Nonlinear Functional Analysis and Applications
Vol. 21, No. 2 (2016), pp. 249-261

http://nfaa.kyungnam.ac.kr/journal-nfaa

REMARKS ON ABSTRACT CONVEXITY SPACES OF XIANG ET AL.

Sehie Park

The National Academy of Sciences
Seoul 137–044, Republic of Korea
and
Department of Mathematical Sciences
Seoul National University, Seoul 151–747, Korea
e-mail: park35@snu.ac.kr; sehiepark@gmail.com

Abstract. Xiang, Xia and Chen in 2013 [12] introduced an abstract convexity structure via an upper semicontinuous multi-valued map and established some generalized versions of the KKM lemma. By employing these general KKM lemmas, they derived some generalizations of minimax inequalities. Our aim in the present article is, as in our previous one [4], to show that most results in [12] are either consequences of known ones or can be stated in more general forms in the frame of abstract convex spaces in the sense of Park.

1. Introduction

Recently all of the known convexity spaces appeared in the KKM theory are unified to the class of abstract convex spaces in 2006 by the present author; see [5] with some corrections in [9].

Recall that, in 2007, Xiang and Yang [13] and Xiang and Xia [11] established some relationships among the abstract convexity, the selection property, and the fixed point property. They showed that if a convexity structure $\mathcal{C}$ defined on a topological space has the selection property [resp. the weak selection property] then $\mathcal{C}$ satisfies the H-condition [resp. H_0-condition]. Moreover, they showed that, in an *l.c.* compact metric space, the selection property

[0]Received September 16, 2015. Revised March 31, 2016.

[0]2010 Mathematics Subject Classification: 47H04, 47H10, 46A16, 46A55, 49J27, 49J35, 52A07, 54C60, 54H25, 55M20, 91B50.

[0]Keywords: KKM theorem, abstract convex space, G-convex space, ϕ_A-space, (partial) KKM space, Fan-Browder fixed point property.

implies the fixed point property. Note that their terminology has their own meaning.

Moreover, in our previous work [4], we showed that all results in [11,13] are either consequences of known ones or can be stated in more general forms in the frame of G-convex spaces. Furthermore, Xiang, Xia and Chen in 2013 [12] introduced an abstract convex structure via an upper semicontinuous multi-valued map and established some generalized versions of the KKM lemma. By employing these general KKM lemmas, they claimed to obtain some generalizations of minimax inequalities, which contain several existing ones as special cases.

Our aim in the present article is, as in our previous one [4], to show that most results in [12] are either consequences of known ones or can be stated in more general forms in the frame of abstract convex spaces in the sense of Park.

2. Abstract convex spaces

A multimap $T : X \multimap Y$ between topological spaces X, Y is said to be *lower semicontinuous* if $T^{-}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ is open for each open $B \subset Y$, and *upper semicontinuous* if $T^{+}(B) := \{x \in X : T(x) \subset B\}$ is open for each open $B \subset Y$.

The following is the original Knaster-Kuratowski-Mazurkiewicz theorem in 1929 and its open valued version:

Theorem 2.1. (KKM) *Let $F_i\,(0 \le i \le n)$ be $n+1$ closed [resp. open] subsets of an n-simplex $v_0 v_1 \cdots v_n$. If the inclusion relation*

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [5] with some corrections in [9]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_{\Gamma} D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_{\Gamma} D' \subset X$.

In case $E = D$, let $(E;\Gamma) := (E, E;\Gamma)$.

Remark 2.1. Recently the present author noticed that an abstract convex space was named a Γ-convex space by Zafarani [14] more early in 2004.

Definition 2.2. Let $(E, D;\Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent works [1-3], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example 2.1. We gave known examples of (partial) KKM spaces in [5] and the references therein.

Definition 2.4. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ (or simply $(X, D; \{\phi_A\})$) consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X, D;\Gamma)$ becomes an abstract convex space.

Definition 2.5. For a ϕ_A-space $(X, D; \{\phi_A\})$, any multimap $G : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset G(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

The following shows that every ϕ_A-space is a KKM space:

Lemma 2.1. *Let $(X, D; \Gamma)$ be a ϕ_A-space and $G : D \multimap X$ a multimap with nonempty closed [resp. open] values. Suppose that G is a KKM map. Then $\{G(a)\}_{a\in D}$ has the finite intersection property.*

Proof. Let $A = \{a_0, a_1, \ldots, a_n\} \in \langle D\rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_A(\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n).$$

Since G is a KKM map, it follows that

$$\begin{aligned}\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} &\subset \phi_A^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n)) \\ &\subset \bigcup_{j=0}^{k} \phi_A^{-1}(G(a_{i_j}) \cap \phi_A(\Delta_n)).\end{aligned}$$

Since $G(a_{i_j}) \cap \phi_A(\Delta_n)$ is closed [resp. open] in the compact subset $\phi_A(\Delta_n)$ of Γ_A, $\phi_A^{-1}(G(a_{i_j}) \cap \phi_A(\Delta_n))$ is closed [resp. open] in Δ_n. Note that $e_i \multimap \phi_A^{-1}(G(a_i) \cap \phi_A(\Delta_n))$ is a KKM map on $\{e_0, e_1, \ldots, e_n\}$. Hence, by the original KKM theorem, we have

$$\bigcap_{i=0}^{n} \phi_A^{-1}(G(a_i) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} G(a_i) \neq \emptyset$. This completes the proof. □

The following diagram for triples $(E, D; \Gamma)$ is well-known:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Consider the following well-known related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.
(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).
(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

Note that (d)$\Longrightarrow$(c)$\Longrightarrow$(b)$\Longrightarrow$(a) due to Luc *et al.*

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem 2.2. ([8]) *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
  (i) *$\bigcap\{G(y) : y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*
  (ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*

$$F(L_N) \cap \bigcap_{y\in D'} G(y) \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Later we adopt a simplified case of Theorem 2.2.

3. Definitions in [12]

Throughout [12], it is assumed that Y is nonempty. In this section, we introduce the definitions given in [12]:

Definition 3.1. A pair $(Y, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of Y, called a convex structure if

(1) $\emptyset$ and Y belong to $\mathcal{C}$;
(2) $\mathcal{C}$ is closed for arbitrary intersections; $\bigcap_{A\in\mathcal{D}} A \in \mathcal{C}$ for each family of subsets $\mathcal{D} \subset \mathcal{C}$.

Then the pair $(Y, \mathcal{C})$ is called an abstract convexity space.

The convex hull $\mathrm{co}_{\mathcal{C}}$ is defined as

$$\mathrm{co}_{\mathcal{C}}(A) = \bigcap\{D \in \mathcal{C} : A \subset D\},\ \ \forall\, A \subset Y.$$

A subset C of Y is said to be convex if $C \in \mathcal{C}$. It is clear that C is convex if and only if $\mathrm{co}_{\mathcal{C}}(C) = C$.

Definition 3.2. Let $(Y, \mathcal{C})$ be an abstract convexity space, let X be a subset of Y, and let $F : X \multimap X$ be a multimap. F is said to be weakly convex-valued if for each $x \in X$ and any finite subset $\{y_0, y_1, \ldots, y_n\} \subset F(x)$, $\mathrm{co}_{\mathcal{C}}\{y_0, y_1, \ldots, y_n\} \subset F(x)$, whenever $x \in \mathrm{co}_{\mathcal{C}}\{y_0, y_1, \ldots, y_n\}$.

Remark 3.1. It is clear that F is convex-valued, then F is weakly convex-valued.

Let $(Y, \mathcal{C})$ be an abstract convexity space and X be a subset of Y.

(i) $F : X \multimap Y$ is said to be a KKM mapping if for each $A \in \langle X \rangle$, F satisfies
$$\mathrm{co}_{\mathcal{C}}(A) \subset \bigcup_{x \in A} F(x).$$
(ii) $F : X \multimap Y$ is said to be a Fan-Browder mapping if F is convex-valued and has relatively open preimages in X (*i.e.*, $F(x)$ is convex for each $x \in X$ and $F^{-}(y)$ is open in X for each $y \in X$).
(iii) $F : X \multimap Y$ is said to be a weak Fan-Browder mapping if F is weakly convex-valued and has relatively open preimages in X.

Let $(Y, \mathcal{C})$ be an abstract convexity space, and let X be a subset of Y. X is said to be of KKM property (briefly KKMP) if every KKM mapping $F : X \multimap Y$ with closed values has a finite intersection property (*i.e.*, $\bigcap_{x \in A} F(x) \neq \emptyset$ for each $A \in \langle X \rangle$).

X is said to be of Fan-Browder fixed point property (briefly FBFP) if every Fan-Browder mapping $F : X \multimap X$ with nonempty values has a fixed point.

X is said to be of a strong Fan-Browder fixed point property (briefly SF-BFP) if every weak Fan-Browder mapping $F : X \multimap X$ with nonempty values has a fixed point.

Comments.

(1) Y should be better to assume a topological space. See the definitions of the KKM mapping and the Fan-Browder mapping.
(2) A convexity space $(Y, \mathcal{C})$ is a particular form of our abstract convex space $(E, D; \Gamma)$ with $Y = E = D$ and $\Gamma_A := \mathrm{co}_{\mathcal{C}} A = \bigcap \{B \in \mathcal{C} : A \subset B\}$ for $A \in \langle Y \rangle$. Then $(Y, \mathcal{C})$ becomes our abstract convex space $(Y; \Gamma)$; see [4].
(3) "X is said to be of KKM property (briefly KKMP)" is not adequate. In this case $(Y, X; \Gamma)$, where $\Gamma_A := \mathrm{co}_{\mathcal{C}} A$ for $A \in \langle X \rangle$, is a partial KKM space in our sense; see [4].
(4) The concepts of weakly convex-valued mappings, weak Fan-Browder mapping, and strongly Fan-Browder fixed point property (briefly SF-BFP) are first introduced in [12] without giving any proper examples.

(5) We define the convex hull $\mathrm{co}_{\mathcal{C}}$ as

$$\mathrm{co}_{\mathcal{C}}(A) = \bigcup\{\mathrm{co}_{\mathcal{C}} N : N \in \langle A \rangle\} \subset Y, \ \ \forall\, A \subset X.$$

4. KKM property and Fan-Browder fixed point property in [12]

In this section, we list several results in [12] which are simple consequences of known ones.

For an abstract convex space $(E, D; \Gamma)$, let us consider the following statements in [5]:

(0) The KKM principle. *For any closed-valued* [*resp. open-valued*] *KKM map* $G : D \multimap E$, *the family* $\{G(z)\}_{z \in D}$ *has the finite intersection property.*

(I) The Fan matching property. *Let* $S : D \multimap E$ *be a map satisfying*

(1.1) $S(z)$ *is open* [*resp. closed*] *for each* $z \in D$*; and*
(1.2) $E = \bigcup_{z \in M} S(z)$ *for some* $M \in \langle D \rangle$.

Then there exists an $N \in \langle M \rangle$ *such that*

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

(V) The Fan-Browder fixed point property. *Let* $S : E \multimap D$, $T : E \multimap E$ *be maps satisfying*

(5.1) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*;*
(5.2) $S^-(z)$ *is open* [*resp. closed*] *for each* $z \in D$*; and*
(5.3) $E = \bigcup_{z \in M} S^-(z)$ *for some* $M \in \langle D \rangle$.

Then T *has a fixed point* $x_0 \in E$*; that is,* $x_0 \in T(x_0)$.

Theorem 4.1. ([5], Characterizations of the KKM spaces) *For an abstract convex space* $(E, D; \Gamma)$*, the statements* (0), (I) *and* (V) *are mutually equivalent.*

Similarly, the following enlarges [12, Theorem 3.1] and [12, Corollary 3.2]:

Theorem 4.2. *Let* $(Y, \mathcal{C})$ *be an abstract convexity space. For a compact subset* X *of* $(Y, \mathcal{C})$*, the following properties are equivalent:*

(i) *KKMP,*
(ii) *SFBFP,*
(iii) *FBFP.*

Proof. (i) $\Longrightarrow$ (ii). See [12].
(ii) $\Longrightarrow$ (iii). Clear.
(iii) $\Longrightarrow$ (i). We follow our method in [5]. Let $G : X \multimap X$ be a KKM map with closed values as in [12]. Suppose that $\{G(x)\}_{x \in X}$ does not have the finite intersection property, that is, there exists an $M \in \langle X \rangle$ such that $\bigcap_{x \in M} G(x) =$

$\emptyset$. Define two maps $S, T : X \multimap X$ by $S^-(y) = G^c(y) := X \setminus G(y)$ for $y \in X$ and $T(x) := \mathrm{co}_{\mathcal{C}} S(x)$ for $x \in X$. Note that $S^-(y)$ is open for each $y \in X$ and X is covered by a finite number of $S^-(y)$s for $y \in M$. Therefore, by FBFP, T has a fixed point $x_0 \in T(x_0) = \mathrm{co}_{\mathcal{C}} S(x_0)$. By our definition of the convex hull $\mathrm{co}_{\mathcal{C}}$, there exists an $N \in \langle S(x_0) \rangle$ such that $x_0 \in \mathrm{co}_{\mathcal{C}} N \subset \mathrm{co}_{\mathcal{C}} S(x_0)$. Then for each $y \in N$, we have $y \in S(x_0)$ or $x_0 \in S^-(y)$, that is, $x_0 \notin G(y)$. Hence $\mathrm{co}_{\mathcal{C}} N \not\subset G(N)$ and G is not a KKM map. This contradiction implies the conclusion. □

5. An abstract convexity structure in [12]

Let $N = \{0, 1, 2, \ldots, n\}$, $\Delta_n = e^0 e^1 \cdots e^n$ be the standard n-simplex, and for $J \subset N$, let $\Delta_J = \mathrm{co}\{e^j : j \in J\}$ be a face of Δ_n.

Lemma 5.1. *Let $D = \{e^0, e^1, \ldots, e^n\}$, Y be a topological space, and $(\Delta_n, D; \mathrm{co})$ be the abstract convex space as in the original KKM theorem. Let $G : D \multimap Y$ be a closed [resp. an open] valued multimap. If G is a KKM map with respect to an upper [a lower] semicontinuous map $F : \Delta_n \multimap Y$, then $\bigcap_{i=0}^n G(e^i) \neq \emptyset$.*

Proof. Since $F(\Delta_J) \subset G(J)$ for each $J \subset D$, we have $\Delta_J \subset F^+G(J)$ for each $J \subset D$. Then $F^+G : D \multimap \Delta_n$ is a KKM map. Moreover, it is closed-valued [resp. open-valued] since F is upper [resp. lower] semicontinuous. Therefore, by the original KKM theorem, we have $\bigcap_{i=0}^n F^+G(e^i) = F^+(\bigcap_{i=0}^n G(e^i)) \neq \emptyset$. Consequently, $\bigcap_{i=0}^n G(e^i)) \neq \emptyset$. □

Note that our Lemma 5.1 subsumes Lemmas 4.2 and 4.3 in [12] as follows:

Definition 5.1. ([12], Definition 4.1.) Let Y be a compact set of a topological space, and let $q : \Delta_n \multimap Y$ be a multimap. If for each continuous map $p : Y \to \Delta_n$ (called a simplex mapping), there exists some point $x_0 \in p \cdot q(x_0)$ then we say that q has a fixed point property with respect to Δ_n and simplex mappings.

Lemma 5.2. ([12], Lemma 4.2.[4.3.]) *Let Y be a metric space [resp. compact space], and let $\{F_0, F_1, \ldots, F_n\}$ be a family of closed subsets of Y. If there exists an upper semicontinuous mapping $q : \Delta_n \multimap Y$ such that*

$$q(\Delta_J) \subset \bigcup_{j \in J} F_j \quad [\text{resp. } q(\Delta_J) \subset \mathrm{co}_{\mathcal{C}}\{y_j : j \in J\}], \quad \forall J \subset N,$$

and q has a fixed point property with respect to Δ_n and simplex mappings. Then $\bigcap_{i=0}^n F_i \neq \emptyset$.

From Lemma 5.1 we have the following:

Theorem 5.1. *Let $(E, D; \Gamma)$ be an abstract convex space, $G : D \multimap E$ be a KKM map with closed [resp. open] values. Suppose that for each finite subset $\{y_0, y_1, \dots, y_n\} \subset D$, there exists an upper [resp. a lower] semicotinuous map $F : \Delta_n \multimap E$ such that*

$$F(\Delta_J) \subset \mathrm{co}_\Gamma\{y_j : j \in J\}, \quad \forall J \subset N = \{0, 1, \dots, n\}.$$

Then $\{G(y) : y \in D\}$ has the finite intersection property.

Proof. Given $\{y_0, y_1, \dots, y_n\} \subset D$, we prove $\bigcap_{i=0}^n G(y_i) \neq \emptyset$. Since G is a KKM map, we have

$$\mathrm{co}_\Gamma\{y_j : j \in J\} \subset \bigcup_{j \in J} G(y_j), \quad \forall J \subset N = \{0, 1, \dots, n\}.$$

Then there exists an upper [resp. a lower] semicontinuous map $F : \Delta_n \multimap E$ such that

$$F(\Delta_j) \subset \mathrm{co}_\Gamma\{y_j : j \in J\} \subset \bigcup_{j \in J} G(y_j), \quad \forall J \subset N = \{0, 1, \dots, n\}.$$

From Lemma 5.1, it follows that $\bigcap_{i=0}^n G(y_i) \neq \emptyset$. □

In view of Theorem 5.1, we can eliminate Definition 4.4, Theorem 4.5, Corollary 4.6, and Theorem 4.12 in [12].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish.

The following is well-known:

Lemma 5.3. *Let Y be a compact topological space. If $p : Y \to \Delta_n$ is continuous and $F : \Delta_n \multimap Y$ is an upper semicontinuous map with closed acyclic values, then there exists some $e \in \Delta_n$ such that $e \in p \cdot F(e)$.*

Recall that $F \in \mathfrak{A}_c^\kappa(\Delta_n, Y) \subset \mathfrak{KC}(\Delta_n, V, Y)$ (see, [6,7,10]) and the references therein. As an immediate corollary, we have the following:

Corollary 5.1. *Let Y be a topological space, and $\{F_0, F_1, \dots, F_n\}$ be a family of closed subsets of Y. If there exists an upper semicontinuous map $F : \Delta_n \multimap Y$ with compact acyclic values such that*

$$F(\Delta_J) \subset \bigcup_{i \in J} F_i, \quad \forall J \subset N = \{0, 1, \dots, n\},$$

then $\bigcap_{i=0}^n F_i \neq \emptyset$.

Note that Corollary 5.1 improves [12, Corollary 4.9].

Note also that there appears in [12] some redundant artificial definitions like the fixed point property with respect to Δ_n and simplex mappings, H_q-property, and H_0^q-property. These are all eliminated in our preceeding arguments.

6. Minimax inequalities in [12]

Lemma 6.1. ([12], Lemma 5.1.) *Let X be a subset of a linear topological space, let Y be a compact topological space, and let $s : X \multimap Y$ be an upper semi-continuous mapping with nonempty, closed and contractible values. Let $F : X \multimap Y$ be a closed valued mapping such that for each finite subset $\{x_0, x_1, \ldots, x_n\} \subset X$,*

$$s(\mathrm{co}\{x_0, x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i).$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This Lemma can be improved as follows:

1. X should be convex.
2. s can be replaced by an acyclic map or, more generally, a map in $\mathfrak{KC}$-map.
3. Instead of the compactness of Y, various coercivity conditions can be used as in Theorem 2.2.

In view of Theorem 2.2, we can adopt the following instead of Lemma 5.1 of [12]:

Lemma 6.2. *Let X be a nonempty convex subset of a linear topological space, let Y be a compact topological space, and let $F : X \multimap Y$ be an upper semicontinuous mapping with nonempty, closed and acyclic values. Let $G : X \multimap Y$ be a closed valued mapping such that for each finite subset $\{x_0, x_1, \ldots, x_n\} \subset X$,*

$$F(\mathrm{co}\{x_0, x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} G(x_i).$$

Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

From Lemma 6.2, we derive some general versions of minimax inequalities.

Theorem 6.1. *Let X, Y be nonempty subsets of linear topological spaces, $\varphi : X \times Y \to \mathbb{R}$ satisfies the following conditions:*

(1) *X is convex and, for any fixed $x \in X$, $\{y \in Y : \varphi(x, y) \leq 0\}$ is lower semicontinuous with respect to y;*

(2) *Y is compact and $F : X \multimap Y$ is an upper semi-continuous mapping with nonempty, closed and acyclic values; and*
(3) *for any finite subset $\{x_0, x_1, \dots, x_n\} \subset X$ and $y \in F(\mathrm{co}\{x_0, x_1, \dots, x_n\})$,*

$$\varphi(x_i, y) \leq 0 \quad \text{for some } i = 0, 1, \dots, n.$$

Then there exists $y^* \in Y$ such that

$$\varphi(x, y^*) \leq 0, \quad \forall\, x \in X.$$

Proof. The multimap $G : X \multimap Y$ is defined as

$$G(x) = \{y \in Y : \varphi(x, y) \leq 0\}, \quad \forall\, x \in X.$$

Condition (1) implies that G is closed-valued. From condition (2), it is easy to check that for each finite subset $\{x_0, x_1, \dots, x_n\} \subset X$, G satisfies

$$F(\mathrm{co}\{x_0, x_1, \dots, x_n\}) \subset \bigcup_{i=0}^{n} G(x_i).$$

By Lemma 6.2, we have $\bigcap_{x \in X} G(x) \neq \emptyset$. Then there exists some $y^* \in Y$ such that $y^* \in \bigcap_{x \in X} G(x)$, so that

$$\varphi(x, y^*) \leq 0, \quad \forall\, x \in X.$$

□

This is a correct form of [12, Theorem 5.2]. Note that other results in Section 5 of [12] can be corrected similarly, and we will stop here.

7. Historical remarks

In this section, we introduce abstracts of papers of Xiang *et al.* [11, 12, 13] and Park [4], and the comments on [12] given by Park [9].

7.1. Xiang and Yang 2007 ([13])

Abstract: We establish some relationships among abstract convexity, the selection property and the fixed point property. We show that if a convexity structure $\mathcal{C}$ defined on a topological space has the selection property then $\mathcal{C}$ satisfies the H-condition. Moreover, in an l.c. compact metric space, the selection property implies the fixed point property.

7.2. Xiang and Xia 2007 ([11])

Abstract: We give a characteristic of abstract convexity structures on topological spaces with selection property. We show that if a convexity structure $\mathcal{C}$ defined on a topological space has the weak selection property then $\mathcal{C}$ satisfies H_0-condition. Moreover, in a compact convex subset of a topological space with convexity structure, the weak selection property implies the fixed point property.

7.3. Park 2010 ([4])

Abstract: All results of S.-w. Xiang and H. Yang [12] and S.-w. Xiang and S. Xia [11] are shown to be consequences of known ones or can be stated in more general forms.

7.4. Xiang et al. 2013 ([12])

Abstract: The purpose of this paper is to give some further results in a type of generalized convexity spaces. First, we prove that an abstract convex space has KKM property if and only if it has a strong Fan-Browder property. Then we introduce an abstract convex structure via an upper semi-continuous multi-valued mapping and establish some generalized versions of KKM lemma. By employing our general KKM lemmas, we derive some generalizations of minimax inequalities, which contain several existing ones as special cases.

7.5. Park 2014 ([9])

Comments on Xiang *et al.* [12]: The authors' abstract convexity space is particular to the abstract convex spaces in the sense of Park. Their weakly convex-valued multimap has a fixed point whenever it has a nonempty value. Hence their new definitions (without giving any proper examples) seem to be incorrect. Moreover, the correct form of their first claim of equivalence was already known for a long time ago.

References

[1] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc., **45**(1) (2008), 1–27.

[2] S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex Anal., **9**(3) (2008), 331–350.

[3] S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal., **69** (2008), 4352–4364.

[4] S. Park, *Comments on abstract convexity structures on topological spaces*, Nonlinear Anal., **72** (2010), 549–554.

[5] S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Anal., **73** (2010), 1028–1042.

[6] S. Park, *Applications of the KKM theory to fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **50**(1) (2011), 21–69.

[7] S. Park, *Applications of multimap classes in abstract convex spaces*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **51**(2) (2012), 1–27.

[8] S. Park, *A genesis of general KKM theorems for abstract convex spaces: Revisited*, J. Nonlinear Anal. Optim., **4**(1) (2013), 127–132.

[9] S. Park, *Review of recent studies on the KKM theory II*, Nonlinear Funct. Anal. Appl., **19**(1) (2014), 143–155.

[10] S. Park, *Recent results in analytical fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **53**(2) (2014), 1–21.

[11] S.-w. Xiang and S. Xia, *A further characteristic of abstract convexity structures on topological spaces*, J. Math. Anal. Appl., **335** (2007), 716–723.

[12] S. Xiang, S. Xia and J. Chen, *KKM lemmas and minimax inequality theorems in abstract convexity spaces*, Fixed Point Theory and Applications **2013**(209) (2013), doi:10.1186/1687-1812-2013-209.

[13] S.-w. Xiang and H. Yang, *Some properties of abstract convexity structures on topological spaces*, Nonlinear Anal., **67** (2007), 803–808.

[14] J. Zafarani, *KKM property in topological spaces*, Bull. Soc. Royale Sci. Liège, **73**(2-3) (2004), 171–185.