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LOCAL CONVERGENCE FOR A FAMILY OF CUBICALLY CONVERGENT METHODS IN BANACH SPACE

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Abstract. We present a local convergence analysis for a family of cubically convergent methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. We only use hypotheses on the first Fréchet-derivative. The local convergence analysis in [6, 15] used hypotheses up to the second Fréchet derivative. Hence, the application of the methods is extended under less computational cost. This work also provides computable convergence ball and computable error bounds. Numerical examples are also provided in this study.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [2, 5, 14, 16]. Closed form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1)

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are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1]-[20]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1]-[20].

We present a local convergence analysis for the cubically convergent family methods defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \alpha A_n^{-1}(I - A_n)F'(x_n)^{-1}F(x_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, α, $\theta \in (-\infty, \infty)$ are given parameters and

$$A_n = \theta F'(x_n)^{-1}F(y_n) + (1 - 2\theta)F'(x_n)^{-1}F'\left(\frac{y_n + x_n}{2}\right) + \theta I.$$

If $\alpha = 1$, method (1.2) specializes to the method studied in [6, 15]. Moreover, if we choose $\theta = 0, \frac{1}{2}, \frac{1}{6}$ and $\frac{1}{4}$ in turn, we obtain, respectively, the midpoint Newton method proposed by Traub [18], the arithmetic Newton's method introduced by Weerakoon and Fernando [20], Hasanov's, Nedzhibov's method [12]. We shall use the conditions $(\mathcal{C})$:

$(\mathcal{C}_1)$ $F(x^*) = 0$, $F'(x^*)^{-1} \in L(Y, X)$;

$(\mathcal{C}_2)$ $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|$ for each $x \in D$;

$(\mathcal{C}_3)$ $\|F'(x^*)^{-1}(F'(x) - F'(y)\| \leq L\|x - y\|$ for each $x, y \in D$;

$(\mathcal{C}_4)$ $\|F'(x^*)^{-1}(F''(x) - F''(y))\| \leq K\|x - y\|$ for each $x, y \in D$.

In particular, the local convergence of method (1.2) for $\alpha = 1$ was studied under condition $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ in [15]. It was then later improved in [6] under conditions $(\mathcal{C}_1)$–$(\mathcal{C}_4)$. Notice that in the earlier results the very restrictive condition $(\mathcal{C}_4)$ is used although the second derivative is not used in the method. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2$$

and

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x.$$

Then, obviously, function F cannot satisfy condition $(\mathcal{C}_4)$. In the present paper, we only use hypotheses on the first Fréchet derivative (see conditions ((2.9)–(2.12)). This way, we extend the applicability of method (1.2) and under less computational cost than in [6], [15] (and for α not necessarily only 1).

The rest of the paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are presented in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Let $U(v,\rho)$ and $\overline{U}(v,\rho)$ denote the open and closed ball in X, respectively, with center $v \in X$ and of radius $\rho > 0$. Let $L_0 > 0$, $L > 0$, $M > 0$, $\beta > 0$, $\gamma \geq 0$ and α, $\theta \in (-\infty, +\infty)$ be given parameters. It is convenient for the local convergence analysis that follows to define some functions on the interval $[0, \frac{1}{L_0})$ by

$$g_1(r) = \frac{Lr}{2(1-L_0 r)},$$

$$g_2(r) = \gamma + |1-\theta| + \frac{|\theta| L_0 (1+3g_1(r))r}{2(1-L_0 r)} + \frac{|1-\theta|M}{1-L_0 r},$$

$$h_2(r) = \beta g_2(r) - 1,$$

$$g_3(r) = \frac{(|\theta| + |1-2\theta|)M}{1-L_0 r} + |\theta|,$$

$$g_4(r) = g_1(r) + \frac{|\alpha| M \beta (1+g_3(r))}{(1-\beta g_2(r))(1-L_0 r)}$$

and

$$h_4(r) = g_4(r) - 1.$$

Define

$$r_A := \frac{2}{L+2L_0}. \tag{2.1}$$

It follows from the definition of function g_1 and (2.1) that

$$0 \leq g_1(r) < 1 \text{ for each } r \in [0, r_A).$$

Suppose that

$$\beta(\gamma + (1+M)|1-\theta|) < 1. \tag{2.2}$$

We have that $h_2(0) < 0$ by (2.2) and $h_2(t) \to \infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that there exist zeros of function h_2 in the interval $(0, \frac{1}{L_0})$. Denote by r_2 the smallest such zero.
Suppose that

$$\beta|\alpha|M((1+(|\theta|+|1-2\theta|)M+|\theta|)+\beta(\gamma+(1+M)|1-\theta|)) < 1. \quad (2.3)$$

Notice that (2.3) implies (2.2). We get by (2.3) that $h_4(0) < 0$ and $h_4(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that function h_4 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_4 the smallest such zero. Then, we have that

$$0 \leq g_4(r) < 1 \quad \text{for each} \quad r \in [0, r_4).$$

Define

$$r^* = \min\{r_A, r_2, r_4\}. \quad (2.4)$$

It follows that

$$0 \leq g_1(r) < 1, \quad (2.5)$$

$$0 \leq \beta g_2(r) < 1, \quad (2.6)$$

$$0 \leq g_3(r), \quad (2.7)$$

and

$$0 \leq g_4(r) < 1, \quad (2.8)$$

for each $r \in [0, r^*)$.

Next, we present the local convergence analysis of method (1.2).

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M > 0$, $\beta > 0$, $\gamma \geq 0$ and α, $\theta \in (-\infty, +\infty)$ such that condition (2.3) holds and for each $x \in D$*

$$F(x^*) = 0,\ F'(x^*)^{-1} \in L(Y, X),\ \|F'(x^*)^{-1}\| \leq \beta,\ \|I - F'(x^*)\| \leq \gamma, \quad (2.9)$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \quad (2.10)$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*))\| \leq \frac{L}{2}\|x - x^*\|^2, \quad (2.11)$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \quad (2.12)$$

and

$$\bar{U}(x^*, r^*) \subseteq D, \quad (2.13)$$

where r^* is defined by (2.4). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*) - \{x^*\}$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \quad (2.14)$$

$$\|F'(x^*) - A(x_n)\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \frac{1}{\beta}, \tag{2.15}$$

$$\|A(x_n)\| \leq g_3(\|x_n - x^*\|) \tag{2.16}$$

and

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.17}$$

where the "g" functions are defined above and

$$\begin{aligned} A(x) &= \theta F'(x)^{-1}F'(x - F'(x)^{-1}F(x)) \\ &\quad +(1-2\theta)F'(x)^{-1}F'\left(\frac{2x - F'(x)^{-1}F(x)}{2}\right) \\ &\quad +\theta I \quad \text{for each } \ x \in D. \end{aligned}$$

Furthermore, suppose that there exists $T \in [r, \frac{2}{L_0})$ *such that* $\bar{U}(x^*, T) \subset D$, *then the limit point* x^* *is the only solution of equation* $F(x) = 0$ *in* $\bar{U}(x^*, R)$.

Proof. Using (2.10), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*) - \{x^*\}$ we get that

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.18}$$

It follows from (2.18) and the Banach Lemma on invertible operators [3, 16] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.19}$$

Hence, from the first substep of method (1.2) for $n = 0$, (2.5), (2.11) and (2.19) we get that

$$y_0 - x^* = -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^a)],$$

so,

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*, r^*)$. We have by the definition of A_0

$$\begin{aligned} \|F'(x^*) - A_0\| &= \Big\| F'(x^*) - I + I - \theta I \\ &\quad -\theta F'(x_0)^{-1}F'(y_0) - (1-2\theta)F'(x_0)^{-1}F'\left(\frac{y_0+x_0}{2}\right)\Big\| \\ &= \Big\| (F'(x^*) - I) + (1-\theta)I \\ &\quad +\theta F'(x_0)^{-1}\left(F'\left(\frac{y_0+x_0}{2}\right) - F'(y_0)\right) \\ &\quad -(1-\theta)F'(x_0)^{-1}F'\left(\frac{y_0+x_0}{2}\right)\Big\|. \end{aligned}$$

Using (2.6), (2.9), (2.10), (2.12) and the definition of r^* (since $\|\frac{y_0+x_0}{2} - x^*\| \leq \frac{1}{2}(\|y_0 - x^*\| + \|x_0 - x^*\| < \frac{1}{2}(r^* + r^*) = r^*)$, we get that

$$\begin{aligned} &\|F'(x^*) - A_0\| \\ &\leq \|F'(x^*) - I\| + \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\ &\quad +|1-\theta| + |\theta|\|F'(x_0)^{-1}F'(x^*)\|\Big\|F'(x^*)^{-1}\Big(F'\left(\frac{y_0+x_0}{2}\right) - F'(x^*)\Big)\Big\| \\ &\quad +|1-\theta|\|F'(x_0)^{-1}F'(x^*)\|\Big\|F'(x^*)^{-1}F'\left(\frac{y_0+x_0}{2}\right)\Big\| \\ &\leq \gamma + |1-\theta| + \frac{|\theta|}{1 - L_0\|x_0 - x^*\|}\Big(\frac{L_0}{2}(\|y_0 - x^*\| + \|x_0 - x^*\|) \\ &\quad +L_0\|y_0 - x^*\|\Big) + \frac{|1-\theta|M}{1 - L_0\|x_0 - x^*\|} \\ &\leq \gamma + |1-\theta| + \frac{|\theta|L_0}{2(1 - L_0\|x_0 - x^*\|)}(3\|y_0 - x^*\| + \|x_0 - x^*\|) \\ &\quad +\frac{M|1-\theta|}{1 - L_0\|x_0 - x^*\|} \\ &\leq \gamma + |1-\theta| + \frac{|\theta|L_0(3g_1(\|x_0 - x^*\|) + 1)\|x_0 - x^*\|}{2(1 - L_0\|x_0 - x^*\|)} + \frac{M|1-\theta|}{1 - L_0\|x_0 - x^*\|} \\ &= g_2(\|x_0 - x^*\|) < \frac{1}{\beta}, \end{aligned} \tag{2.20}$$

which shows (2.15) for $n = 0$. It follows from (2.20) and the Banach lemma on invertible operators that A_0^{-1} exists and

$$\|A_0^{-1}\| \leq \frac{\beta}{1-\beta g_2(\|x_0-x^*\|)}. \tag{2.21}$$

Moreover from the definition of A_0, g_3, (2.19) and (2.12) we have that

$$\begin{aligned}
\|A_0\| &\leq \theta\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(y_0)\| \\
&\quad +|1-2\theta|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(\frac{y_0+x_0}{2})\| + |\theta| \\
&\leq \frac{|\theta|M}{1-L_0\|x_0-x^*\|} + \frac{|1-2\theta|M}{1-L_0\|x_0-x^*\|} + |\theta| \\
&= g_3(\|x_0-x^*\|), \qquad (2.22)
\end{aligned}$$

which shows (2.16) for $n=0$. Furthermore, using the last substep of method (1.2) for $n=0$, we get from (2.12), (2.14), (2.18), (2.20), (2.21) and the definition of function g_4 and r^* that

$$\begin{aligned}
&\|x_1-x^*\| \\
&\leq \|y_0-x^*\| + |\alpha|\|A_0^{-1}\|(1+\|A_0\|) \\
&\quad \times\|F'(x_0)^{-1}F'(x^*)\|\left\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*))(x_0-x^*)dt\right\| \\
&\leq \left[g_1(\|x_0-x^*\|) + \frac{\beta M|\alpha|(1+g_3(\|x_0-x^*\|))}{(1-\beta g_2(\|x_0-x^*\|))(1-L_0\|x_0-x^*\|)}\right]\|x_0-x^*\| \\
&= g_4(\|x_0-x^*\|)\|x_0-x^*\| < r^*,
\end{aligned}$$

which shows (2.17) for $n=0$ and $x_1 \in U(x^*,r^*)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at (2.14)-(2.17). Then, from the estimate $\|x_{k+1}-x^*\| < \|x_k-x^*\| < r^*$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*,r^*)$.

Finally, to show the uniqueness part, let $Q = \int_0^1 F'(y^*+t(x^*-y^*)dt$ for some $y^* \in \bar{U}(x^*,T)$ with $F(y^*)=0$. Using (2.10), we get that

$$\begin{aligned}
\|F'(x^*)^{-1}(Q-F'(x^*))\| &\leq \int_0^1 L_0\|y^*+t(x^*-y^*)-x^*\|dt \\
&\leq \int_0^1 (1-t)\|x^*-y^*\|dt \\
&\leq \frac{L_0}{2}T < 1. \qquad (2.23)
\end{aligned}$$

It follows from (2.23) that Q is invertible. Then, from the identity $0 = F(x^*) - F(y^*) = Q(x^*-y^*)$, we deduce that $x^* = y^*$. □

Remark 2.2. 1. In view of (2.10) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\ &\leq 1+L_0\|x-x^*\|\end{aligned}$$

condition (2.12) can be dropped and be replaced by

$$M(r)=1+L_0 r.$$

Moreover, condition (2.11) can be replaced by the popular but stronger condition $(\mathcal{C}_3)$ or

$$\|F'(x^*)^{-1}(F'(x^*+t(x-x^*))-F'(x))\| \quad \leq \quad L(1-t)\|x-x^*\|$$

for each $x, y \in D$ and $t \in [0,1]$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3, 5, 14] of the form

$$F'(x)=T(F(x)),$$

where T is a continuous operator. Then, since $F'(x^*)=T(F(x^*))=T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x)=e^x-1$. Then, we can choose: $T(x)=x+1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2, 5].

4. The parameter r_A given by (2.1) was shown by us to be the convergence radius of Newton's method [2, 6]

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n) \quad \text{for each} \quad n=0,1,2,\cdots \tag{2.24}$$

under the conditions (2.10) and $(\mathcal{C}_3)$. It follows from (2.2) that the convergence radius r^* of method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.24). As already noted in [2, 5] r_A is at least as large as the convergence ball given by Rheinboldt [18]

$$r_R=\frac{2}{3L}.$$

In particular, for $L_0<L$ we have that

$$r_R<r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [19].

5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger $(\mathcal{C})$ conditions used in [6, 15]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Numerical Examples

We present numerical examples in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = \left(e^x - 1, \, \frac{e-1}{2}y^2 + y, \, z\right)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{C})$ conditions, we get $L_0 = e - 1, L = M = e$. Then, for $\alpha = 0.1121$, $\theta = -1.6388$, $\beta = 1$, $\gamma = 0$, we have $r_A = 0.3249$, $r_R = 0.2453$, $r_2 = 0.5067$, $r_4 = 0.2736, r^* = 0.2736$

$$\xi_1 = 0.9991, \, \xi = 0.9992.$$

Example 3.2. Returning back to the motivational example at the introduction of this study, we have using the $(\mathcal{C})$ conditions, $L_0 = L = 146.6629073$, $M = 101.5578008$. Then, for $\alpha = 0.0025$, $\theta = -1.1039$, $\beta = \frac{1}{3}$, $\gamma = 0$, we have $r_A = 0.0045$, $r_R = 0.0045$, $r_2 = 0.0069$, $r_4 = 0.0021, r^* = 0.0021$

$$\xi_1 = 1.0000, \, \xi = 1.0000.$$

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