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SOME COUPLED FIXED POINT WITHOUT MIXED MONOTONE MAPPINGS

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Abstract. The aim of this paper is to derive existence and uniqueness coupled fixed point results under generalized contractive condition without monotone mappings. Our results extend, generalize, unify and improve the existing results; in particular, results of Radenovi.

1. Introduction

In recent years, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering $\preceq$ in the literature. The contraction mapping theorem and the abstract monotone iterative technique are well known and are applicable to a variety of situations. Recently, there is a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. In the context of ordered metric

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spaces, the usual contraction is weakened but at the expense that the operator is monotone. It is of interest to determine if it is still possible to establish the existence of a unique fixed point assuming that the operator considered is monotone in such a setting. The first result in this direction was given by Ran and Rearing [13, Theorem 3.1] who presented an analogue of Banach's fixed point theorem in partially ordered sets. It was applied to the resolution of matrix equations. Subsequently many works have been done in this line.

On the other hand, Guo and Laksmikantham [2] introduced the notion of coupled fixed point. In 2006 Bhaskar and Lakshmikantham [1] reconsidered the concept of a coupled fixed point of the mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Bhaskar and Lakshmikantham [1] also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. As, an application, they studied the existence and uniqueness of the solution for a periodic boundary value problem associated with first order differential equation. For detail one can refer [4]-[12], [15]-[17]. Recently, Radenovic [13] introduced a notion of monotone mappings and derived coupled fixed point results without use of mixed monotone property. In this paper, we use the concept of monotone property [3, 13] and prove coupled fixed point results for relational type contraction conditions. Our result generalizes the result of Radenovic [13] and many similar types of results.

2. Preliminaries

We start out with listing some notations and preliminaries that we shall need to express our results. In this paper $(X, d, \preceq)$ denotes a partially ordered metric space where $(X, \preceq)$ is a partially ordered set and (X, d) is a metric space.

Definition 2.1. ([2]) An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping F if $F(x, y) = x$ and $F(x, y) = y$. It is clear that (x, y) is a coupled fixed point of F if and only if (y, x) is a coupled fixed point of F.

Definition 2.2. ([13]) Let $(X, d, \preceq)$ be a partial order set and $F : X \times X \to X$ be a mapping. Then a map F is said to have the monotone property if $F(x, y)$ is monotone nondecreasing in both variables x and y, that is for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1) \preceq F(x, y_2).$$

Definition 2.3. ([13]) Let $(X, \preceq)$ be an ordered set and d be a metric space on X. We say that $(X, d, \preceq)$ is regular if it has the following properties:

(1) If a non-decreasing sequence $\{x_n\}$ holds $d(x_n, x) \to 0$, then $x_n \preceq x$ for all n.
(2) If a non-increasing sequence $\{y_n\}$ holds $d(y_n, y) \to 0$, then $y_n \succeq y$ for all n.

Lemma 2.4. ([13])

(1) *Let $(X, d, \preceq)$ be a partially ordered metric space. If relation $\sqsubseteq$ is defined on $X^2 = X \times X$ by,*

$$Y \sqsubseteq V \quad \Leftrightarrow \quad x \preceq u \wedge y \preceq v,\ Y = (x, y),\ V = (u, v) \in X^2$$

and $d_+ : X^2 \times X^2 \to R^2$ is given by

$$d_+(Y, V) = d(x, u) + d(y, v),\ Y = (x, y),\ V = (u, v) \in X^2.$$

Then $(X^2, \sqsubseteq, d_+)$ is an ordered metric space. The space (X^2, d_+) is a complete if and only if (X, d) is a complete. Also, the space $(X^2, \sqsubseteq, d_+)$ is a regular if and only if $(X, d, \preceq)$ is a regular.

(2) *If $F : X \times X \to X$ then the mapping $T_F : X \times X \to X \times X$ given by*

$$T_F(Y) = (F(x, y), F(y, x)),\ Y = (x, y) \in X^2,$$

is non-decreasing with respect to $\sqsubseteq$, that is

$$Y \sqsubseteq V \quad \Rightarrow \quad T_F(Y) \sqsubseteq T_F(V).$$

(3) *The mapping F is continuous if and only if T_F is continuous.*
(4) *Mapping $F : X^2 \to X$ has a coupled fixed point if and only if mapping T_F has a fixed point in X^2.*

Lemma 2.5. ([13]) *Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in (X, d), then there exist $\varepsilon > 0$ and two sequences $m\{k\}$ and $n\{k\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d(y_{m(k)}, y_{n(k)}),\ d(y_{m(k)}, y_{n(k)+1}),\ d(y_{m(k)-1}, y_{n(k)}),\ d(y_{m(k)-1}, y_{n(k)+1}).$$

3. Main results

Our first result is the following :

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ be a continuous mapping having the monotone property on X and satisfying*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \frac{\alpha}{2}[d(x,u)+d(y,v)] &+ \beta N((x,y),(u,v)) \\ &+ \frac{\gamma}{2}[d(x,F(x,y))+d(u,F(u,v)) \\ &+ d(y,F(y,x))+d(v,F(v,u))] \end{aligned} \tag{3.1}$$

for all $(x,y),(u,v) \in X \times X$ with $x \preceq u$ and $y \preceq v$, when

$$D_1 = d(x,F(u,v)) + d(u,F(x,y)) \neq 0$$

and

$$D_2 = d(y,F(v,u)) + d(v,F(y,x)) \neq 0,$$

where

$$\begin{aligned} &N((x,y),(u,v)) \\ &= \min\left\{\frac{d^2(x,F(u,v))+d^2(u,F(x,y))}{d(x,F(u,v))+d(u,F(x,y))}, \frac{d^2(y,F(v,u))+d^2(v,F(y,x))}{d(y,F(u,v))+d(v,F(y,x))}\right\}, \end{aligned} \tag{3.2}$$

and $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$. Further,

$$d(F(x,y),F(u,v)) = 0 \quad \textit{if} \quad D_1 = 0 \quad \textit{and} \quad D_2 = 0. \tag{3.3}$$

We assume that there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \quad \textit{and} \quad y_0 \preceq F(y_0,x_0). \tag{3.4}$$

Then, F has a coupled fixed point $(\overline{x}, \overline{y}) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \preceq F(y_0,x_0)$. Since $F : X \times X \to X$, we can choose $x_1, y_1 \in X$ such that $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Again from $F : X \times X \to X$ we can choose $x_2, y_2 \in X$ such that $x_2 = F(x_1,y_1)$ and $y_2 = F(y_1,x_1)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(y_n,x_n), \text{ for all } n \geq 0.$$

We shall show that,

$$x_n \preceq x_{n+1} \text{ and } y_n \preceq y_{n+1}, \text{ for all } n \geq 0. \tag{3.5}$$

We will use the mathematical induction. For $n = 0$, since $x_0 \preceq F(x_0,y_0)$ and $y_0 \preceq F(y_0,x_0)$, and as $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$, we have that $x_0 \preceq x_1$ and $y_0 \preceq y_1$. Thus (3.5) holds for $n = 0$.

Suppose now that (3.5) holds for $n \geq 0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \preceq y_{n+1}$ and so by the monotone property of F, we have

$$x_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = x_{n+2} \tag{3.6}$$

and

$$y_{n+1} = F(y_n, x_n) \preceq F(y_{n+1}, x_n) \preceq F(y_{n+1}, x_{n+1}) = y_{n+2}.$$

That is (3.5) holds for all $n \geq 0$. If $x_{n+1} = x_{n+2}$ and $y_{n+1} = y_{n+2}$ for some n, then $F(x_{n+1}, y_{n+1}) = x_{n+1}$ and $F(y_{n+1}, x_{n+1}) = y_{n+1}$, hence (x_{n+1}, y_{n+1}) is a coupled fixed point of F. Suppose further that $x_{n+1} \neq x_{n+2}$ or $y_{n+1} \neq y_{n+1}$ for each $n \in \mathbb{N}_0$. Now, we claim that, for $n \in \mathbb{N}_0$,

$$d(x_{n+1}, x_n)+d(y_{n+1}, y_n) \leq ((\alpha+\beta+\gamma)/(1-\beta-\gamma))^n[d(x_1, x_0)+d(y_1, y_0)] \tag{3.7}$$

Indeed, for $n = 1$, consider the following possibilities.

Case I. Suppose $x_0 \neq x_2$ and $y_0 \neq y_2$. Then

$$d(x_1, F(x_0, y_0)) + d(x_0, F(x_1, y_1)) \neq 0$$

and

$$d(y_1, F(y_0, x_0)) + d(y_0, F(y_1, x_1)) \neq 0.$$

Hence using $x_1 \succeq x_0$, $y_1 \succeq y_0$ and (3.1), we get

$$\begin{aligned}
&d(x_2, x_1)\\
&= d(F(x_1, y_1), F(x_0, y_0))\\
&\leq \frac{\alpha}{2}[d(x_1, x_0) + d(y_1, y_0)] + \beta N(x_1, y_1), (x_0, y_0)\\
&\quad + \frac{\gamma}{2}[d(x_1, F(x_1, y_1)) + d(x_0, F(x_0, y_0)) + d(y_1, F(y_1, x_1)) + d(y_0, F(y_0, x_0))]
\end{aligned}$$

or,

$$\begin{aligned}
d(x_2, x_1) \leq{}& \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)] + \beta\frac{d^2(x_1, F(x_0, y_0)) + d^2(x_1, F(x_1, y_1))}{d(x_1, F(x_0, y_0)) + d(x_0, F(x_1, y_1))}\\
&+ \frac{\gamma}{2}[d(x_1, x_2) + d(x_0, x_1) + d(y_1, y_2) + d(y_0, y_1)]
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(x_2, x_1) \leq{}& \frac{\alpha}{2}[d(x_0, x_1) + d(y_0, y_1)] + \beta[d(x_0, x_1) + d(x_1, x_2)]\\
&+ \frac{\gamma}{2}[d(x_0, x_1) + d(y_0, y_1) + d(x_1, x_2) + d(y_1, y_2)].
\end{aligned} \tag{3.8}$$

Similarly, using that

$$d(y_2, y_1) = d(F(y_1, x_1), F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$$

and

$$N((y_1,x_1),(y_0,x_0)) \le \frac{d^2(y_1,F(y_0,x_0))+d^2(y_0,F(y_1,x_1))}{d(y_1,F(y_0,x_0))+d(y_0,F(y_1,x_1))} = d(y_0,y_2) \le d(y_0,y_1)+d(y_1,y_2),$$

we get,

$$d(y_2,y_1) \le \frac{\alpha}{2}[d(x_0,x_1)+d(y_0,y_1)] + \beta[d(y_0,y_1)+d(y_1,y_2)] + \frac{\gamma}{2}[d(x_0,x_1)+d(y_0,y_1)+d(x_1,x_2)+d(y_1,y_2)]. \tag{3.9}$$

Adding (3.8) and (3.9) we have,

$$d(x_2,x_1)+d(y_2,y_1) \le \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}[d(x_0,x_1)+d(y_0,y_1)]. \tag{3.10}$$

Case II. Suppose $x_0 = x_2$ and $y_0 \neq y_2$. The first equality implies that $d(x_1,F(x_0,y_0))+d(x_0,F(x_1,y_1)) \neq 0$, and hence

$$d(x_1,x_2) = d(F(x_0,y_0),F(x_1,y_1)) = 0,$$

by (3.3). This means that $x_0 = x_1 = x_2$. From $y_0 \neq y_2$, as in the first case, we get that (3.7) holds true. As a consequence

$$d(y_1,y_2) \le \left(\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}}\right) d(y_0,y_1) \le \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} d(y_0,y_1),$$

since $\left(\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}}\right) < \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. But then $d(x_0,x_1) = d(x_1,x_2) = 0$ implies that (3.8) holds. The case $x_0 \neq x_2$ and $y_0 = y_2$ is treated analogously.

Case III. Suppose $x_0 = x_2$ and $y_0 = y_2$. Then

$$d(x_1,F(x_0,y_0))+d(x_0,F(x_1,y_1)) = 0$$

and

$$d(y_1,F(y_0,x_0))+d(y_0,F(y_1,x_1)) = 0.$$

Hence, (3.3) implies that $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, and so (3.7) holds trivially. Thus (3.7) holds for $n = 1$. In a similar way, proceeding by induction, if we assume that (3.6) holds, we get that

$$d(x_{n+2},x_{n+1})+d(y_{n+2},y_{n+1}) \le \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(x_{n+1},x_n)+d(y_{n+1},y_n)] \le \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n [d(x_0,x_1)+d(y_0,y_1)].$$

Hence, by induction, (3.7) is proved. Set, $h_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}), n \in \mathbb{N}$ and $\rho = \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) < 1$. Then, the sequence $\{h_n\}$ is decreasing and, $h_n \leq \rho^n h_0$. Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence. Then (by Lemma 2.6) there exists $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d_+(z_{m(k)}, z_{n(k)}),\ d_+(z_{m(k)}, z_{n(k)+1}),\ d_+(z_{m(k)-1}, z_{n(k)}),\ d_+(z_{m(k)-1}, z_{n(k)+1}),$$

where $z_n = (x_n, y_n)$ is a sequence in (X^2, d_+). Putting

$$(x, y) = (x_{m(k)-1}, y_{m(k)-1})$$

and

$$(u, v) = (x_{n(k)}, y_{n(k)})$$

in (3.1), we have

$$\begin{aligned}
&d(F(x_{m(k)-1}, y_{m(k)-1}), F(x_{n(k)}, y_{n(k)})) \\
&\leq \frac{\alpha}{2}[d(x_{m(k)-1}, x_n(k)) + d(y_{m(k)-1}, y_{n(k)})] \\
&\quad + \beta N((x_{m(k)-1}, y_{m(k)-1}), (x_{n(k)}, y_{n(k)})) \\
&\quad + \frac{\gamma}{2}[d(x_{m(k)-1}, F(x_{m(k)-1}, y_{m(k)-1})) + d(x_{n(k)}, F(x_{(k)}, y_{n(k)})) \\
&\quad + d(y_{m(k)-1}, F(y_{m(k)-1}, x_{m(k)-1})) + d(y_{n(k)}, F(y_{n(k)}, x_{n(k)}))
\end{aligned}$$

i.e.,

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k+1)}) \\
&\leq \frac{\alpha}{2}[d(x_{m(k)-1}, x_{n(k)}) + d(y_{m(k)-1}, y_{n(k)})] \\
&\quad + \beta N((x_{m(k)-1}, y_{m(k)-1}), (x_{n(k)}, y_{n(k)})) \\
&\quad + \frac{\gamma}{2}[d(x_{m(k)-1}, F(x_{m(k)-1}, y_{m(k)-1})) + d(x_{n(k)}, F(x_{n(k)}), y_{n(k)}))) \\
&\quad + d(y_{m(k)-1}, F(y_{m(k)-1}, x_{m(k)-1})) + d(y_{n(k)}), F(y_{n(k)}), x_{n(k)})))].
\end{aligned} \tag{3.11}$$

Similarly, putting $(y, x) = (y_{m(k)-1}, x_{m(k)-1})$ and $(v, u) = (y_{n(k)}), x_{n(k)}))$ in (3.1), we obtain

$$\begin{aligned}
&d(F(y_{m(k)-1}, x_{m(k)-1}, F(y_{n(k)}, x_{n(k)})) \\
&\leq \frac{\alpha}{2}[d(y_{m(k)-1}, y_{n(k)}) + d(x_{m(k)-1}, x_{n(k)})] \\
&\quad + \frac{\gamma}{2}[d(y_{m(k)-1}, F(y_{m(k)-1}, x_{m(k)-1})) + d(y_{n(k)}, F(y_{n(k)}, x_{n(k)})) \\
&\quad + d(x_{m(k)-1}, F(x_{m(k)-1}, y_{m(k)-1})) + d(x_{m(k)}, F(x_{m(k)}, y_{n(k)}))],
\end{aligned}$$

i.e.,

$$\begin{aligned} & d(y_{m(k)}, y_{n(k)+1}) \\ & \leq \frac{\alpha}{2}[d(y_{m(k)-1}, y_{n(k)}) + d(x_{m(k)-1}, x_{n(k)})] \\ & \quad + \beta N((x_{n(k)}, y_{n(k)}), (x_{m(k)-1}, y_{m(k)-1})) \\ & \quad + \frac{\gamma}{2}[d(y_{m(k)-1}, F(y_{m(k)-1}, x_{m(k)-1})) + d(y_{n(k)}, F(y_{n(k)}, x_{n(k)})) \\ & \quad + d(x_{m(k)-1}, F(x_{m(k)-1}, y_{m(k)-1})) + d(x_{m(k)}, F(x_{m(k)}, y_{n(k)}))]. \end{aligned} \tag{3.12}$$

Adding (3.11) and (3.12), we get

$$\begin{aligned} & d(x_{m(k)}, x_{n(k)+1}) + d(y_{(m(k)}, y_{n(k)+1}) \\ & \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(x_{m(k)-1}, x_{n(k)}) + d(y_{m(k)-1}, y_{n(k)})], \end{aligned}$$

or equivalently,

$$d(z_{m(k)}, z_{n(k)+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d_+(z_{m(k)-1}, z_{n(k)}). \tag{3.13}$$

Passing to the limit as $k \to \infty$ in (3.12) we obtain that $\varepsilon \leq \varepsilon\rho < \varepsilon$, a contradiction. Hence, both sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in complete metric space (X, d). Since (X, d) is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = x$$

and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = y.$$

Further, from the continuity of mapping F we have that $F(x, y) = x$ and $F(y, x) = y$. Thus, the proof is complete. □

We note that previous result is still valid for F not necessarily continuous. We have the following result.

Theorem 3.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ be a mapping having the monotone property. Assume that there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that* (3.1)-(3.4) *satisfy for all $(x, y), (u, v) \in X \times X$ with $x \preceq u$ and $y \preceq v$. Finally assume that X has following properties:*

(1) *If a non-decreasing sequence $\{x_n\} \in X$ converges to $x \in X$, then $x_n \preceq x$ for all n.*
(2) *If a non-decreasing sequence $\{y_n\} \in X$ converges to $y \in X$, then $y_n \preceq y$ for all n.*

Then, F has a coupled fixed point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 3.1, we only have to show that $(\overline{x}, \overline{y})$ is a coupled fixed point of F. Suppose this is not the case, *i.e.*, $F(\bar{x}, \bar{y}) \neq \bar{x}$ or $F(\bar{y}, \bar{x}) \neq \bar{y}$ (e.g., let the first one of these holds). We have

$$\begin{aligned} d(F(\bar{x}, \bar{y}), \bar{x}) &\leq d(F(\bar{x}, \bar{y}), x_{n+1}) + d(x_{n+1}, \bar{x}) \\ &= d(F(\bar{x}, \bar{y}), F(x_n, y_n)) + d(x_{n+1}, \bar{x}). \end{aligned} \tag{3.14}$$

Since the nondecreasing sequence $\{x_n\}$ converges to X and the nondecreasing sequence $\{y_n\}$ converges to Y, by (i)-(ii), we have

$$x_n \preceq \bar{x} \quad \text{and} \quad y_n \preceq \bar{y}, \ \ \forall\, n.$$

Now, from the contractive condition, we have

$$\begin{aligned} &d(F(\bar{x}, \bar{y}), F(x_n, y_n)) \\ &\leq \frac{\alpha}{2}[d(\bar{x}, x_n) + d(\bar{y}, y_n)] + \beta N((\bar{x}, \bar{y}), (x_n, y_n)) \\ &\quad + \frac{\gamma}{2}[d(\bar{x}, F(\bar{x}, \bar{y})) + d(x_n, F(x_n, y_n)) + d(\bar{y}, F(\bar{x}, \bar{y})) + d(y_n, F(y_n, x_n))] \\ &\leq \frac{\alpha}{2}[d(\bar{x}, x_n) + d(\bar{y}, y_n)] + \beta \frac{d^2(\bar{x}, x_{n+1}) + d^2(x_n, F(\bar{x}, \bar{y}))}{d(\bar{x}, x_{n+1}) + d(x_n, F(\bar{x}, \bar{y}))} \\ &\quad + \frac{\gamma}{2}[d(\bar{x}, F(\bar{x}, \bar{y})) + d(x_n, x_{n+1}) + d(\bar{y}, F(\bar{x}, \bar{y})) + d(y_n, y_{n+1})]. \end{aligned}$$

We note that case $d(\bar{x}, x_{n+1}) + d(x_n, F(\bar{x}, \bar{y})) = 0$ is impossible, since otherwise the condition (3.3) would imply $\bar{x} = F(\bar{x}, \bar{y})$, which is excluded. Then, from (3.13), we get

$$\begin{aligned} d(F(\bar{x}, \bar{y}), \bar{x}) \leq d(\bar{x}, x_{n+1}) &+ \frac{\alpha}{2}[d(\bar{x}, x_n) + d((\bar{y}, y_n)] \\ &+ \beta \frac{d^2(\bar{x}, x_{n+1}) + d^2(x_n, F(\bar{x}, \bar{y}))}{d(\bar{x}, x_{n+1}) + d(x_n, F(\bar{x}, \bar{y}))} \\ &+ \frac{\gamma}{2}[d(\bar{x}, F(\bar{x}, \bar{y})) + d(x_n, x_{n+1}) + d(\bar{y}, F(\bar{x}, \bar{y})) + d(y_n, y_{n+1})]. \end{aligned}$$

Taking limit as $n \to \infty$ (and again using that $F(\bar{x}, \bar{y}) \neq \bar{x}$), we have

$$d(F(\bar{x}, \bar{y}) \leq \beta d(\bar{x}, F(\bar{x}, \bar{y})) + \frac{\gamma}{2}[d(\bar{x}, F(\bar{x}, \bar{y}), d(\bar{y}, F(\bar{y}, \bar{x}))]. \tag{3.15}$$

Now, if $\bar{y} = F(\bar{y}, \bar{x})$, using that $\beta + \frac{\gamma}{2} < 1$, it follows immediately that $\bar{x} = F(\bar{x}, \bar{y})$, a contradiction, if this is not the case, we similarly get

$$d(\bar{y}, F(\bar{y}, \bar{x})) \leq \beta d(\bar{y}, F(\bar{y}, \bar{x})) + \frac{\gamma}{2}[d(\bar{x}, F(\bar{x}, \bar{y})) + d(\bar{y}, F(\bar{y}, \bar{x}))]. \tag{3.16}$$

Adding (3.14) and (3.15), we have

$$\begin{aligned} d(\bar{x}, F(\bar{x}, \bar{y}) + d(\bar{y}, F(\bar{y}, \bar{x})) &\leq (\beta + \gamma)[d(\bar{x}, F(\bar{x}, \bar{y}), d(\bar{y}, F(\bar{y}, \bar{x}))] \\ &\leq (\alpha + 2\beta + 2\gamma)[d(\bar{x}, F(\bar{x}, \bar{y}), d(\bar{y}, F(\bar{y}, \bar{x}))]. \end{aligned}$$

Since $0 \leq (\alpha+2\beta+2\gamma) < 1$, we obtain $d(F(\bar{x},\bar{y}),\bar{x}) = 0$ and $d(\bar{y}, F(\bar{y},\bar{x})) = 0$, *i.e.*, $F(\bar{x},\bar{y}) = \bar{x}$ and $F(\bar{x},\bar{y}) = \bar{y}$, again a contradiction. This completes the proof of the theorem. □

4. Uniqueness Theorem

Now we shall prove a uniqueness theorem for the coupled fixed point.

Theorem 4.1. *Assume that for all $(x,y),(x^*,y^*) \in X \times X$, there exists $(z_1,z_2) \in X \times X$ that is comparable to (x,y) and (x^*,y^*). Adding above hypotheses with Theorem* 3.1*, we obtain the uniqueness of the coupled fixed point of F.*

Proof. From Theorem 3.1 we know that there exists a coupled fixed point $(\bar{x},\bar{y})$ of F, which is obtained as $x = \lim_{n\to\infty} F^n(x_0,y_0)$ and $\bar{y} = \lim_{n\to\infty} F^n(y_0,x_0)$. Suppose that (x^*,y^*) is another coupled fixed point, that is, $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$. Let us prove that

$$d_+((x,y),(x^*,y^*)) = d(\bar{x},x^*) + d(\bar{y},y^*) = 0. \tag{4.1}$$

Considering two cases:

Case I. $(\bar{x},\bar{y})$ is comparable with (x^*,y^*) with respect to ordering in $X \times X$. Let $(\bar{x}) \succeq x^*$ and $(\bar{y}) \succeq y^*$. Then we can apply the contractive condition (2.1) to obtain

$$\begin{aligned} d(\bar{x},x^*) &= d(F(\bar{x},\bar{y}),F(x^*,y^*)) \\ &\leq \frac{\alpha}{2}[d(\bar{x},x^*) + d(\bar{y},y^*)] + \beta d(\bar{x},x^*) \end{aligned}$$

and

$$\begin{aligned} d(\bar{y},y^*) &= d(F(\bar{y},\bar{x}),F(y^*,x^*)) = d(F(y^*,x^*),F(\bar{y},\bar{x})) \\ &\leq \frac{\alpha}{2}[d(\bar{x},x^*) + d(\bar{y},y^*)] + \beta d(\bar{y},y^*). \end{aligned}$$

By adding, we get

$$d(\bar{x},x^*) + d(\bar{y},y^*) \leq (\alpha+\beta)[d(\bar{x},x^*) + d(\bar{y},y^*)],$$

that is,

$$d_+((x,y),(x^*,y^*)) \leq (\alpha+\beta)d_+((x,y),(x^*,y^*)).$$

Since $0 \leq \alpha+\beta < 1$, (4.1) holds.

Case II. $(\bar{x},\bar{y})$ is not comparable with (x^*,y^*). In this case, there exists $(z_1,z_2) \in X \times X$ that is comparable both to $(\bar{x},\bar{y})$ and (x^*,y^*). Then for all

$n \in N$, $(F^n(z_1, z_2), F^n(z_2, z_1))$ is comparable both to $(F^n(\bar{x}, \bar{y}), F^n(\bar{y}, \bar{x})) = (\bar{x}, \bar{y})$ and $(F^n(x^*, y^*), F^n(y^*, x^*)) = (x^*, y^*)$. We have

$$\begin{aligned} d(\bar{x}, x^*) + d(\bar{y}, y^*) &\leq d(F^n(\bar{x}, \bar{y}), F^n(x^*, y^*)) + d(F^n(\bar{y}, \bar{x}), F^n(y^*, x^*)) \\ &\leq d(F^n(\bar{x}, \bar{y}), F^n(z_1, z_2)) + d(F^n(z_1, z_2), F^n(x^*, y^*)) \\ &\quad + d(F^n(\bar{y}, \bar{x}), F^n(z_2, z_1)) + d(F^n(z_2, z_1), F^n(y^*, x^*)) \\ &\leq (\alpha^n + \beta^n)[d(\bar{x}, z_1) + d(\bar{y}, z_2) + d(x^*, z_1) + d(y^*, z_2)]. \end{aligned}$$

That is

$$d_+((x, y), (x^*, y^*)) = (\alpha^n + \beta^n)[d_+((x, y), (z_1, z_2)) + d_+((z_1, z_2), (x^*, y^*))].$$

Since $0 < \alpha, \beta < 1$, (4.1) holds. We deduce that in all cases (4.1) holds. This implies that $(\bar{x}, \bar{y}) = (x^*, y^*)$ and the uniqueness of the coupled fixed point of F is proved. □

Assuming that every pair of elements of X have either an upper bound or a lower bound in X, one can in fact show that even the components of the coupled fixed points are equal. The following theorem establishes this fact.

Theorem 4.2. *In addition to the hypotheses of Theorem* 3.1 (*resp. Theorem* 3.2)*, suppose that x_0, y_0 in X are comparable. Then $\bar{x} = \bar{y}$.*

Proof. It is clear that (y, x) is a coupled fixed point of F if and only if (x, y) is coupled fixed point. Therefore, by previous Theorem we obtain that $(x, y) = (y, x)$, that is $x = y$. □

Example 4.3. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, $x \preceq y$ if and only if $x \leq y$ and a mapping $F : X \times X \to X$, defined by $F(x, y) = \frac{2x+3y}{15}$ with the standard metric and ordered by the relation $\preceq$. Suppose that $x \preceq u$ and $y \preceq v$.

Let α, β, γ be nonnegative numbers satisfying $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$, and denote by $\mathbf{L}$ and $\mathbf{R}$, respectively, the left-hand and right-hand side of contraction condition (3.1). It is easy to check that all the condition of Theorem 3.1 and 3.2 are satisfied for $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ and that $(0, 0)$ is a unique coupled fixed point of F. We note that function F has not mixed monotone property, but F is a monotone, that is, $F(x, y)$ is monotone nondecreasing in x and y.

Consider the example

$$\mathbf{L} \leq \frac{2x + 3y}{15} \leq \frac{\alpha}{2}[d(x, u) + d(y, v)] \leq \mathbf{R}.$$

For example, if $(x,y)=(1,2)$, $(u,v)=(2,3)$ for all $(x,y),(u,v)\in X\times X$ and $x\preceq u$, $y\preceq v$ then,

$$\mathbf{L}=d(F(x,y),F(u,v))=d\left(\frac{2x+3y}{15},\frac{2u+3v}{15}\right)=d\left(\frac{8}{15},\frac{13}{15}\right)=\frac{1}{3};$$

$$\begin{aligned}\mathbf{R}=&\frac{\alpha}{2}[d(x,u)+d(y,v)]+\beta N((x,y),(u,v)\\&+\frac{\gamma}{2}[d(x,F(x,y))+d(u,F(u,v))+d(y,F(y,x))+(v,F(v,u))].\end{aligned}$$

Suppose $\alpha=\frac{1}{15}$, $\beta=0$, $\gamma=\frac{15}{36}$, then $\alpha,\beta,\gamma\geq 0$ with $\alpha+2\beta+2\gamma<1$, we get $\mathbf{R}=\frac{9}{10}$. This implies that $\mathbf{L}\leq\mathbf{R}$ and the given contraction condition is satisfied.

References

[1] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partial ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393.

[2] D. Guo and V. Lakhsmikantham, *Coupled fixed point of nonlinear operator with application*, Nonlinear Anal. TMA., **11** (1987), 623–632.

[3] Z. Kadelburg, P. Kumam, S. Radenović and W. Sintunavarat, *Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property*, Fixed Point Theory Appl., **2015**(27) (2015).

[4] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contraction in partial ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349.

[5] N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., **74** (2011), 983–992.

[6] N.V. Luong and N.X. Thuan, *Coupled fixed points theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl., **62** (2011), 4238–4248.

[7] N.V. Luong and N.X. Thuan, *Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces*, Fixed Point Theory Appl., **2011**(46) (2011).

[8] H.K. Nashine, Z. Kadelburg and S. Radenovíc, *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, Mathe. Comput. Model., **57**(9-10) (2013), 2355–2365.

[9] H.K. Nashine and W. Shatanawi, *Coupled common fixed point theorems for pair of commuting mappings in partially ordered complete metric spaces*, Comput. Math. Appl., **62** (2011), 1984–1993.

[10] H.K. Nashine and Z. Kadelburg, *Partially ordered metric spaces, rational contractive expressions and coupled fixed points*, Nonlinear Funct. Anal. Appl., **17**(4) (2012), 471–489.

[11] J.J. Nieto and R.R. Lopez, *Contractive mapping theorems in partially ordered set and applications to ordinary differential equations*, Order, **22** (2005), 223–239.

[12] J.J. Nieto and R.R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and application to ordinary differential equations*, Acta Math. Sinica, **23** (2007), 2205–2212.

[13] S. Radenović, *Bhaskar-Lakshmikantham type results for monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal., **5** (2014), 37–49.
[14] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered set and some applications to matrix equation*, Proc. Amer. Math. Soci., **132** (2004), 1435–1443.
[15] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contractions in a partially ordered metric spaces*, Nonlinear Analysis, **72**(12) (2010), 4508–4517.
[16] B. Samet and H. Yazidi, *Coupled fixed point theorems in partial ordered ϵ-chainable metric spaces*, J. Math. Comp. Sci., **1**(3) (2010), 142–151.
[17] B. Samet and H. Yazidi, *Coupled fixed point theorems for contraction involving rational expressions in partially ordered metric spaces,* preprint, arXiv:1005.3142v1 [math.GN].