\Big] + sd(Tx_n, u) \\
&= s\Big[\alpha\, d(u, x_n) + \frac{\beta\, d(x_n, x_{n+1})[1 + d(u, Tu)]}{1 + d(u, x_n)} \\
&\quad + \frac{\gamma\, [d(x_n, x_{n+1}) + d(x_n, Tu)]}{1 + d(x_n, x_{n+1})d(x_n, Tu)}\Big] + sd(x_{n+1}, u).
\end{aligned}$$

As $x_n \to u$ and $x_{n+1} \to u$ as $n \to \infty$, we get

$$(1 - \gamma)\|d(Tu, u)\| \leq M\Big[s\alpha\|d(u, x_n)\| + s\|d(x_{n+1}, u)\|\Big] \to 0 \text{ as } n \to \infty.$$

Hence $(1 - \gamma)\|d(Tu, u)\| = 0 \Rightarrow \|d(Tu, u)\| = 0$, since $(1 - \gamma) > 0$. Thus we get $Tu = u$, that is, u is a fixed point of T.

Uniqueness. Let v be another fixed point of T, that is, $Tv = v$ such that $u \neq v$. Then from (2.4), we have

$$\begin{aligned}
d(u, v) &= d(Tu, Tv) \\
&\leq \alpha\, d(u, v) + \frac{\beta\, d(v, Tv)[1 + d(u, Tu)]}{1 + d(u, v)} \\
&\quad + \frac{\gamma\, [d(v, Tv) + d(v, Tu)]}{1 + d(v, Tv)d(v, Tu)} \\
&\leq \alpha\, d(u, v) + \frac{\beta\, d(v, v)[1 + d(u, u)]}{1 + d(u, v)} \\
&\quad + \frac{\gamma\, [d(v, v) + d(v, u)]}{1 + d(v, v)d(v, u)} \\
&\leq (\alpha + \gamma)d(u, v) \\
&< d(u, v),
\end{aligned}$$

since $0 < (\alpha + \gamma) < 1$, which is a contradiction. Hence $\|d(u, v)\| = 0$ and so $u = v$. Thus u is a unique fixed point of T. This completes the proof. □

From Theorem 2.1 and 2.3, we obtain the following result as corollary.

Corollary 2.4. *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P be a normal cone with normal constant M. Suppose that the mapping $T\colon X \to X$ satisfies the contraction contraction*

$$d(Tx,Ty) \leq \alpha\, d(x,y),$$

for all $x, y \in X$ and $\alpha \in [0,1)$ with $s\alpha < 1$. Then T has a unique fixed point in X.

Remark 2.5. (i) Our results extend and generalize the corresponding results of Arshad et al. [2], Dass and Gupta [7] and Uthayakumar and Arockia prabakar [19] from complete partially ordered metric space, complete metric space and cone metric space to that setting of cone b-metric space considered in this paper.
(ii) Theorem 2.3 extends Theorem 3.3 of Sarwar and Rahman [16] from b-metric space to that setting of cone b-metric space considered in this paper.
(iii) Corollary 2.4 extends well known Banach contraction principle from complete metric space to that setting of complete cone b-metric space considered in this paper.
(iv) Our results also extends and generalizes several known results from the existing literature.

Example 2.6. Let $E = C_{\mathbb{R}}[0,1]$, $P = \{f \in E : f \geq 0\} \subset E$, $X = [0,\infty)$ and $d(x,y) = |x-y|^2\, e^t$. Then (X,d) is a cone b-metric space with the coefficient $s = 2$. But it is not a cone metric space. We consider the mappings $T\colon X \to X$ defined by $T(x) = \frac{1}{2}x$. Hence

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{1}{2}x - \frac{1}{2}y\right|^2 e^t = \frac{1}{4}\,|x-y|^2\, e^t \\ &\leq \frac{1}{2}\,|x-y|^2\, e^t = \frac{1}{2}\, d(x,y). \end{aligned}$$

Clearly $0 \in X$ is the unique fixed point of T.

3. Conclusion

In this paper, we establish some unique fixed point theorems for rational contractions in the setting of cone b-metric spaces. Our results extend and generalize several results from the existing literature (see, e.g., [2, 7, 16, 19] and many others).

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