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ON SOME BOUNDARY VALUE PROBLEMS FOR A FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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Abstract. We study the existence of solutions for a fractional integro-differential inclusion with non-separated local and non-separated integral-flux boundary conditions. We establish Filippov type existence results in the case of nonconvex set-valued maps.

1. Introduction

In this paper, we study the following fractional integro-differential inclusion

$$D_C^\alpha x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,1]) \tag{1.1}$$

subject to the following boundary conditions

$$x(0) + x(1) = a\int_0^1 x(s)ds, \quad x'(0) = bD_C^\beta x(1), \tag{1.2}$$

$$x(0) + x(1) = aI^\gamma x(\eta), \quad x'(0) = bD_C^\beta x(1), \tag{1.3}$$

where $\alpha \in (1,2]$, D_C^α is the Caputo fractional derivative of order α, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map, $V : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a given function, $a, b \in \mathbb{R}$, $\beta, \gamma \in (0,1]$, $\eta \in (0,1)$ and $I^\gamma x(.)$ is the fractional integral of order $\gamma > 0$.

If F does not depend on the last variable, inclusion (1.1) reduces to

$$D_C^\alpha x(t) \in F(t, x(t)) \quad a.e. \ ([0,1]). \tag{1.4}$$

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The present paper is motivated by a recent paper of Ahmad and Ntouyas ([1]) where existence results for problems (1.2)-(1.4) and (1.3)-(1.4) are established for convex as well as nonconvex set-valued maps. The existence results in [1] are based on a nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory.

Our aim is to extend the study in [1] to the more general problem (1.1) and to show that Filippov's ideas ([8]) can be suitably adapted in order to obtain the existence of solutions for problems (1.1)-(1.2) and (1.1)-(1.3). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([8]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

Finally, we note that differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations and inclusions of fractional order ([9, 10, 11] etc.). Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [3] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I = [0,1]$, we denote by $C(I,\mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I,\mathbb{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbb{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$.

Definition 2.1. (a) The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.
(b) The Caputo fractional derivative of order $\alpha>0$ of a function $f:[0,\infty)\to\mathbb{R}$ is defined by

$$\mathrm{D}_c^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{-\alpha+n-1}f^{(n)}(s)\mathrm{d}s,$$

where $n=[\alpha]+1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

We recall (e.g., [9]) that if $\alpha>0$ and $f\in C(I,\mathbb{R})$ or $f\in L^\infty(I,\mathbb{R})$ then $(\mathrm{D}_c^\alpha I^\alpha f)(t)\equiv f(t)$.

The next two technical lemmas are proved in [1].

Lemma 2.2. *Assume that $a\neq 2$, $b\neq\Gamma(2-\beta)$ and consider $f(.)\in C(I,\mathbb{R})$. The unique solution $x(.)\in C(I,\mathbb{R})$ of problem*

$$D_C^\alpha x(t)=f(t)\quad a.e.\ ([0,T]), \tag{2.1}$$

with boundary conditions (1.2) *is given by*

$$\begin{aligned} x(t)=&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s)ds\\ &-\frac{1}{2-a}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\frac{a}{2-a}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}f(s)ds.\end{aligned} \tag{2.2}$$

Remark 2.3. If we denote

$$\begin{aligned} G_1(t,s):=&\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{[0,t]}(s)+\frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)}\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\\ &-\frac{1}{2-a}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{a}{2-a}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)},\end{aligned}$$

where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(.)$ in Lemma 2.2 may be written as $x(t)=\int_0^1 G_1(t,s)f(s)ds$. Moreover, if $\alpha-\beta\geq 1$, then for every $t,s\in I$, we have

$$\begin{aligned} |G_1(t,s)|\leq&\frac{1}{\Gamma(\alpha)}+\frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta)}\\ &+\frac{1}{|2-a|\Gamma(\alpha)}+\frac{|a|}{|2-a|\Gamma(\alpha+1)}=:M_1.\end{aligned}$$

Definition 2.4. A function $x(.) \in C^2(I,\mathbb{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $f(.) \in L^1(I,\mathbb{R})$ that satisfies $f(t) \in F(t,x(t),V(x)(t))$ *a.e.* (I) and $x(.)$ is given by (2.2).

Lemma 2.5. *Assume that* $2\Gamma(\gamma+1)-a\eta^\gamma \neq 0$, $b \neq \Gamma(2-\beta)$ *and consider* $f(.) \in C(I,\mathbb{R})$. *The unique solution* $x(.) \in C(I,\mathbb{R})$ *of problem* (1.3)-(2.1) *is given by*

$$\begin{aligned} x(t) = &\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\ &+\tfrac{\Gamma(\gamma+1)}{2\Gamma(\gamma+1)-a\eta^\gamma}\left[a\int_0^\eta \tfrac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s)ds - \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds\right] \\ &+\left(t-\tfrac{\Gamma(\gamma+2)-a\eta^{\gamma+1}}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^\gamma)}\right)\tfrac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b}\int_0^1 \tfrac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s)ds. \end{aligned} \tag{2.3}$$

Remark 2.6. If we denote

$$\begin{aligned} G_2(t,s) := &\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{[0,t]}(s) \\ &+\tfrac{a\Gamma(\gamma+1)}{2\Gamma(\gamma+1)-a\eta^\gamma}\tfrac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\chi_{[0,\eta]}(s) - \tfrac{a\Gamma(\gamma+1)}{2\Gamma(\gamma+1)-a\eta^\gamma}\tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &+\left(t-\tfrac{\Gamma(\gamma+2)-a\eta^{\gamma+1}}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^\gamma)}\right)\tfrac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b}\tfrac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \end{aligned}$$

then the solution $x(.)$ in Lemma 2.5 may be written as

$$x(t) = \int_0^1 G_2(t,s)f(s)ds.$$

Moreover, if $\alpha-\beta \geq 1$, then for every $t,s \in I$, we have

$$\begin{aligned} |G_2(t,s)| \leq &\tfrac{1}{\Gamma(\alpha)} + \tfrac{|a|\Gamma(\gamma+1)}{|2\Gamma(\gamma+1)-a\eta^\gamma|}\tfrac{\eta^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} + \tfrac{|a|\Gamma(\gamma+1)}{|2\Gamma(\gamma+1)-a\eta^\gamma|}\tfrac{1}{\Gamma(\alpha)} \\ &+(1+|\tfrac{\Gamma(\gamma+2)-a\eta^{\gamma+1}}{(\gamma+1)(2\Gamma(\gamma+1)-a\eta^\gamma)}|)\tfrac{|b|\Gamma(2-\beta)}{|\Gamma(2-\beta)-b|}\tfrac{1}{\Gamma(\alpha-\beta)} =: M_2. \end{aligned}$$

Definition 2.7. A function $x(.) \in C^2(I,\mathbb{R})$ is called a solution of problem (1.1)-(1.3) if there exists a function $f(.) \in L^1(I,\mathbb{R})$ that satisfies $f(t) \in F(t,x(t),V(x)(t))$ *a.e.* (I) and $x(.)$ is given by (2.3).

3. Main results

First we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3.1. *Consider* X *a separable Banach space,* B *is the closed unit ball in* X, $H : I \to \mathcal{P}(X)$ *is a set-valued map with nonempty closed values and* $g : I \to X, L : I \to \mathbb{R}_+$ *are measurable functions. If*

$$H(t) \cap (g(t)+L(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to H(t) \cap (g(t)+L(t)B)$ *has a measurable selection.*

In order to prove our results we need the following hypotheses.

Hypothesis H1. (i) *$F(.,.): I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ measurable.*
(ii) *There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I, F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|), \ \ \forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(iii) *$k(.,.,.): I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $\forall x \in \mathbb{R},\ (t,s) \to k(t,s,x)$ is measurable.*
(iv) $|k(t,s,x) - k(t,s,y)| \leq L(t)|x - y| \quad$ *a.e.* $(t,s) \in I \times I, \ \forall\, x, y \in \mathbb{R}$.

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad M_0 = \int_0^T M(t)dt.$$

Theorem 3.2. *Assume that Hypothesis* H1 *is satisfied, $a \neq 2$, $b \neq \Gamma(2-\beta)$, $\alpha - \beta \geq 1$ and $M_1 M_0 < 1$. Let $y(.) \in C(I, \mathbb{R})$ be such that $y(0) + y(1) = a\int_0^1 y(s)ds$, $y'(0) = bD_C^\beta y(1)$ and there exists $p(.) \in L^1(I, \mathbb{R}_+)$ with*

$$d(D_c^q y(t), F(t, y(t), V(y)(t))) \leq p(t) \ \ a.e. \ \ (I).$$

Then there exists $x(.) \in C(I, \mathbb{R})$ a solution of problem (1.1)-(1.2) *satisfying for all $t \in I$,*

$$|x(t) - y(t)| \leq \frac{M_1}{1 - M_1 M_0} \int_0^1 p(t)dt. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{D_C^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - D_C^q y(t)| \leq p(t) \quad a.e.\ (I). \tag{3.2}$$

Define $x_1(t) = \int_0^1 G_1(t,s) f_1(s) ds$ and one has

$$|x_1(t) - y(t)| \leq M_1 \int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbb{R})$, $f_n(.) \in L^1(I, \mathbb{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G_1(t,s) f_n(s) ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e. \ \ (I), \tag{3.4}$$

$$
\begin{aligned}
&|f_{n+1}(t)-f_n(t)| \\
&\leq L(t)\left(|x_n(t)-x_{n-1}(t)|+\int_0^t L(s)|x_n(s)-x_{n-1}(s)|ds\right) \quad a.e.\,(I).
\end{aligned} \tag{3.5}
$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t\in I$

$$
|x_{n+1}(t)-x_n(t)|\leq M_1(M_1M_0)^n\int_0^1 p(t)dt,\ \ \forall\, n\in\mathbb{N}.
$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$
\begin{aligned}
&|x_{n+1}(t)-x_n(t)|\leq \int_0^1 |G_1(t,t_1)|.|f_{n+1}(t_1)-f_n(t_1)|dt_1 \\
&\leq M_1\int_0^1 L(t_1)\left[|x_n(t_1)-x_{n-1}(t_1)|+\int_0^{t_1} L(s)|x_n(s)-x_{n-1}(s)|ds\right]dt_1 \\
&\leq M_1\int_0^1 L(t_1)\left(1+\int_0^{t_1}L(s)ds\right)dt_1.M_1^nM_0^{n-1}\int_0^1 p(t)dt \\
&= M_1(M_1M_0)^n\int_0^1 p(t)dt.
\end{aligned}
$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$, hence converging uniformly to some $x(.)\in C(I,\mathbb{R})$. Therefore, by (3.5), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$. Moreover, one has

$$
\begin{aligned}
&|x_n(t)-y(t)|\leq |x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \\
&\leq M_1\int_0^1 p(t)dt+\sum_{i=1}^{n-1}\left(M_1\int_0^1 p(t)dt\right)(M_1M_0)^i \\
&= \frac{M_1\int_0^1 p(t)dt}{1-M_1M_0}.
\end{aligned} \tag{3.6}
$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t\in I$

$$
\begin{aligned}
&|f_n(t)-D_C^q y(t)| \\
&\leq \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-D_C^q y(t)| \\
&\leq L(t)\frac{M_1\int_0^1 p(t)dt}{1-M_1M_0}+p(t).
\end{aligned}
$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.)\in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $x(.)$ is a solution of (1.1)-(1.2). Finally, passing to the limit in (3.6) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I, \mathbb{R})$ and $f_n(.) \in L^1(I, \mathbb{R})$, $n = 1, 2, ..., N$ satisfying (3.3), (3.5) for $n = 1, 2, ..., N$ and (3.4) for $n = 1, 2, ..., N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the Lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1, 1]\} \neq \emptyset.$$

Lemma 3.1 yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq L(t)\left(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds\right).$$

We define $x_{N+1}(.)$ as in (3.3) with $n = N+1$. Thus $f_{N+1}(.)$ satisfies (3.4) and (3.5) and the proof is complete. □

The assumptions in Theorem 3.3 are satisfied, in particular, for $y(.) = 0$ and therefore with $p(.) = L(.)$. We obtain the following consequence of Theorem 3.3.

Corollary 3.3. *Assume that Hypothesis* 3.2 *is satisfied,* $a \neq 2$, $b \neq \Gamma(2-\beta)$, $\alpha - \beta \geq 1$, $M_1 M_0 < 1$ *and* $d(0, F(t, 0, V(0)(t)) \leq L(t)$ *a.e.* (I). *Then there exists* $x(.)$ *a solution of problem* (1.1)-(1.2) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{M_1}{1 - M_1 M_0} \int_0^1 L(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 became

Hypothesis H2. (i) $F(.,.) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R})$ *measurable.*
(ii) *There exists* $L(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I$, $F(t, .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t, x_1), F(t, x_2)) \leq L(t)|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

Denote $L_0 = \int_0^1 L(t)dt$.

Corollary 3.4. *Assume that Hypothesis* H2 *is satisfied,* $a \neq 2$, $b \neq \Gamma(2-\beta)$, $\alpha - \beta \geq 1$, $M_1 M_0 < 1$ *and* $d(0, F(t,0) \leq L(t)$ *a.e.* (I). *Then there exists* $x(.)$ *a solution of problem* (1.2)-(1.4) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{M_1 L_0}{1 - M_1 L_0}.$$

Remark 3.5. A similar result to the one in Corollary 3.4 may be found in [1], namely Theorem 3.7. Assuming that $L(.) \in C(I, \mathbb{R})$ the result in [1] provides the existence of solutions of problem (1.2)-(1.4) under the condition $||L(.)||_C \Lambda < 1$, where

$$\Lambda = \frac{1 + |2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta) - b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}.$$

In our approach we assume only that $L(.) \in L^1(I, \mathbb{R})$. The price we pay is that we suppose that $\alpha - \beta \geq 1$ and our contraction qualification is $||L(.)||_1 M_1 < 1$.

On the other hand, the approach in [1], apart from the requirement that the values of $F(.,.)$ are compact, does not provides a priori bounds as in (3.7).

In the case when we consider nonlocal integral-flux boundary conditions, namely problem (1.1)-(1.3), with a similar proof as the one of Theorem 3.3 we obtain the following existence result.

Theorem 3.6. *Assume that Hypothesis* H1 *is satisfied,* $2\Gamma(\gamma+1) - a\eta^\gamma \neq 0$, $b \neq \Gamma(2-\beta)$, $\alpha - \beta \geq 1$ *and* $M_2 M_0 < 1$. *Let* $y(.) \in C(I, \mathbb{R})$ *be such that* $y(0) + y(1) = aI^\gamma y(\eta)$, $y'(0) = bD_C^\beta y(1)$ *and there exists* $p(.) \in L^1(I, \mathbb{R}_+)$ *with* $d(D_c^q y(t), F(t, y(t), V(y)(t))) \leq p(t)$ *a.e.* (I). *Then there exists* $x(.) \in C(I, \mathbb{R})$ *a solution of problem* (1.1)-(1.3) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_2}{1 - M_2 M_0} \int_0^1 p(t)dt.$$

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