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SET-VALUED FIXED POINT THEOREM BASED ON THE SUPER-TRAJECTORY IN COMPLETE SUPER HAUSDORFF METRIC SPACE

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Abstract. In this paper, a super Hausdorff metric is introduced and constructed, and the completeness of super Hausdorff metric space is studied. A new concept, the trajectory of set valued mapping is introduced, and by using the trajectory condition, the existence theorems for the set-valued fixed point and the fixed point of a new class of set valued mappings is proved. The obtained results seem to be general in nature.

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1. Introduction

Set-valued analysis [1] which was studied by Jean-Pierre Aubin and Hèlène Frankowska, is a useful extension of the mathematics analysis, have wide applications to many fields including, for example, control, differential games, game theory, variational inclusions, optimization, nonlinear programming, economics, and engineering sciences [2]-[7]. Specially, various variational inclusions which have been intensively developed by Bella [8], Huang-Tang-Liu [9] and Jeong [10] and studied by Ding [11], Verma [12], Huang [13], Fang and Huang [14], Lan-Cho-Verma [15], Fang-Huang-Thompson [16], Zhang et al. [17] and the authors [18]-[32] in recent years, is an important context in the set-valued inclusions problem, and is an important application of set-valued analysis.

On the other hand, the influence and function of Hausdorff metric spaces theory is extensive and profound in many fields such as topology, geometric, topology, set-valued analysis, variational inclusions, optimization, game theory, nonlinear programming and economics [33]-[42]. Further, to develop a theory of supper Hausdorff metric spaces for expressing the distance between the two set classes, set classes convergence and fixed point, is inevitable and useful in mathematics theory and application because of needing to solve the practical something.

For example, in the age of big data, that how to intelligently and efficiently discover domain users knowledge has been becoming a key problem in current research in the today, the age of big data. Since big data always is huge, heterogeneous, sparse, high dimensional, dynamic and real-time so that aiming at the above features of big data, some noes propose a multi-granularity data mining model for processing big data based on both data feature, users requirements and the multi-granularity cognitive mechanism of human being. Aiming at the above features of big data, this project will propose a multi-granularity data mining model for processing big data based on both data feature, users requirements and the multi-granularity cognitive mechanism of human being. The main contents of this research project include [43]-[47]. A multi-granularity automatic preliminary classification model be established for multi-source heterogeneous data is important. The preliminary classification process for big data can be expressed as follows:

Let $\Re$ be real, and $E = \{x_i | x_i \in \Re, 1 \leq i \leq n\}$ be a big data set(the n is a enough big natural number).

1. To make class of subset of E, $\widehat{F_1}$ =$\{\alpha^{(1)} \subseteq E | \alpha^{(1)}$ be family of nonempty closed subset of $E\}$ for founding domain users knowledge, where according to a metric of multi-granularity framework in big data, every set $\alpha^{(1)}$ is produced by the element $x_1, x_2, \cdots, x_n$.

2. Further, to make set class of some subsets of $\widehat{F_1}$, $\widehat{F_2} = \bigcup_{\alpha^{(1)} \subseteq E}\{\{\alpha^{(2)}|\alpha^{(2)} \subseteq \alpha^{(1)}\}$, $\alpha^{(2)}$ be family of nonempty closed subset of $\alpha^{(1)}$ } for mining knowledge in big data, where according to a supper Hausdorff metric of set class in $\widehat{F_2}$, every set $\alpha^{(2)}$ is produced by the element in $\alpha^{(1)}$.

3. To discover domain users knowledge by analyzing a element β_k in every set class $\{\alpha^{(2)}|\alpha^{(2)} \subseteq \alpha^{(1)}\}_k$ where some metric $\mathbf{h}(\beta_k, \{\alpha^{(2)}|\alpha^{(2)} \subseteq \alpha^{(1)}\}_k) < \delta_k$ and $\delta_k > 0$ is a granularity valued.

It is meaningful to deal with the above process and to change one into a mathematical model.

Let (X, d) be a metric space as d metric, $2^X = \{\alpha|\alpha \subseteq X, \alpha \neq \emptyset\}$ be the family of all nonempty subset of X, $CB(X) \subset 2^X$ be family of all nonempty bounded closed subset of X, $p(2^X) = \{A|A \subseteq 2^X, A \neq \emptyset\}$ be family of all nonempty class of subset of X, $\Omega(X) = \{A|A \subseteq CB(X)\} \subseteq p(2^X)$ be the family of nonempty bounded closed subset of X and $(CB(X), H)$ be a Hausdorff metric space induced by (X, d) as H Hausdorff metric, then it is very necessary and interesting to establish and study super Hausdorff metric spaces $(\Omega(X), \hbar)$ on the Hausdorff metric space $(CB(X), H)$.

Therefore, we introduce super Hausdorff metric spaces $(\Omega(X), \hbar)$ and study basic properties of the $(\Omega(X), \hbar)$ in this work, and refer to [1]-[49] and references contained therein.

Inspired and motivated by recent research work in this field, in this paper, a super Hausdorff metric is introduced and constructed, and the completeness of super Hausdorff metric space is studied. a new concept, the trajectory of set valued mapping is introduced, and by using the trajectory condition, the existence theorems for the set-valued fixed point and the fixed point of a new class of set valued mappings is proved. The obtained results seem to be general in nature.

2. Super Hausdorff metric spaces and completeness of ones

Let (X, d) be a metric space, $2^X = \{\alpha|\alpha \subseteq X, \alpha \neq \emptyset\}$ be the family of all nonempty subset of X, $CB(X) \subset 2^X$ be family of all nonempty bounded closed subset of X, $C(X) \subset CB(X)$ be family of all nonempty compact subsets of X, $p(2^X) = \{A|A \subseteq 2^X, A \neq \emptyset\}$ be family of all nonempty class of subset of X and denote the closure of a set $\alpha(\subseteq X)$ as $\overline{\alpha}$. For $x \in X$ and $\alpha \in 2^X$, let $d(x, \alpha) = \inf_{y \in \alpha} d(x, y)$. We denoted by H a Hausdorff metric induced by the metric d of X:

$$H(\alpha,\beta)=\max\left\{\sup_{x\in\alpha}d(x,\overline{\beta}),\sup_{y\in\beta}d(y,\overline{\alpha})\right\} \tag{2.1}$$

for $\alpha,\beta\in 2^X$ and $\overline{\alpha},\overline{\beta}\in CB(X)$.

Through this paper we will use the following concepts and notations.

Definition 2.1. Let $(CB(X),H)$ be a Hausdorff metric space induced by (X,d). Let $\alpha_n\in CB(X)(n=1,2,...)$, $\alpha\in CB(X)$. If $\lim_{n\to\infty}H(\alpha_n,\alpha)=0$, then the $\{\alpha_n\}_1^\infty$, the subset sequence of X is side to be convergence to α in Hausdorff metric $H(.,.)$.

Let $\{\alpha_n\}_1^\infty\subseteq 2^X$, that's $\alpha_n\subseteq X(n=1,2,\ldots)$ or $\{\alpha_n\}_1^\infty\in p(2^X)$, then we write the following concepts introduced by Kuratowski:

Definition 2.2. Let the $\mathbf{P}$ be a cone of X. $\mathbf{P}$ is said to be a normal cone if and only if there exists a constant $N>0$ such that for $\theta\le x\le y$, holds $\|x\|\le N\|y\|$, where the N is called normal constant of $\mathbf{P}$.

Definition 2.3. ([33]) Let $\{\alpha_n\}_1^\infty\subseteq 2^X$, then

(i) The Kuratowski limit superior of $\{\alpha_n\}_1^\infty$ is
$Kls(\alpha_n)=\left\{x\;\middle|\;x\in X,\ \liminf_{n\to\infty}d(x,\alpha_n)=0\right\}$;

(ii) The Kuratowski limit inferior of $\{\alpha_n\}_1^\infty$ is
$Kli(\alpha_n)=\left\{x\;\middle|\;x\in X,\ \limsup_{n\to\infty}d(x,\alpha_n)=0\right\}$;

(iii) The Kuratowski limit of $\{\alpha_n\}_1^\infty$ is
$Kl(\alpha_n)=Kls(\alpha_n)$ if and only if $Kls(\alpha_n)=Kli(\alpha_n)$.

Lemma 2.4. ([34]) *Let* $\{\alpha_n\}_1^\infty\subseteq 2^X$, *then*

(i) $Kls(\alpha_n)=\bigcap_{1\le i\le\infty}\left(\overline{\bigcup_{i\le k<\infty}\alpha_k}\right)$
$=\{x|x\in X,\ \forall$ *open neighbourhoods* U *of* x, *exist infinitely many natural numbers* n *such that* $U\cap\alpha_n\ne\emptyset\}$;

(ii) $Kli(\alpha_n)=\{x|x\in X,\ \ x=\lim_{n\to\infty}x_n,\ \ x_n\in\alpha_n(n=1,2,...)\}$
$=\{x|x\in X,\forall$ *open neighbourhoods* U *of* x, *exists a natural number* N *such that* $U\cap\alpha_n\ne\emptyset$ *for* $n\ge N\}$.

We mention that for sequence of closed sets, convergence in Hausdorff metric implies convergence in the sense of Kuratoski. But for sequence of bounded closed sets, both types of convergence are equivalent provided the limit set is nonempty. Therefore, the following result holds:

Lemma 2.5. ([38]) *Let* (X, d) *is complete metric space,* $\alpha_n \in C(X)(n = 1, 2, ...)$ *and* $\alpha \in C(X)$. $\lim_{n\to\infty} \alpha_n = \alpha$ *if and only if* $Kli(\alpha_n) = Kls(\alpha_n) = Kl(\alpha_n) = \alpha$.

Lemma 2.6. ([38]) *The metric space* $(CB(X), H)$ *is complete provided* (X, d) *is complete.*

Now, we will introduce some new concepts for $p(2^X)$, the nonempty set class of X.

Definition 2.7. Let (X, d) be a metric space, $2^X = \{\alpha | \alpha \subseteq X, \alpha \neq \emptyset\}$ be the family of all nonempty subset of X, $CB(X) \subset 2^X$ be family of all nonempty bounded closed subset of X, $C(X) \subset CB(X)$ be family of all nonempty compact subsets of X, $p(2^X) = \{A | A \subseteq 2^X, A \neq \emptyset\}$ be family of all nonempty class of subset of X. Let $A \in p(2^X)$, then

(i) Let $A, B \in p(2^X)$, then

$$A \cup B = \{\alpha | \alpha \in A \;\; or \;\; \alpha \in B\},$$
$$A \cap B = \{\alpha | \alpha \in A \;\; and \;\; \alpha \in B\}, and$$
$$A - B = \{\alpha | \alpha \;\; to \;\; belong \;\; to \;\; A \;\; and \;\; \alpha \;\; not \;\; to \;\; belong \;\; to \;\; B\},$$

is said to be union, intersection and difference for A and B, respectively;

(ii) the set A' is said to be a set-valued limit point set of A, if $A' = \left\{\alpha \;\middle|\; if \quad \alpha = \lim_{k\to\infty} \alpha_k \;\; exists, \quad \forall \, \{\alpha_k\}_1^\infty \subseteq A\right\}$, where the subset α of X is called a set-valued limit point of A;

(iii) the set $\overline{A}$ is said to be the closure of A, if $\overline{A} = A \bigcup A'$;

(iv) A is said to be a closed subset class of X, or a closed subset of 2^X, if $A = \overline{A}$;

(v) A is said to be a open subset class of X, or a open subset of 2^X, if $A = X - B$ for $B = \overline{B}$ to be a closed subset class of X;

(vi) A is said to be a bounded subset class of X, or a bounded subset of 2^X, if

$$a = \sup_{\alpha\in A} \left\{b \;\middle|\; b = \sup_{x\in\alpha} d(0, x) < +\infty, \quad 0 \;\; is \;\; zero \;\; element \;\; in \;\; X\right\}$$

and $a < +\infty$;

(vii) A is said to be a nonempty bounded closed subset class of X, or a bounded closed subset of 2^X, if A is bounded and closed;

(viii) $\Omega(X) = \{A|A \subseteq CB(X)\} \subseteq p(2^X)$ is said to be the family of nonempty bounded closed subset class of X, if A is a nonempty bounded closed subset of 2^X for any $A \in \Omega(X)$;

(ix) A is said to be a compact subset class of X, or a compact subset of 2^X, if $A \subseteq C(X)$ and for any $\{\alpha_n\}_1^\infty \subseteq A$, there exists subsequence $\{\alpha_{n_k}\}_1^\infty \subseteq \{\alpha_n\}_1^\infty$ such that $\alpha = \lim \alpha_{n_k}$ and $\alpha \in A$;

(x) $\Sigma(X) = \{A|A \subseteq C(X)\} \subseteq p(2^X)$ is said to be the family of nonempty compact subset class of X, if A is a nonempty compact subset of 2^X for any $A \in \Sigma(X)$;

(xi) Let $B \in \Omega(X)$, $\{A_k\}_1^\infty \subseteq \Omega(X)$ be a sequence, $\alpha \in CB(X)$ and $\mathbf{h}(\alpha, B) = \inf\limits_{\beta \in B} H(\alpha, \beta)$. Then

the Kuratowski Li limit superior of $\{A_k\}_1^\infty$ is

$$KLls(A_k) = \left\{\alpha \;\middle|\; \alpha \in CB(X),\ \liminf_{k\to\infty} \mathbf{h}(\alpha, A_k) = 0\right\}; \tag{2.2}$$

the Kuratowski Li limit inferior of $\{A_k\}_1^\infty$ is

$$KLli(A_k) = \left\{\alpha \;\middle|\; \alpha \in CB(X),\ \limsup_{n\to\infty} \mathbf{h}(\alpha, A_k) = 0\right\}; \tag{2.3}$$

the Kuratowski Li limit of $\{A_k\}_1^\infty$ is

$$KLl(A_k) = KLli(A_k) = KLls(A_k), \tag{2.4}$$

if and only if

$$KLls(A_k) = KLli(A_k). \tag{2.5}$$

The following conclusions are obvious.

Lemma 2.8. *Let (X, d) be a metric space, $CB(X) \subset 2^X$ be family of all nonempty bounded closed subset of X, $C(X) \subset CB(X)$ be family of all nonempty compact subsets of X, $\{A_k\}_1^\infty \subseteq \Omega(X)$, then*

(i) $C(X) \subseteq CB(X)$ *and* $\Sigma(X) \subseteq \Omega(X)$;

(ii) $KLls(A_k) = \bigcap\limits_{1\le i\le\infty} \left(\overline{\bigcup\limits_{i\le k<\infty} A_k}\right)$
$= \{\alpha|\alpha \in CB(X),\ \forall$ *open neighbourhoods U of x, exist infinitely many natural numbers n such that* $U \cap A_k \neq \emptyset\}$;

(iii) $KLli(A_k) = \{\alpha|\alpha \in CB(X),\ \alpha = \lim\limits_{k\to\infty} \alpha_k,\ \alpha_k \in A_k\ (k = 1, 2, ...)\}$
$= \{\alpha|\alpha \in CB(X),\ \forall$ *open neighbourhoods U of $CB(X)$, exists a natural number N such that $U \cap A_k \neq \emptyset$ for $k \geq N$*$\}$;

(iv) $KLli(A_k) \subseteq KLls(A_k)$.

Proof. The proof directly follows from Definition 2.2, Lemma 2.2 and Definition 2.6 (xi). □

In the future, we will introduce super Hausdorff metric and study the completeness of super Hausdorff metric space.

Definition 2.9. Let (X, d) be a metric space, $2^X = \{\alpha | \alpha \subseteq X, \alpha \neq \emptyset\}$ be the family of all nonempty subset of X, $CB(X) \subset 2^X$ be family of all nonempty bounded closed subset of X, $p(2^X) = \{A | A \subseteq 2^X, A \neq \emptyset\}$ be family of all nonempty class of subset of X, $\Omega(X) = \{A | A \subseteq CB(X)\} \subseteq p(2^X)$ is said to be the family of nonempty bounded closed subset of X. If $\mathbf{h}(\alpha, B) = \inf_{\beta \in B} H(\alpha, \beta)$ for $\alpha \in CB(X)$ and $B \in p(2^X)$, denote super Hausdorff metric induced by $\mathbf{h}$ as $\hbar(.,.)$:

$$\hbar(A, B) = \max\left\{\sup_{\alpha \in A} \mathbf{h}(\alpha, B), \sup_{\beta \in B} \mathbf{h}(A, \beta)\right\} \tag{2.6}$$

for $A, B \in \Omega(X)$. Then $(\Omega(X), \hbar)$ is side to be a super Hausdorff metric space induced by $CB(X)$.

Obviously, $\mathbf{h}(\alpha, B) \leq \hbar(A, B)$ for any $\alpha \in A$ and $A, B \in \Omega(X)$.

Definition 2.10. Let $(\Omega(X), \hbar)$ be a super Hausdorff metric space induced by $CB(X)$. Let $A_n \in \Omega(X)(n = 1, 2, ...)$, $A \in \Omega(X)$. If $\lim_{n\to\infty} \hbar(A_n, A) = 0$, that's $\forall \varepsilon > 0$, $\exists N$ such that $\hbar(A_n, A) < \varepsilon$ for any $n > N$, then the subset class sequence $\{A_n\}_1^\infty$ of X is side to be convergence to A in super Hausdorff metric $\hbar(.,.)$.

Lemma 2.11. *Let (X, d) be a metric space, $\Omega(X)$ be the family of nonempty bounded closed subset class of X. For any $\alpha \in CB(X)$ and any $B \in \Omega(X)$, then $\mathbf{h}(\alpha, B)$=0 if and only if $\alpha \in B$.*

Proof. For any $B \in \Omega(X)$ and any $\alpha \in CB(X)$, if $\alpha \in B$, then $H(\alpha, \alpha) = 0$ for (2.1), and so $\mathbf{h}(\alpha, B) = \inf_{\beta \in B} H(\alpha, \beta)$=0.

On the other hand, since $\mathbf{h}(\alpha, B) = \inf_{\beta \in B} H(\alpha, \beta) = 0$ for $B \in \Omega(X)$ so that for $n = 1, 2, ...$, exists a $\beta_n \in B$, respectively, such that

$$0 < H(\alpha, \beta_n) < \frac{1}{n} + \mathbf{h}(\alpha, B) < \frac{1}{n}.$$

Therefor, $\lim_{n\to\infty} H(\alpha, \beta_n) = 0$ and $\lim_{n\to\infty} \beta_n = \alpha$ in $H(.,.)$ metric. It follows that $\alpha \in B \in \Omega(X)$ from Definition 2.6 (ii)-(iv). □

Theorem 2.12. *Let (X, d) be a metric space, $\Omega(X)$ be the family of nonempty bounded closed subset of X, H be a Hausdorff metric induced by the metric d of X. If $\hbar$ is a super Hausdorff metric of X induced by (2.5), then for any $A, B, C \in \Omega(X)$, the following propositions hold:*

(i) $\hbar(A,B) \geq 0$;
(ii) $\hbar(A,B) = \hbar(B,A)$;
(iii) $\hbar(A,B) = 0$, *if and only if* $A = B$;
(iv) $\hbar(A,B) \leq \hbar(A,C) + \hbar(C,B)$;
(v) $|\mathbf{h}(A,\beta) - \mathbf{h}(\beta,B)| \leq \hbar(A,B), \quad \forall\, \beta \in CB(X)$.

Therefore, $(\Omega(X),\hbar)$*, super Hausdorff metric space induced by* $CB(X)$ *is a metric space.*

Proof. (i) and (ii) are clear. For (iii), $\hbar(A,B) = 0$ if and only if $\sup_{\alpha\in A} \mathbf{h}(\alpha,B) = 0$ and $\sup_{\beta\in B} \mathbf{h}(\beta,B) = 0$, and if and only if $A \subseteq B$ and $B \subseteq A$ hold for Lemma 2.8. Therefore, $A = B$.

Let $A,B,C \in \Omega(X)$. For any $\alpha \in A, \beta \in B, \gamma \in C$, by H Hausdorff metric of X, we have $H(\alpha,\beta) \leq H(\alpha,\gamma) + H(\gamma,\beta)$ and

$$\inf_{\beta\in B}\{H(\alpha,\beta)\} \leq \inf_{\beta\in B}\{H(\alpha,\gamma) + H(\gamma,\beta)\} \leq \inf_{\beta\in B}\{H(\gamma,\beta)\} + H(\alpha,\gamma).$$

It follows that $\mathbf{h}(\alpha,B) - \mathbf{h}(\gamma,B) \leq H(\alpha,\gamma)$ and

$$\inf_{\gamma\in C}\{\mathbf{h}(\alpha,B) - \mathbf{h}(\gamma,B)\} \leq \inf_{\gamma\in C}\{H(\alpha,\gamma)\},$$

and

$$\inf_{\gamma\in C}\{\mathbf{h}(\alpha,B)\} - \sup_{\gamma\in C}\{\mathbf{h}(\gamma,B)\} \leq \inf_{\gamma\in C}\{H(\alpha,\gamma)\}.$$

Therefore,

$$\mathbf{h}(\alpha,B) \leq \sup_{\gamma\in C}\{\mathbf{h}(\gamma,B)\} + \mathbf{h}(\alpha,C).$$

Then holds

$$\sup_{\alpha\in A} \mathbf{h}(\alpha,B) \leq \sup_{\gamma\in C}\{\mathbf{h}(\gamma,B)\} + \sup_{\alpha\in A} \mathbf{h}(\alpha,C). \tag{2.7}$$

In as same,

$$\sup_{\beta\in B} \mathbf{h}(A,\beta) \leq \sup_{\gamma\in C}\{\mathbf{h}(A,\gamma)\} + \sup_{\beta\in B} \mathbf{h}(C,\beta). \tag{2.8}$$

It follows from (2.5)-(2.8) that (iv) holds. Therefore, $\hbar$ is a metric in $\Omega(X)$ and $(\Omega(X),\hbar)$, super Hausdorff metric space induced by $CB(X)$ is a metric space for (i)-(iv).

For any $\beta \in CB(X)$, since

$$\mathbf{h}(\beta,A) = \inf_{\alpha\in A}\{H(\beta,\alpha)\} \leq H(\beta,\alpha) \leq H(\beta,\gamma) + H(\alpha,\gamma)$$

for any $\gamma \in B$, so that

$$\mathbf{h}(\beta,A) - H(\beta,\gamma) = H(\alpha,\gamma),$$

and hence

$$\mathbf{h}(\beta,A) - H(\beta,\gamma) \leq \mathbf{h}(A,\gamma)$$

or

$$\mathbf{h}(\beta, A) \leq \mathbf{h}(A, \gamma) + H(\beta, \gamma).$$

Then

$$\mathbf{h}(\beta, A) \leq \mathbf{h}(A, \gamma) + H(\beta, \gamma) \leq \hbar(A, B) + H(\beta, \gamma).$$

We can have

$$\mathbf{h}(\beta, A) \leq \hbar(A, B) + \mathbf{h}(\beta, \gamma)$$

for $\beta \in B$, that is

$$\mathbf{h}(\beta, A) \leq \hbar(A, B) + \mathbf{h}(\beta, B).$$

In same way,

$$\mathbf{h}(\beta, B) \leq (A, B) + \mathbf{h}(\beta, A)$$

holds. It follows that

$$|\mathbf{h}(\beta, A) - \mathbf{h}(\beta, B)| \leq \hbar(A, B)$$

for any $\beta \in B$. □

Corollary 2.13. *Let (X, d) be a metric space, $C(X)$ be the family of nonempty compact subset of X, then $(\sum(X), \hbar)$, super Hausdorff metric space induced by $C(X)$ is a metric space.*

The following result holds obviously.

Lemma 2.14. *Let $A \in p(2^X)$, then A is a bounded set class if and only if exists a constant $r > 0$ such that $\hbar(A, \{\{0\}\}) < r$, where 0 is zero element in X.*

Theorem 2.15. *Let $A_n \in \Omega(X)(n = 1, 2, ...)$ and $A \in \Omega(X)$, if $\lim_{n\to\infty} A_n = A$ exists, then the limit A of the sequence $\{A_n\}_1^\infty$ is unique.*

Proof. Let $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} A_n = B$ are limits of the sequence $\{A_n\}_1^\infty$, by Theorem 2.9 (i)-(iv) and Definition 2.7, we have

$$0 \leq \hbar(A, B) \leq \hbar(A_n, A) + \hbar(A_n, B) \to 0 (n \to \infty).$$

Then $A = B$. □

Theorem 2.16. *Let $A_n \in \Omega(X)(n = 1, 2, ...)$ and $\lim_{n\to\infty} A_n = A \in \Omega(X)$, let $B_n \in \Omega(X)(n = 1, 2, ...)$ and $\lim_{n\to\infty} B_n = B \in \Omega(X)$. Then $\lim_{n\to\infty} \hbar(A_n, B_n) = \hbar(A, B)$, that is $\hbar(\cdot, \cdot)$ is continuous in the $\Omega(X) \times \Omega(X)$.*

Proof. By using Theorem 2.9, we have

$$\hbar(A_n, B_n) \leq \hbar(A_n, A) + \hbar(A, B) + \hbar(B_n, B)$$

and

$$\hbar(A, B) \leq \hbar(A, A_n) + \hbar(A_n, B_n) + \hbar(B_n, B).$$

Therefore,

$$|\hbar(A_n, B_n) - \hbar(A, B)| \leq \hbar(A, A_n) + \hbar(B_n, B).$$

Then $\lim_{n\to\infty} \hbar(A_n, B_n) = \hbar(A, B)$ holds, that is, $\hbar(\cdot, \cdot)$ is continuous in the $\Omega(X) \times \Omega(X)$. □

Lemma 2.17. *If* $\lim_{n\to\infty} A_n = A$, *then* $\{A_k\}_1^\infty \subseteq \Omega(X)$ *is a bounded set class sequence.*

Proof. The proof directly follows from Definition 2.6. □

Definition 2.18. Let $(\Omega(X), \hbar)$ be a super Hausdorff metric space induced by $CB(X)$. If for every $\varepsilon > 0$, there exists an open covering of $\Omega(X)$ by finitely many ε-balls, then the space $\Omega(X)$ is called totally bounded.

Lemma 2.19. *If* $(CB(X), H)$ *is totally bounded, then* $\Omega(X)$, *super Hausdorff metric space induced by* $CB(X)$ *is totally bounded.*

Proof. Select $\varepsilon > 0$. Take a finite open cover $\widehat{C}$ of $CB(X)$ by ε-balls $C_k = C_k(\alpha_k, \varepsilon) \subseteq CB(X)$ which center is $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, respectively, for $k = 1, 2, \cdots, n$. Then

$$\widehat{C} = \{C_k(\alpha_k, \varepsilon)\}_{k=1}^n$$

and

$$\bigcup \widehat{C} = \bigcup \{C_k(\alpha_k, \varepsilon)\}_{k=1}^n = CB(X).$$

Note, $C_k = C_k(\alpha_k, \varepsilon) \in \Omega(X)$ for $k = 1, 2, \cdots, n$. For any element $A \in \Omega(X)$, $\emptyset \neq A \subseteq CB(X)$, then there exist some C_{k_j} such that $A \subseteq \cup\{C_{k_j}\}_{j=1}^N$ and $C_{k_j} \cap A \neq \emptyset$ for $j = 1, 2, \cdots, N \leq n$. Let $D_A = \{C_{k_j}\}_{j=1}^N \in \Omega(X)$, then $A \in \mathbf{B}(D_A, \varepsilon)$ which $\mathbf{B}(D_A, \varepsilon)$ is a ε-ball of $\Omega(X)$ and $\Omega(X)$ is covered by set class of X, open ε-ball class $\{\mathbf{B}(D_A, \varepsilon) | A \in \Omega(X)\}$. Since $D_A = \{C_{k_j}\}_{j=1}^N \subseteq \{C_k(\alpha_k, \varepsilon)\}_{k=1}^n$ so that the number of different D_A is not more than 2^n. Therefore, $\{\mathbf{B}(D_A, \varepsilon) | A \in \Omega(X)\}$ is a finite open cover of $\Omega(X)$, and obviously, $\hbar(A, \mathbf{B}(D_A, \varepsilon)) < \varepsilon$. □

It follows that the following theorem from Lemma 2.18.

Theorem 2.20. *If* (X, d) *is totally bounded, then* $\Omega(X)$, *super Hausdorff metric space induced by* $CB(X)$ *is totally bounded.*

Proof. Since if X is totally bounded, then the induced Hausdorff space $CB(X)$ is totally bounded [34], and Lemma 2.18 so that the result holds. □

Theorem 2.21. *Let (X, d) be a complete metric space, then $\Omega(X)$ is a complete space for the super Hausdorff metric $\hbar(\cdot,\cdot)$.*

Proof. Let (X, d) be a complete metric space, then $(CB(X), H)$ is a complete metric space by Lemma 2.5.

Let $\{A_n\}_{n=1}^{\infty} \subseteq \Omega(X)$ be a Caushy sequence in $\hbar(\cdot,\cdot)$, that is $\forall\varepsilon > 0$, $\exists N > 0$, $\hbar(A_n, A_m) < \varepsilon$ holds as $n, m > N$, where

$$\hbar(A_n, A_m) = \max\left\{ \sup_{\beta_m \in A_m} \mathbf{h}(A_n, \beta_m), \sup_{\beta_n \in A_n} \mathbf{h}(\beta_n, A_m)\right\}.$$

Obviously, Caushy sequence $\{A_n\}_{n=1}^{\infty}$ be bounded, and then $\mathbf{h}(A_n, \beta_m) = \inf_{\alpha_n \in A_n}\{H(\alpha_n, \beta_m)\} < \varepsilon$ and $\exists\ \tilde{\alpha}_n \in A_n$ such that $H(\tilde{\alpha}_n, \beta_m) < 2\varepsilon$ for $n, m > N$ and any $\beta_m \in A_m$. In the same theory, $\exists\ \tilde{\beta}_m \in A_m$ such that $H(\tilde{\beta}_m, \alpha_n) < 2\varepsilon$ for $n, m > N$ and any $\alpha_n \in A_n$, then $H(\tilde{\beta}_m, \tilde{\alpha}_n) < 2\varepsilon$ holds for $\tilde{\alpha}_n \in A_n$ and $\tilde{\beta}_m \in A_m$. $\{\tilde{\alpha}_n\}_{n=1}^{\infty}$ selected from the sequence $\{A_n\}_{n=1}^{\infty}$ is a Caushy sequence for the Hausdorff metric $H(\cdot,\cdot)$ in $CB(X)$, and there exists a $\tilde{\alpha} = \lim_{n\to\infty} \tilde{\alpha}_n \in C(X)$, because $CB(X)$ is the complete space for the Hausdorff metric $H(\cdot,\cdot)$. Therefore,

$$A = KLli(A_n) = \{\alpha | \alpha = \lim_{n\to\infty} \tilde{\alpha}_n | \tilde{\alpha}_n \in A_n\} \neq \emptyset$$

and $A \in \Omega(X)$.

Next, we show that $\lim_{n\to\infty} \hbar(A, A_n) = 0$.

Let $\hbar(A_n, A) = \max\{\sup_{\alpha\in A} \mathbf{h}(A_n, \alpha), \sup_{\alpha_n \in A_n} \mathbf{h}(\alpha_n, A)\}$, and $A = KLli(A_n)$. Then $\forall\varepsilon > 0$ and $\forall\alpha \in A$, $\exists\ \tilde{\alpha}_n \in A_n$ such that $\alpha = \lim_{n\to\infty} \tilde{\alpha}_n$, and so that there exists a $N > 0$, $H(\tilde{\alpha}_n, \alpha) < \varepsilon$ holds as $n > N$. Then $\mathbf{h}(A_n, \alpha) < \varepsilon$ and

$$\mathbf{h}(\alpha_n, \hbar(A_n, A)) = \max\left\{ \sup_{\alpha \in A} \mathbf{h}(A_n, \alpha), \sup_{\alpha_n \in A_n} \mathbf{h}(\alpha_n, A)\right\} < \varepsilon$$

as $n > N$. Therefore, $\lim_{n\to\infty} \hbar(A_n, A) = 0$. □

It follows that the following corollaries from the proof process of Theorem 2.20.

Corollary 2.22. *Let X be a complete metric space, let $\{A_n\}_{n=1}^{\infty} \subseteq \Omega(X)$ be a Cauchy sequence for super Hausdorff metric $\hbar(\cdot,\cdot)$, then $\{A_n\}_{n=1}^{\infty}$ is bounded.*

Corollary 2.23. *Let X be a complete space and $\lim_{n\to\infty} A_n = A \in \Omega(X)$, then $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\hbar(\cdot,\cdot)$.*

Theorem 2.24. *Let $(\Omega(X), \hbar)$ be a complete space, $A_n \in \Omega(X)(n = 1, 2, ...)$ and $A \in \Omega(X)$, then $\lim_{n\to\infty} A_n = A$ in $\hbar$ if and only if $KLli(A_n) = KLls(A_n) = A$.*

Proof. Let $A_n \in \Omega(X)(n = 1, 2, ...)$ and $A = \lim_{n\to\infty} A_n$ in $\hbar$, then $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\Omega(X), \hbar)$ for Corollary 2.22, and $\lim_{n\to\infty} A_n = A = KLli(A_n) \subseteq KLls(A_n)$, because of Theorem 2.19 and Lemma 2.7 (iv). Let

$$B = KLls(A_n) = \bigcap_{1\le k\le\infty} \left(\overline{\bigcup_{k\le n<\infty} A_n} \right).$$

We will show that $B = A$. Since $\{A_n\}_{n=1}^{\infty} \subseteq \Omega(X)$ in $\hbar(\cdot,\cdot)$, so that $\forall\varepsilon > 0$, $\exists N > 0$, $\hbar(A_n, A_m) < \varepsilon$ holds as $n, m > N$, where

$$\hbar(A_n, A_m) = \max\left\{ \sup_{\beta_m\in A_m} \mathbf{h}(A_n, \beta_m), \sup_{\beta_n\in A_n} \mathbf{h}(\beta_n, A_m) \right\}.$$

Then $\forall\varepsilon > 0$, $\exists N_1 > 0$, $\sup_{\alpha_n\in A_n} \mathbf{h}(\alpha_n, A_m) < \varepsilon$ holds as $n, m > N_1$, and $\forall\alpha_n \in A_n$, $\mathbf{h}(\alpha_n, A_m) = \inf_{\alpha_m\in A_m}\{H(\alpha_n, \alpha_m)\} < \varepsilon$ holds as $n, m > N$. It follows that $\forall\alpha_n \in A_n$, $\exists\tilde{\alpha}_m \in A_m$ such that $H(\alpha_n, \tilde{\alpha}_m) < \varepsilon$ holds as $n, m > N_1$ and $\{\tilde{\alpha}_m\}_{m>N_1}^{\infty}$ is a Caushy sequence in H on $CB(X)$ if which is complete. Setting $\alpha = \lim_{m\to\infty} \tilde{\alpha}_m$, then $\alpha \in KLli(A_n) \subseteq B$ and $\lim_{m\to\infty} H(\tilde{\alpha}_m, \alpha) = 0$, and $\forall\alpha_n \in A_n$, $H(\alpha_n, \alpha) < \varepsilon$ holds as $n > N_1$. Therefore, $\forall\alpha_n \in A_n$, $\inf_{\alpha\in B}\{H(\alpha_n, \alpha)\} = \mathbf{h}(\alpha_n, B) < \varepsilon$, and $\sup_{\alpha_n\in A_n} \mathbf{h}(\alpha_n, B) < \varepsilon$ holds as $n > N_1$.

On the other hand, for any $\alpha \in B$, then exists a sequence $\{\alpha_p\}_{p\ge k}^{\infty} \subseteq \bigcup_{k\le n<\infty} A_n$ for $k \ge 1$ such that $\alpha = \lim_{p\to\infty} \alpha_p$, where $\alpha_p \in A_{n_p}$. For any $\varepsilon > 0$ and any $\alpha_p \in A_{n_p}$, exists $N_2 > 0$, and exists $\alpha_n \in A_n$ such that $H(\alpha_n, \alpha_p) < \varepsilon$ and $H(\alpha_n, \alpha) < H(\alpha_n, \alpha_p) + H(\alpha_p, \alpha) < 2\varepsilon$ holds as $n, p > N_2$. Therefore, for any $\varepsilon > 0$ and any $\alpha \in B$, exists $N_2 > 0$, and exists $\alpha_n \in A_n$ such that $H(\alpha_n, \alpha) < \varepsilon$ holds as $n > N_2$, then $\sup_{\alpha\in B} \mathbf{h}(A_n, \alpha) < \varepsilon$ holds as $n > N_2$. We have prove that for any $\varepsilon > 0$, exists $N > 0$ such that $\hbar(A_n, B) < \varepsilon$ holds as $n > N = \max\{N_1, N_2\}$, that is $B = \lim_{n\to\infty} A_n$. Therefore, if $A = \lim_{n\to\infty} A_n$ in $\hbar$, then $KLli(A_n) = KLls(A_n) = A$.

If $KLli(A_n) = KLls(A_n)$, Kuratoski Li limit of sequence $\{A_k\}_{k=1}^{\infty}$ exists, then let $A = KLli(A_n)$, and belong to $\Omega(X)$ for $KLls(A_n) \in \Omega(X)$. For any $\alpha \in A$, there exists $\alpha_n \in A_n$ such that $\alpha = \lim_{n\to\infty} \alpha_n$, it follows that $\lim_{n\to\infty} \hbar(A_n, A) = 0$. □

Corollary 2.25. *If X is a complete space, then nonempty compact subset space $C(X)$ of X is a complete space, and hence, nonempty compact subset class space $\Sigma(X)$ of X is a complete space.*

3. Set-valued fixed point theorem based on the super-trajectory in $\Sigma(X)$

Definition 3.1. Let (X, d) be a metric space, $(C(X), H)$ be a nonempty compact subset metric space of X, $\Sigma(X) = \{A|A \subseteq C(X)\} \subseteq p(2^X)$ be the super Hausdorff metric space of X in $\hbar$, and 0 be zero element in X. Let $P : 2^X \to p(2^X)$ defined by $P(x) = \alpha \in 2^X(\forall x \in X)$ be a set-valued mapping. Then

(i) A set class sequence $\{A_n\}_{n=1}^{\infty}$ in X is bounded, if

$$a = \sup_{A_n \in \{A_n\}_{n=1}^{\infty}} \left\{ a_n \,\middle|\, a_n = \sup_{\alpha_n \in A_n} H(\{0\}, \alpha_n) < +\infty, \;\; n = 1, 2, \cdots \right\}$$

and $a < +\infty$;

(ii) P is said to be a α-compact mapping, if $P\alpha = \{P(y)|y \in \alpha\} \in \Sigma(X)$ for $\alpha \in C(X)$;

(iii) P is said to be a class compact mapping, if $P\alpha = \{P(y)|y \in \alpha\} \in \Sigma(X)$ for any $\alpha \in C(X)$;

(iv) x is said to be a fixed point of the set valued mapping P in X, if $x \in P(x)$ for $x \in X$;

(v) α is said to be a set valued fixed point of the set valued mapping P in X if $\alpha \in P(\alpha)$ for $\alpha \in 2^X$;

(vi) $\{\alpha_n\}_{n=1}^{\infty}$ is side to be supper-trajectory of set valued mapping P at α_0 in 2^X if $\alpha_{n+1} \in P(\alpha_n)$ and $H(\alpha_{n+1}, \alpha_n) = \mathbf{h}(P(\alpha_n), \alpha_n)$ for $n = 1, 2, \cdots$, denote by $O_P(\alpha_0, 0, \infty)$;

(vii) P is side to be continuous at α in 2^X if $\lim_{n\to\infty} \hbar(P(\alpha_n), P(\alpha)) = 0$, or $\lim_{n\to\infty} P(\alpha_n) = P(\alpha)$ in $\hbar$;

(viii) α is side to be set-valued limit point of $\{\alpha_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \hbar(\alpha_{n_k}, \alpha) = 0$, or $\lim_{k\to\infty} \alpha_{n_k} = \alpha$ in H.

Lemma 3.2. *Let $(C(X), H)$ be a complete metric space. If a sequence $\{\alpha_n\}_{n=1}^{\infty}$ is bounded, then there exists a set-valued limit point of $\{\alpha_n\}_{n=1}^{\infty}$ at last.*

Proof. Let a set sequence $\{\alpha_n\}_{n=1}^{\infty}$ be bounded for the Hausdorff metric H, then $\bigcup_{k \le n < \infty} \alpha_n$ is bounded, and $\{x_n\}_{n=1}^{\infty}(x_n \in \alpha_n, n = 1, 2, \cdots)$ is bounded.

Therefore, there exists a limit(cluster) point y of $\{x_n\}_{n=1}^{\infty}$ at last, that is there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{k\to\infty} x_{n_k} = y$. Let

$$\alpha = \left\{ y \;\middle|\; y = \lim_{k\to\infty} x_{n_k},\ \ x_{n_k} \in \alpha_{n_k},\ \ k = 1, 2, \cdots \right\},$$

then $\alpha \neq \emptyset$ and $\alpha \in C(X)$. Then α is a set-valued limit point of $\{\alpha_n\}_{n=1}^{\infty}$. □

Lemma 3.3. *Let $(\Sigma(X), \hbar)$ be a complete metric space. If a set class sequence $\{A_n\}_{n=1}^{\infty}$ in $\Sigma(X)$ is bounded, then there exists a set class limit point of $\{A_n\}_{n=1}^{\infty}$ at last.*

Proof. Let a sequence $\{A_n\}_{n=1}^{\infty}$ be bounded for the Hausdorff metric H, then $\bigcup\limits_{k\leq n<\infty} A_n$ is bounded by the Definition 3.1, and $\{\alpha_n\}_{n=1}^{\infty}(\alpha_n \in A_n, n = 1, 2, \cdots)$ is bounded. Therefore, there exists a cluster point β of $\{\alpha_n\}_{n=1}^{\infty}$ at last, that is there is a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that $\lim\limits_{k\to\infty} x_{n_k} = y$. Let

$$A = \left\{ \beta \;\middle|\; \beta = \lim_{k\to\infty} \alpha_{n_k},\ \ \alpha_{n_k} \in A_{n_k},\ \ k = 1, 2, \cdots \right\},$$

then $A \neq \emptyset$ and $A \in C(X)$. Then A is a set-valued limit point of $\{A_n\}_{n=1}^{\infty}$. □

It follows easily that the result from the conditions for $C(X)$ made up by compact subset of X and completeness of $(C(X), H)$.

Theorem 3.4. *Let $(C(X), H)$ be a complete metric space, $(\Sigma(X), \hbar)$ be a super Hausdorff metric space induced by $(C(X), H)$. Let $P : C(X) \to \Sigma(X)$ induced by $P : X \to C(X)$ be a continuous set-valued mapping at any $\alpha \in C(X)$. For any $\alpha, \beta \in C(X)$, let*

$$\begin{aligned} &\hbar(P(\alpha), P(\beta)) \\ &< \max\left\{ H(\alpha,\beta), \boldsymbol{h}(\alpha, P(\alpha)), \boldsymbol{h}(\beta, P(\beta)), \frac{1}{2}[\boldsymbol{h}(\alpha, P(\beta)) + \boldsymbol{h}(\beta, P(\alpha))] \right\}. \end{aligned} \tag{3.1}$$

hold as $\alpha \neq \beta$. If there exists a $\alpha_0 \in C(X)$ such that the supper-trajectory $O_P(\alpha_0, 0, \infty)$ has a set-valued limit point α, then α is a set-valued fixed point of set-valued mapping P.

Proof. Let $\{\alpha_n\}_{n=1}^{\infty}$ be super-trajectory of set-valued mapping P at α_0 in 2^X. If for k_0, $\alpha_{k_0} = \alpha_{k_0+1}$, then $\alpha_{k_0} = \alpha_{k_0+1} \in P(\alpha_{k_0})$, that is α_{k_0} is a set-valued fixed point of the mapping P.

Without loss of generality, let $\alpha_n \neq \alpha_{n+1}$ for $n \neq 0, 1, 2, \cdots$, by the conditions of super-trajectory of set-valued mapping P at α_0, and (3.1), then by using the Definition 2.8 and the Definition 3.1 (vi), we can have for $n \geq 1$,

$$\begin{aligned} H(\alpha_n, \alpha_{n+1}) &= \mathbf{h}(P(\alpha_n), \alpha_n) \\ &\leq \hbar(P(\alpha_{n-1}), P(\alpha_n)) \\ &< \max\{H(\alpha_{n-1}, \alpha_n), \mathbf{h}(\alpha_{n-1}, P(\alpha_{n-1})), \mathbf{h}(\alpha_n, P(\alpha_n)), \\ &\qquad \frac{1}{2}[\mathbf{h}(\alpha_{n-1}, P(\alpha_n)) + \mathbf{h}(\alpha_n, P(\alpha_{n-1}))]\} \\ &\leq \max\{H(\alpha_{n-1}, \alpha_n), H(\alpha_n, \alpha_{n+1})\}. \end{aligned}$$

Since $\alpha_{n+1} \in P(\alpha_n)$ and $H(\alpha_{n+1}, \alpha_n) = \mathbf{h}(P(\alpha_n), \alpha_n)$ for $n = 1, 2, \cdots$, so that $H(\alpha_{n+1}, \alpha_n) \leq H(\alpha_n, \alpha_{n-1})$. Therefore, $\lim_{n\to\infty} H(\alpha_{n+1}, \alpha_n) = a \geq 0$ exists.

Because the super-trajectory sequence $\{\alpha_n\}_{n=1}^{\infty}$ has a set-valued limit point for the Lemma 3.2, denote by α, then there exists subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty} \subseteq \{\alpha_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} \alpha_{n_k} = \alpha$. We need to prove $\alpha \in P(\alpha)$. By using Theorem 2.11 (vi), we have

$$\begin{aligned} &|H(\alpha_{n_k}, \alpha_{n_k+1}) - \mathbf{h}(\alpha, P(\alpha))| \\ &\leq |H(\alpha_{n_k}, \alpha_{n_k+1}) - \mathbf{h}(\alpha_{n_k}, P(\alpha))| + |\mathbf{h}(\alpha_{n_k}, P(\alpha)) - \mathbf{h}(\alpha_n, P(\alpha_n))| \\ &= |\hbar(\alpha_{n_k}, P(\alpha_{n_k})) - \mathbf{h}(\alpha_{n_k}, P(\alpha))| + |\mathbf{h}(\alpha_{n_k}, P(\alpha)) - \mathbf{h}(\alpha, P(\alpha))| \\ &\leq \mathbf{h}(P(\alpha_{n_k}, P(\alpha)) + |\mathbf{h}(\alpha_{n_k}, P(\alpha)) - \mathbf{h}(\alpha, P(\alpha))|, \end{aligned}$$

that is

$$\begin{aligned} &|H(\alpha_{n_k}, \alpha_{n_k+1}) - \mathbf{h}(\alpha, P(\alpha))| \\ &\leq \mathbf{h}(P(\alpha_{n_k}, P(\alpha)) + |\mathbf{h}(\alpha_{n_k}, P(\alpha)) - \mathbf{h}(\alpha, P(\alpha))|, \end{aligned} \tag{3.2}$$

$\lim_{k\to\infty} \mathbf{h}(P(\alpha_{n_k}, P(\alpha)) = 0$ for the continuousness of P and $\lim_{k\to\infty} |\mathbf{h}(\alpha_{n_k}, P(\alpha)) - \mathbf{h}(\alpha, P(\alpha))| = 0$, then

$$\lim_{k\to\infty} H(\alpha_{n_k}, \alpha_{n_k+1}) = \mathbf{h}(\alpha, P(\alpha)) = a \tag{3.3}$$

holds.

On the other hand, $\lim_{k\to\infty} \mathbf{h}(P(\alpha), \alpha_{n_k+1}) \leq \lim_{k\to\infty} \hbar(P(\alpha), P(\alpha_{n_k})) = 0$ and $P(\alpha) \in \Sigma(X)$ is compact, must be hounded and clouded for completeness of $\Sigma(X)$, then exists a subsequence $\{\gamma_{n_k}\}_{k=1}^{\infty} \subseteq P(\alpha) \in \Sigma(X)$ such that for each $k = 1, 2, \cdots$, exists a γ_{n_k},

$$H(\alpha_{n_k}, \gamma_{n_k}) \leq \frac{1}{k} + \mathbf{h}(P(\alpha), \alpha_{n_k+1})$$

holds for $\mathbf{h}(P(\alpha), \alpha_{n_k+1}) = \inf_{\gamma\in P(\alpha)} \{H(\alpha_{n_k}, \gamma)\}$, then $\lim_{k\to\infty} H(\alpha_{n_k+1}, \gamma_{n_k}) = 0$ and $\{\gamma_{n_k}\}_{k=1}^{\infty}$ is hounded, and then there exists a subsequence $\{\gamma_{n_{k_j}}\}_{k=1}^{\infty} \subseteq \{\gamma_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{j\to\infty} \gamma_{n_{k_j}} = \xi \in P(\alpha)$ for compactness of P. Therefore, $\lim_{j\to\infty} H(\alpha_{n_{k_j}+1}, \xi) = 0$.

Redoing (3.2) and (3.3), we have

$$\lim_{j\to\infty} H(\alpha_{n_{k_j}+1}, \alpha_{n_{k_j}+2}) = \mathbf{h}(\xi, P(\xi)) = a \tag{3.4}$$

and it follows that

$$a = \mathbf{h}(\xi, P(\xi)) = \mathbf{h}(\alpha, P(\alpha)) = H(\xi, \alpha) \tag{3.5}$$

from (3.3)-(3.5), and $\lim_{n\to\infty} H(\alpha_{n+1}, \alpha_n) = a$ and $\lim_{j\to\infty} H(\alpha_{n_{k_j}+1}, \alpha_{n_{k_j}+2}) = H(\xi, \alpha) = a$.

Further, for $\xi \in P(\alpha)$, if $\alpha = \xi \in P(\alpha)$, then it is all right, but it is not, by using (3.1), we consider

$$\begin{aligned}&\mathbf{h}(P(\xi), \xi) \le \hbar(P(\xi), P(\alpha))\\ &< \max\left\{H(\alpha, \xi), \mathbf{h}(\alpha, P(\alpha)), \mathbf{h}(\xi, P(\xi)), \frac{1}{2}[\mathbf{h}(\alpha, P(\xi)) + \mathbf{h}(\xi, P(\alpha))]\right\}\\ &\le \max\{H(\alpha_{n-1}, \alpha_n), H(\alpha_n, \alpha_{n+1})\}.\end{aligned}$$

Since $\alpha \neq \xi \in P(\alpha)$ so that $\mathbf{h}(P(\xi), \xi) \le \mathbf{h}(\alpha, P(\alpha))$, and $a = \mathbf{h}(P(\xi), \xi) < \mathbf{h}(\alpha, P(\alpha)) = a$. This is a contradiction. Hence, $\alpha \in P(\alpha)$. □

Lemma 3.5. *Let $(CB(X), H)$ be a metric space, $(\Omega(X), \hbar)$ be a super Hausdorff metric space induced by $(CB(X), H)$. Let $P : CB(X) \to \Omega(X)$ induced by $P : X \to CB(X)$ be a continuous set-valued mapping. Then $\alpha \in P(\alpha)$ if and only if exists a $x_0 \in \alpha$ such that $\alpha = P(x_0)$ for $\alpha \in \Omega(X)$.*

Proof. The proof directly follows from Definition 3.1, so it is omitted. □

Theorem 3.6. *Let (X, d) be a complete metric space, $(\Sigma(X), \hbar)$ be a super Hausdorff metric space induced by $(C(X), H)$ which is induced by (X, d). Let $P : C(X) \to \Sigma(X)$ induced by $P : X \to C(X)$ be a continuous set-valued mapping at any $\alpha \in C(X)$. For any $\alpha, \beta \in C(X)$, let*

$$\begin{aligned}&\hbar(P(\alpha), P(\beta))\\ &< \max\left\{H(\alpha, \beta), \boldsymbol{h}(\alpha, P(\alpha)), \hbar(\beta, P(\beta)), \frac{1}{2}[\boldsymbol{h}(\alpha, P(\beta)) + \boldsymbol{h}(\beta, P(\alpha))]\right\}\end{aligned} \tag{3.6}$$

hold as $\alpha \neq \beta$. If there exists a $\alpha_0 \in C(X)$ such that the super-trajectory $O_P(\alpha_0, 0, \infty)$ has a set-valued limit point α, then there is a fixed point of set-valued mapping P in X. And there is a fixed point of set-valued mapping P in the set-valued limit point set of the super-trajectory $O_P(\alpha_0, 0, \infty)$ at lest.

Proof. It follows directly that the result from Theorem 3.3 and Lemma 3.4. □

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