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GENERALIZED ρ-(η, θ)-INVEXITY AND KKT CONDITIONS FOR OPTIMALITY

N. Behera[1], C. Nahak[2] and S. Nanda[3]

[1]Department of Mathematics, School of Applied Sciences
KIIT University, Bhubaneswar-751024, India
e-mail: narmada.iitkgp@gmail.com

[2]Department of Mathematics
IIT Kharagpur, India
e-mail: cnahak@maths.iitkgp.ernet.in

[3]Department of Mathematics, School of Applied Sciences
KIIT University, Bhubaneswar-751024, India
e-mail: snanda@kiit.ac.in

Abstract. In this work, we introduce the notion of ρ-(η, θ)-invex function between Banach spaces. By using these functions, we obtain the Karush-Kuhn-Tucker (KKT) sufficient conditions for an optimization problem in a Banach space. A pair of duality results are also studied under the generalized ρ-(η, θ)-invexity assumptions on the functions involved.

1. Introduction and formulation of the problem

In the classical theory of optimization, the theorems on sufficient optimality conditions and duality are based on convexity assumptions, which are rather restrictive in applications. Many attempts have been made to weaken these assumptions by introducing generalized convexity. In 1965, Mangasarian [4] proved that if the objective function is pseudo-convex and the constraint functions are quasi-convex, then Karush Kuhn Tucker conditions are sufficient for a global solution. Sixteen years later, Hanson [2] introduced the invex function and showed that the Karush Kuhn Tucker conditions are sufficient for

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[0]Corresponding author: N. Behera.

global solution if the objective function and constraint functions are invex with respect to same η for detail (see, [6, 7, 8] etc).

Consider the following problem of optimization

$$(P)\ \ \min_{x\in C}\ f(x) \tag{1.1}$$

$$\text{subject to } \ g(x) \leq 0, \tag{1.2}$$

C is a convex subset of a Banach space X, f is a real-valued Frechet differentiable function on C, g is a Frechet differentiable function from C into a Banach space Y having a positive cone P.

According to generalized Karush-Kuhn-Tucker theorem [3] it is necessary, under certain constraint qualification, that for x_0 to be minimal in the problem, there is a $y_0^* \in Y^*$, $y_0^* \geq 0$ such that the Lagrangian

$$f(x) + \langle y_0^*, g(x)\rangle$$

is satisfied at x_0 *i.e.*,

$$f'(x_0) + y_0^*\ g'(x_0) = 0, \tag{1.3}$$

$$\langle y_0^*, g(x_0)\rangle\ = 0, \tag{1.4}$$

$$y_0^* \geq 0. \tag{1.5}$$

We also write the Lagrangian in functional notation

$$L(x, y_0^*) = f(x) + y_0^* g(x),$$

since this is similar to the convenient for the finite dimensional theory.

We denote X^*, be the space of continuous linear functionals on X. For $x^* \in X^*$, define

$$\langle x^*, x\rangle = x^*(x) = x^*x.$$

2. Generalized ρ-(η, θ)-invexity

For $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, Hanson [2] proved that if all functions $f, f_1, f_2, \cdots,$ f_m are invex with respect to same η, then conditions (1.3)-(1.5) are also sufficient. It will be shown that these conditions are sufficient for another wider classes of functions (generalized ρ-(η, θ)-invex).

Definition 2.1. Let P be a convex cone with nonempty interior in a vector space X. For $x, y \in X$, we write $x \geq y$ (with respect to P) if $x - y \in P$. This cone P defining this relation is called a positive cone in X. The cone $N = -P$ is called the negative cone in X and we write $y \leq x$ for $y - x \in N$.

Definition 2.2. A Frechet differentiable function $f : X \to \mathbb{R}$ is said to be ρ-(η, θ)-invex with respect to η, θ, if there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that

$$f(x_0) - f(x_1) \geq \langle f'(x_1), \eta(x_0, x_1)\rangle + \rho \parallel \theta(x_0, x_1) \parallel^2, \tag{2.1}$$

for all $x_0, x_1 \in X$.

Definition 2.3. A Frechet differentiable function $f : X \to \mathbb{R}$ is said to be ρ-(η, θ)-pseudo-invex with respect to η, θ, if there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that

$$\langle f'(x_1), \eta(x_0, x_1)\rangle \geq -\rho \parallel \theta(x_0, x_1) \parallel^2 \Rightarrow f(x_0) \geq f(x_1), \tag{2.2}$$

for all $x_0, x_1 \in X$.

Definition 2.4. A Frechet differentiable function $f : X \to \mathbb{R}$ is said to be ρ-(η, θ)-quasi-invex with respect to η, θ, if there exist $\eta : X \times X \to X$ and $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that

$$f(x_0) \leq f(x_1) \Rightarrow \langle f'(x_1), \eta(x_0, x_1)\rangle \leq -\rho \parallel \theta(x_0, x_1) \parallel^2, \tag{2.3}$$

for all $x_0, x_1 \in X$.

Definition 2.5. Let X and Y be two Banach spaces. A Frechet differentiable function $g : X \to Y$ is said to be ρ_1-(η, θ)-invex if for all $y^* \in Y^*$, there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho_1 \in \mathbb{R}$ such that

$$y^* \circ g(x) - y^* \circ g(y) \geq y^*\langle g'(y), \eta(x, y)\rangle + \rho_1 \parallel \theta(x, y) \parallel^2, \tag{2.4}$$

for all $x, y \in X$.

Definition 2.6. A Frechet differentiable function $g : X \to Y$ is said to be ρ_1-(η, θ)-pseudo-invex if for all $y^* \in Y^*$, there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho_1 \in \mathbb{R}$ such that

$$y^*\langle g'(y), \eta(x, y)\rangle \geq -\rho_1 \parallel \theta(x, y) \parallel^2 \Rightarrow y^* \circ g(x) \geq y^* \circ g(y), \tag{2.5}$$

for all $x, y \in X$.

Definition 2.7. A Frechet differentiable function $g : X \to Y$ is said to be ρ_1-(η, θ)-quasi-invex if for all $y^* \in Y^*$, there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho_1 \in \mathbb{R}$ such that

$$y^* \circ g(x) \leq y^* \circ g(y) \Rightarrow y^*\langle g'(y), \eta(x, y)\rangle \leq -\rho_1 \parallel \theta(x, y) \parallel^2, \tag{2.6}$$

for all $x, y \in X$.

Lemma 2.8. *Every Frechet differentiable invex function f is ρ-(η,θ)-invex for $\rho \leq 0$. The converse is true $\rho \geq 0$, but for $\rho < 0$, this is false.*

Proof. It is clear that every invex function f is ρ-(η,θ)-invex. The converse is not true, which follows from the following counter example. □

Example 2.1. Let $f : (0, \frac{1}{2}) \to \mathbb{R}$ be a mapping defined by

$$f(x) = -x^3.$$

Let the maps η and θ be defined by

$$\eta(y,x) = \begin{cases} y - x, & \text{if } y > x, \\ 0, & \text{if } y \leq x, \end{cases}$$

and

$$\theta(y,x) = \begin{cases} \sqrt{y - x}, & \text{if } y > x, \\ 0, & \text{if } y \leq x. \end{cases}$$

Taking $\rho = -1$, we have to show

$$f(y) - f(x) \geq \langle f'(x), \eta(y,x)\rangle + \rho \parallel \theta(y,x) \parallel^2,$$

i.e.,

$$f(y) - f(x) - \left[\langle f'(x), \eta(y,x)\rangle + \rho \parallel \theta(y,x) \parallel^2\right] \geq 0, \tag{2.7}$$

the LHS of the expression (2.7) becomes

$$x^3 - y^3 - \left[- 3x^2(y - x) - (y - x)\right],$$

then

$$(x - y)(x^2 + xy + y^2) + 3x^2(y - x) + (y - x) \geq 0.$$

If $x \geq y$ we obtain

$$(x - y)(x^2 + xy + y^2) \geq 0$$

and if $x < y$ we get

$$(y - x)\left[(3x^2 + 1) - (y^2 + xy + x^2)\right] > 0.$$

Thus f is ρ-(η,θ)-invex but not invex at $x = \frac{1}{4}$, $y = \frac{1}{3}$ because

$$f(y) - f(x) < \langle f'(x), \eta(y,x)\rangle.$$

Lemma 2.9. *Every Frechet differentiable quasi-invex function f is ρ-(η,θ)-quasi-invex for $\rho \leq 0$ but the converse is not true.*

Proof. It is clear that every quasi-invex function f is ρ-(η,θ)-quasi-invex. The converse is not true, which follows from the following counter example. □

Example 2.2. Let $f:(0,\frac{\pi}{2})\to\mathbb{R}$ be a mapping defined by

$$f(x)=\sin x.$$

Let the maps η and θ be defined by

$$\eta(y,x)=x-y$$

and

$$\theta(y,x)=\begin{cases}\sqrt{x-y}, & \text{if } y\le x,\\ 0, & \text{if } y>x.\end{cases}$$

Taking $\rho=-1$, we have to prove

$$f(y)\le f(x)\quad\Rightarrow\quad\langle f'(x),\eta(y,x)\rangle+\rho\parallel\theta(y,x)\parallel^2\le 0.$$

Now

$$f(y)-f(x)=\sin y-\sin x\le 0\quad\text{if}\quad y\le x.$$

It follows that

$$\begin{aligned}\langle f'(x),\eta(y,x)\rangle+\rho\ \parallel\theta(y,x)\parallel^2 &= \cos x(x-y)-(x-y)\\ &= (\cos x-1)(x-y)\\ &\le 0\end{aligned}$$

if $y\le x$. For $y>x$, it is true vacuously. But f is not quasi-invex at $x_1=\frac{\pi}{3}$ and $y_1=\frac{\pi}{6}$, as

$$f(y_1)-f(x_1)=\sin y_1-\sin x_1=\frac{(1-\sqrt{3})}{2}<0$$

and

$$\langle f'(x_1),\eta(y_1,x_1)\rangle=\cos x_1(x_1-y_1)=\frac{\pi}{12}>0.$$

Lemma 2.10. *Every pseudo-invex function f is ρ-(η,θ)-pseudo-invex but the converse is not true.*

Proof. It is clear that every pseudo-invex function f is ρ-(η,θ)-pseudo-invex. The converse is not true, which follows from the following counter example. □

Example 2.3. Let $f:(0,\frac{\pi}{2})\to\mathbb{R}$ be a mapping defined by

$$f(x)=\sin x-1.$$

Let the maps η and θ be defined by

$$\eta(y,x)=\begin{cases}\sin y-\sin x, & \text{if } y>x,\\ 0, & \text{if } y\le x,\end{cases}$$

and

$$\theta(y,x)=\begin{cases}\sqrt{\sin x-\sin y}, & \text{if } x>y,\\ 0, & \text{if } x\le y.\end{cases}$$

Taking $\rho = -1$, we have to show

$$\langle f'(x), \eta(y,x)\rangle + \rho \parallel \theta(y,x) \parallel^2 \geq 0 \quad \Rightarrow \quad f(y) \geq f(x).$$

Now

$$\begin{aligned}\langle f'(x), \eta(y,x)\rangle + \rho \ \parallel \theta(y,x) \parallel^2 &= \cos x(\sin y - \sin x) \\ &\geq 0\end{aligned}$$

if $y \geq x$. It follows that

$$f(y) - f(x) = \sin y - \sin x \geq 0$$

if $y \geq x$. For $y < x$, it is true vacuously. But f is not pseudo-invex at $x_1 = \frac{\pi}{4}$ and $y_1 = \frac{\pi}{6}$, as

$$\langle f'(x_1), \eta(y_1, x_1)\rangle = 0$$

and

$$f(y_1) - f(x_1) = \sin y_1 - \sin x_1 = \frac{1}{2} - \frac{1}{\sqrt{2}} < 0.$$

3. Optimality in the presence of ρ-(η, θ)-invexity

In this section, we show that in problem (1.1), (1.2), if f is ρ-(η,θ)-invex and $g(x)$ is satisfying (2.4), the Karush Kuhn-Tucker conditions are also sufficient conditions for optimality.

Theorem 3.1. *Let f and g be Frechet differentiable functions on $C \subset X$, X is a Banach space and C is a closed convex cone with nonempty interior satisfying* (2.1) *and* (2.4) *on $C \times C$. If there exists $x_0 \in C$ and $v_0^* \in Y^*$ satisfying the Karush-Kuhn-Tucker conditions* (1.3)-(1.5) *and $\rho + \rho_1 \geq 0$ then*

$$f(x_0) = \min_{x \in C}\{f(x) : g(x) \leq 0\}.$$

Proof. For $x \in C$ satisfying $g(x) \leq 0$, we have

$$f(x) - f(x_0) \geq \ \langle f'(x_0), \eta(x, x_0)\rangle + \rho \parallel \theta(x, x_0) \parallel^2 .$$

By (1.3), it follows that

$$f(x) - f(x_0) \geq -v_0^* \langle g'(x_0), \eta(x, x_0)\rangle + \rho \parallel \theta(x, x_0) \parallel^2 . \tag{3.1}$$

As

$$v_0^* g(x) - v_0^* g(x_0) \geq v_0^* \langle g'(x_0), \eta(x, x_0)\rangle + \rho_1 \parallel \theta(x, x_0) \parallel^2,$$

we have

$$-v_0^* \langle g'(x_0), \eta(x, x_0)\rangle \ \geq -v_0^* g(x) + v_0^* g(x_0) + \rho_1 \parallel \theta(x, x_0) \parallel^2 .$$

Hence (3.1) becomes

$$f(x) - f(x_0) \geq -v_0^* g(x) + v_0^* g(x_0) + (\rho + \rho_1) \parallel \theta(x, x_0) \parallel^2 .$$

Since $v_0^* g(x_0) = 0$ and $\rho + \rho_1 \geq 0$, we have

$$f(x) - f(x_0) \geq 0,$$

that is

$$f(x_0) \leq f(x), \ \forall \, x \in C.$$

Hence x_0 is minimal, which proves the theorem. □

4. Duality

We consider the following pair of problems defined on $C \subset X$ (Banach space), C is a closed convex cone with nonempty interior.

$$\textbf{Primal Problem(P)} \ \min_{x \in C} \ f(x) \tag{4.1}$$

$$\text{subject to: } g(x) \leq 0. \tag{4.2}$$

$$\textbf{Dual Problem(D)} \ \max_{(u,v^*)} \ f(u) + \langle v^*, g(u) \rangle \tag{4.3}$$

$$\text{subject to: } f'(u) + \langle v^*, g'(u) \rangle = 0, \tag{4.4}$$

$$v^* \geq 0. \tag{4.5}$$

Theorem 4.1. *Under the conditions of the Karush-Kuhn-Tucker theorem, if x_0 is minimal in primal problem (P), then (x_0, v_0^*) is maximal in the dual problem (D), where v_0^* is given by the Karush-Kuhn-Tucker conditions and f and g are satisfying* (2.1) *and* (2.4) *with $\rho + \rho_1 \geq 0$, then the extreme values are equal in the two problems.*

Proof. Let (u, v^*) be any point satisfying constraints (4.4) and (4.5) of the dual problem (D), then using (2.1) and (4.4), we get

$$\begin{aligned} &\Big[f(x_0) + \langle v_0^*, g(x_0) \rangle \Big] - \Big[f(u) + \langle v^*, g(u) \rangle \Big] \\ &= \ f(x_0) - f(u) - \langle v^*, g(u) \rangle \qquad\qquad (as \ \langle v_0^*, g(x_0) \rangle = 0) \\ &\geq \langle f'(u), \eta(x_0, u) \rangle + \rho \parallel \theta(x_0, u) \parallel^2 - \langle v^*, g(u) \rangle \\ &= \langle -v^* g'(u), \eta(x_0, u) \rangle + \rho \parallel \theta(x_0, u) \parallel^2 - \langle v^*, g(u) \rangle. \end{aligned} \tag{4.6}$$

But

$$v^* g(x_0) - v^* g(u) \geq \langle v^* g'(u), \eta(x_0, u) \rangle + \rho_1 \parallel \theta(x_0, u) \parallel^2 .$$

That is

$$-\langle v^* g'(u), \eta(x_0, u \rangle \geq -v^* g(x_0) + v^* g(u) + \rho_1 \parallel \theta(x_0, u) \parallel^2 .$$

So the RHS of (4.6) becomes

$$\begin{aligned}&\Big[f(x_0)+\langle v_0^*, g(x_0)\rangle\Big]-\Big[f(u)+\langle v^*, g(u)\rangle\Big]\\ &\geq -v^*g(x_0)+v^*g(u)+(\rho+\rho_1)\parallel \theta(x_0,u)\parallel^2 -v^*g(u)\\ &\geq -v^*g(x_0)+(\rho+\rho_1)\parallel \theta(x_0,u)\parallel^2\\ &\geq 0.\end{aligned}$$

So (x_0, v_0^*) is maximal in the dual problem (D), and since $\langle v_0^*, f(x_0)\rangle = 0$, the extrema of the two problems are equal. This completes the proof. □

5. Optimality in the presence of ρ-(η,θ)-pseudo-invexity

In this section, we show that in problem (1.1), (1.2), if $f(x)$ is ρ-(η,θ)-pseudo-invex and $g(x)$ is satisfying (2.6), the Karush Kuhn-Tucker conditions are also sufficient conditions for optimality.

Theorem 5.1. *Let f and g satisfy* (2.2) *and* (2.6) *respectively. If there exists $x_0 \in C$ and $v_0^* \in Y^*$ satisfy the Karush Kuhn-Tucker conditions* (1.3)-(1.5) *and* $(\rho+\rho_1)\geq 0$, *then*

$$f(x_0)=\min_{x\in C}\{f(x): g(x)\leq 0\}.$$

Proof. For $x\in C$ satisfying $g(x)\leq 0$, we have,

$$\begin{aligned}&\langle f'(x_0), \eta(x,x_0)\rangle+\rho\parallel \theta(x,x_0)\parallel^2\\ &= f'(x_0)\eta(x,x_0)+\rho\parallel \theta(x,x_0)\parallel^2\\ &= -v_0^*g'(x_0)\eta(x,x_0)+\rho\parallel \theta(x,x_0)\parallel^2 \quad (\textit{ by } (1.3)).\end{aligned}\tag{5.1}$$

Since $v_0^*g(x)-v_0^*g(x_0)\leq 0$ (as $v_0^*g(x_0)=0$ and $v_0^*g(x)\leq 0$)), by ρ-(η,θ)-quasi-invexity of g, we have

$$\langle v_0^*g'(x_0), \eta(x,x_0)\rangle \ \leq -\rho_1\parallel \theta(x,x_0)\parallel^2,$$

i.e.,

$$-\langle v_0^*g'(x_0), \eta(x,x_0)\rangle \ \geq \rho_1\parallel \theta(x,x_0)\parallel^2 .$$

So from (5.1), we obtain

$$\begin{aligned}\langle f'(x_0), \eta(x,x_0)\rangle+\rho\parallel \theta(x,x_0)\parallel^2 &\geq (\rho+\rho_1)\parallel \theta(x,x_0)\parallel^2\\ &\geq 0\end{aligned}$$

as $(\rho+\rho_1)\geq 0$. Hence

$$\langle f'(x_0), \eta(x,x_0)\rangle\geq -\rho\parallel \theta(x,x_0)\parallel^2 .$$

By ρ-(η,θ)-pseudo-invexity of f, we have

$$f(x)-f(x_0)\geq 0,$$

hence

$$f(x) \geq f(x_0).$$

This completes the proof. □

6. Concluding remarks

We have demonstrated sufficient optimality conditions and duality for optimization problems in Banach space involving generalized ρ-(η, θ)-invex functions. Most of the corresponding results for the invex case have been extended to these problems under several types of generalized ρ-(η, θ)-invexity assumptions. So our results contain many known ones. As generalized ρ-(η, θ)-invexity is a generalization of invexity, variational problem and control problems in Banach space under generalized ρ-(η, θ)-invexity will orient future research of the authors.

Further note that ρ-(η, θ)-invexity is useful if all the needed functions are ρ-(η, θ)-invex with respect to the same functions η, θ. If one of them does not satisfy this condition, then it is hard to assert something even if it is convex.

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