HYBRID STEEPEST-DESCENT METHOD WITH A COUNTABLY INFINITE FAMILY OF NONEXPANSIVE MAPPINGS ON BANACH SPACES

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Abstract. In this paper, we propose an explicit iterative algorithm of hybrid steepest-descent type for finding a solution for variational inequalities over the set of common fixed points of an infinite family of nonexpansive mappings on real reflexive and strictly convex Banach spaces with a uniformly Gâteaux differentiable norm. A numerical example is also given for illustration.

1. Introduction

Let E be a Banach space and let E^* be the dual space of E. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We use the symbol $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. A mapping J from E into E^*, satisfying the condition

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\| \quad \text{and} \quad \|x^*\| = \|x\|\},$$

is called normalized duality mapping of E. Mapping J exists for any Banach space and, in general, is multi-valued. In the sequel, we shall denote the single

valued normalized duality mapping by j. Recall that a mapping $T : E \to E$, satisfying the following conditions:

$$\|T(x) - T(y)\| \leq L\|x - y\|;$$
$$\langle T(x) - T(y), j(x - y) \rangle \geq \eta\|x - y\|^2;$$
$$\langle T(x) - T(y), j(x - y) \rangle \leq \|x - y\|^2 - \gamma\|(I - T)(x) - (I - T)(y)\|^2,$$

for all $x, y \in E$ and some elements $j(x-y) \in J(x-y)$, where $L, \eta > 0, \gamma \in (0,1)$ are the fixed numbers, and I denote the identity mapping in E, is said to be L-Lipschitz continuous, η-strongly accretive and γ-strictly pseudocontractive on E, respectively. When $L = 1$, T is called nonexpansive.

Let C be a closed convex subset of E. In 2006, Aoyama *et al.* [2] introduced firstly the variational inequality problem of finding a point $p^* \in C$ such that

$$\langle F(p^*), j(p^* - p) \rangle \leq 0, \quad \forall\, p \in C, \tag{1.1}$$

for some $j(p^* - p) \in J(p^* - p)$, and show that this problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive mapping and so on.

In this paper, we study the variational inequality problem (1.1) with $C := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$, where T_i is a nonexpansive mapping on E with the fixed point set $\mathrm{Fix}(T_i) = \{x \in E : T_i(x) = x\}$ and F is an η-strongly accretive and γ-strictly pseudocontractive on E with $\eta + \gamma > 1$. Throughout this paper, we assume that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$.

When $E \equiv H$, a Hilbert space, an η-strongly accretive and γ-strictly pseudo-contractive mapping is η-strongly monotone and L-Lipschitz continuous with the coefficient $L = 1 + 1/\gamma$, respectively. Moreover, $j \equiv I$, and hence, (1.1) become the form

$$\langle F(p^*), p^* - p \rangle \leq 0, \quad \forall\, p \in C. \tag{1.2}$$

This problem was introduced by Hartman and Stampacchia in the 1960s [12] as a tool for the study of partial differential equations principally drawn from mechanics. Note that, the variational inequalities over the set of common fixed points of an infinite family of nonexpansive mappings also contains many applications in the theory of signal recovery problems [10], power control problems [13], bandwidth allocation problems [14], optimal control problems [15], and beamforming problems [19].

The well-known iterative method for finding a solution of (1.2) with any closed and convex subset C in H may be the projected gradient method [11,

26],

$$\begin{cases} x_1 \in C, \text{ any element,} \\ x_{k+1} = P_C(I - \mu F)(x_k), \quad k \geq 1, \end{cases} \tag{1.3}$$

where P_C is the metric projection of H onto C and μ is a positive real number. If F is a η-strongly monotone and L-Lipschitz continuous mapping in C, then the sequence $\{x_k\}_{k \geq 1}$, generated by (1.3), converges strongly to unique solution of (1.2). But, the projected gradient method requires the use of P_C, that is difficult to compute when C has a complicated structure. In many practical situations, where no closed form expression of P_C is known, but where the closed form expression of a nonexpansive mapping whose fixed point set is C can be based on fundamentals of fixed point theory. On the base of this observation, in [24] Yamada introduced the hybrid steepest descent method,

$$x_{k+1} = T_{[k+1]}(x_k) - \lambda_{k+1}\mu F(T_{[k+1]}(x_k)), \quad k \geq 1, \tag{1.4}$$

where $T_{[k]} = T_{k \bmod N}$, taking values in $\{1, 2, \ldots, N\}$, x_1 is an arbitrary initial point in H, $\mu \in (0, 2\eta/L^2)$ and $\{\lambda_k\}_{k \geq 1} \subset [0, 1]$, and proved that with the following conditions:

(L1) $\lim_{k \to \infty} \lambda_k = 0$,

(L2) $\sum_{k=1}^{\infty} \lambda_k = \infty$,

(L3) $\sum_{k=1}^{\infty} |\lambda_k - \lambda_{k+N}| < \infty$,

(L4) $C = \bigcap_{i=1}^{N} \text{Fix}(T_i) = \text{Fix}(T_1 T_2 \ldots T_N) = \text{Fix}(T_N T_1 T_2 \ldots T_{N-1}) = \ldots$
$= \text{Fix}(T_2 T_3 \ldots T_N T_1) \neq \emptyset$,

the sequence $\{x_k\}_{k \geq 1}$ in (1.4) converges strongly to p^* in (1.2). Next, in [23] Xu and Kim, by replacing condition (L3) with

(L5) $\lim_{k \to \infty} (\lambda_k - \lambda_{k+N})/\lambda_{k+N} = 0$,

proved the same result as Yamada did in [24]. Recently, in [4], Buong and Duong proposed the following strongly convergent algorithm:

$$\begin{cases} x_0 \in H, \\ x_{k+1} = (1 - \beta_k^0)x_k + \beta_k^0 T_0^k \tilde{V}_k(x_k), \\ \tilde{V}_k = T_N^k T_{N-1}^k \ldots T_1^k, \quad k \geq 0, \end{cases} \tag{1.5}$$

where $T_0^k = I - \lambda_k \mu F$ with a fixed real number μ and $\lambda_k \in (0, 1)$ satisfies conditions (L1) and (L2), $T_i^k = (1 - \beta_k^i)I + \beta_k^i T_i$ with $\beta_k^i, i = 0, 1, 2, \ldots, N$ hold

these conditions $\beta_k^i \in (\alpha, \beta)$ for some $\alpha, \beta \in (0, 1)$ and $k \geq 0$, $|\beta_{k+1}^i - \beta_k^i| \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, N$. Very recently, in [27], Zhou and Wang, with the similar conditions on the iterative parameters, proved a strong convergence theorem for a new simpler explicit iterative than (1.5), that is defined by

$$x_{k+1} = (I - \lambda_k \mu F) T_N^k T_{N-1}^k \ldots T_1^k(x_k), \quad k \geq 0. \tag{1.6}$$

In the case that $C = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$, an infinite family of nonexpansive mappings T_i on H, by using Takahashi's W_k-mapping (see, [17, 18]), generated by $T_k, T_{k-1}, \ldots, T_1$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ with $0 < a \leq \alpha_i \leq b < 1$ for $i \geq 1$ as follows:

$$U_{k,k+1} = I,$$
$$U_{k,k} = \alpha_k T_k U_{k,k+1} + (1 - \alpha_k)I,$$
$$U_{k,k-1} = \alpha_{k-1} T_{k-1} U_{k,k} + (1 - \alpha_{k-1})I,$$
$$\vdots$$
$$U_{k,2} = \alpha_2 T_2 U_{k,3} + (1 - \alpha_2)I,$$
$$W_k = U_{k,1} = \alpha_1 T_1 U_{k,2} + (1 - \alpha_1)I.$$

Iemoto and Takahashi (see, [16]) obtained a strong convergence result for the following algorithm

$$x_{k+1} = (I - \lambda_k \mu F) W_k(x_k), \quad k \geq 1, \tag{1.7}$$

where $\mu \in (0, 2\eta/L^2)$ without conditions (L3)-(L5). Further, in [25], Yao et al. obtained the same result under the same conditions on λ_k for the algorithm

$$x_{k+1} = (1 - \gamma_k) F_k(x_k) + \gamma_k W_k F_k(x_k), \quad k \geq 1, \tag{1.8}$$

where $F_k = I - \lambda_k F$ and $\gamma_k \in [\gamma, 1/2]$ for some $\gamma > 0$. Next, in [21], Wang also obtained the same result, when (L1) is replaced by $0 < \lambda_k \leq \eta/L^2 - \varepsilon$ for a small positive constant ε, $k \geq k_0$ with some integer $k_0 > 1$ and $\lambda_k F(x_k) \to 0$ as $k \to \infty$.

In the case that F is an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$ and $\{T_i\}_{i \geq 1}$ is a countably infinite family of nonexpansive mappings on E, a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, in [6, 7], Buong and Phuong proposed some regularization and implicit iterative methods to solve (1.1). In these works, they introduced a new mapping V_k, defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1} \ldots T^k, \quad T^i = (1 - \alpha_i)I + \alpha_i T_i, \tag{1.9}$$

for all $1 \leq i \leq k$, and $\{\alpha_i\}_{i \geq 1}$ satisfies the conditions:

$$\alpha_i \in (0,1) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i < \infty. \tag{1.10}$$

It is clear that computing a value of V_k and W_k at any point in the space E or H is a very complex work. Moreover, it is easy to see that algorithms (1.5)-(1.8) are not parallel ones. Therefore, it takes much time in computation when N is sufficiently large. To overcome the drawbacks, in [8], by using a mapping S_k, defined bellow, instead of V_k and W_k, Buong et al. introduced two following explicit iteration methods: For any $x_1 \in E$, the sequences $\{x_k\}_{k \geq 1}$ are generated by

$$x_{k+1} = (1 - \gamma_k)x_k + \gamma_k S_k F_k(x_k)$$

and

$$x_{k+1} = (1 - \gamma_k)S_k x_k + \gamma_k F_k(x_k)$$

for $k \geq 1$, where

$$0 < \liminf_{k \to \infty} \gamma_k \leq \limsup_{k \to \infty} \gamma_k < 1,$$

$$S_k = \sum_{i=1}^{k} s_i T_i / \tilde{s}_k, \ \ s_i > 0, \ \ \tilde{s}_k = \sum_{i=1}^{k} s_i \quad \text{and} \quad \sum_{i=1}^{\infty} s_i = \tilde{s} < \infty, \tag{1.11}$$

and $F_k = I - \lambda_k F$ with $\lambda_k \in (0,1)$, having the properties (L1) and (L2). It is also clear that from (1.10) and (1.11) it follows that α_i and s_i tend to 0 as $i \to \infty$ and the inverse conclusion is negative.

Motivated by the results above, in this paper, we introduce a new explicit iterative method of hybrid steepest - descent type with the same conditions on the iterative parameter λ_k and we only use the condition in (1.10) for each $i \geq 1$ and $\{s_k\}_{k \geq 0}$ is a strictly decreasing to zero sequence of real numbers. Our method is constructed as follows:

$$\begin{cases} x_1 \in E, \ \text{any element}, \\ x_{k+1} = (I - \lambda_k F)\tilde{S}_k(x_k), \quad k \geq 1, \end{cases} \tag{1.12}$$

where $\tilde{S}_k$ is define by

$$\tilde{S}_k = \frac{1}{s_0 - s_k} \sum_{i=1}^{k} (s_{i-1} - s_i)T^i \ \text{and} \ \tilde{S} = \frac{1}{s_0} \sum_{i=1}^{\infty} (s_{i-1} - s_i)T^i, \tag{1.13}$$

where T^i is defined by (1.9).

In Section 2, we list some related facts, that will be used in the proof of our result. In Section 3, we prove a strong convergence theorem for iterative

method defined by (1.12)-(1.13). Finally, in Section 4, we give a numerical example for illustration.

2. Preliminaries

Let $S_E := \{x \in E : \|x\| = 1\}$ be a unit sphere on Banach space E. The space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_E$. Such an E is called a smooth Banach space. The space E is said to have a uniformly Gâteaux differentiable norm if the limit is attained uniformly for $x \in S_E$. The space E is said to be uniformly smooth if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $x \in S_E$ and $\|y\| < \delta$, the inequality

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 < \varepsilon \|y\|$$

holds. Recall that a Banach space E is said to be

  (i) uniformly convex, if for any ε, $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$;
  (ii) strictly convex, if for $x, y \in S_E$ with $x \neq y$, then

$$\|(1 - \lambda)x + \lambda y\| < 1, \quad \forall \, \lambda \in (0, 1).$$

It is well known that every uniformly smooth real Banach space is reflexive and has a uniformly Gâteaux differentiable norm; Every uniformly convex Banach space E is reflexive and strictly convex; If the norm of E is uniformly Gâteaux differentiable, then J is norm to weak star uniformly continuous on each bounded subset of E; and if E is smooth, then duality mapping is single valued (see, [1]).

These following lemmas will be used frequently throughout the proofs of the main results of the paper in Section 3, in [5, 6, 7] and below.

Lemma 2.1. ([9]) *Let E be a real smooth Banach space and $F : E \to E$ be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$. Then, for any $\lambda \in (0, 1)$, $I - \lambda F$ is contractive with constant $1 - \lambda\tau$, where $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$. Moreover, when $\lambda = 1$, $I - F$ is contractive with constant $\tau_1 = \sqrt{(1 - \eta)/\gamma}$.*

Lemma 2.2. ([9], Lemma 2.2) *Let E be a real smooth Banach space. Then, the following inequalities hold*

$$\|x\|^2 + 2\langle y, J(x)\rangle \leq \|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall \, x, y \in E,$$

where $J : E \to E^*$ is the normalized duality mapping of E.

Lemma 2.3. ([22]) *Let $\{a_k\}_{k\geq 1}$ be a sequence of nonnegative real numbers satisfying the following conditions $a_{k+1} \leq (1 - b_k)a_k + b_k c_k$, where $\{b_k\}_{k\geq 1}$ and $\{c_k\}_{k\geq 1}$ are sequences of real numbers such that*

(i) $b_k \in [0, 1]$ *and* $\displaystyle\sum_{k=1}^{\infty} b_k = \infty$;

(ii) $\displaystyle\limsup_{k\to\infty} c_k \leq 0.$

Then, $\displaystyle\lim_{k\to\infty} a_k = 0.$

Lemma 2.4. ([20]) *Let $\{x_k\}_{k\geq 1}$ and $\{z_k\}_{k\geq 1}$ be bounded sequences in a Banach space E such that $x_{k+1} = \theta_k z_k + (1 - \theta_k)x_k$ for $k \geq 1$, where $\{\theta_k\}_{k\geq 1}$ be a sequence in $[0, 1]$ satisfies the condition*

$$0 < \liminf_{k\to\infty} \theta_k \leq \limsup_{k\to\infty} \theta_k < 1.$$

Assume that

$$\limsup_{k\to\infty} \big(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|\big) \leq 0.$$

Then, $\displaystyle\lim_{k\to\infty} \|x_k - z_k\| = 0.$

Let μ be a continuous linear functional on l^∞ and let $(a_1, a_2, \dots) \in l^\infty$. We write $\mu_k(a_k)$ instead of $\mu((a_1, a_2, \dots))$. We recall that μ is a Banach limit when μ satisfies $\|\mu\| = \mu_k(1) = 1$ and $\mu_k(a_{k+1}) = \mu_k(a_k)$ for every $(a_1, a_2, \dots) \in l^\infty$. For a Banach limit μ, we know that

$$\liminf_{k\to\infty} a_k \leq \mu_k(a_k) \leq \limsup_{k\to\infty} a_k$$

for all $(a_1, a_2, \dots) \in l^\infty$. If $a = (a_1, a_2, \dots) \in l^\infty$, $b = (b_1, b_2, \dots) \in l^\infty$ and $a_k \to c$ (respectively, $a_k - b_k \to 0$), as $k \to \infty$, we have $\mu_k(a_k) = \mu(a) = c$ (respectively, $\mu_k(a_k) = \mu_k(b_k)$).

3. Main results

Lemma 3.1. ([5]) *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$ and let T be a nonexpansive mapping on E, which be a real reflexive and strictly convex Banach space with a uniformly*

Gâteaux differentiable norm. For each $t \in (0,1)$, choose a number $\lambda_t \in (0,1)$ arbitrarily such that $\lambda_t \to 0$ as $t \to 0$ and let $\{y_t\}_{t>0}$ be defined by

$$y_t = (I - \lambda_t F)T(y_t). \tag{3.1}$$

Then, $\{y_t\}$ converges strongly to p^*, solving (1.1) with $C = \text{Fix}(T)$, assumed to be nonempty, as $t \to 0$.

First, we prove the following lemma.

Lemma 3.2. *Let E, F and $\tilde{S}$ be as in Lemma 3.1 and (1.13), respectively. Then, we have*

$$\limsup_{k \to \infty} \langle F(p^*), j(p^* - x_k) \rangle \leq 0 \tag{3.2}$$

if the sequence $\{x_k\}_{k \geq 1}$, defined by (1.12), is bounded.

Proof. Replacing T in Lemma 3.1 by $\tilde{S}$, noting that, $S_k, \tilde{S}$ are nonexpansive mappings and

$$\text{Fix}(\tilde{S}) = \bigcap_{i=1}^{\infty} \text{Fix}(T^i) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i),$$

(see, [3, 5, 6, 8]), and taking $y_m = y_{t_m}$ where $\{t_m\}_{m \geq 1}$ is a sequence in $(0,1)$ that converges to 0 as $m \to \infty$, we have

$$\begin{aligned}
\|x_{k+1} - \tilde{S}(y_m)\| &= \|(I - \lambda_k F)\tilde{S}_k(x_k) - \tilde{S}(y_m)\| \\
&\leq \|\tilde{S}_k(x_k) - \tilde{S}_k(y_m)\| + \|\tilde{S}_k(y_m) - \tilde{S}(y_m)\| + \lambda_k\|F\tilde{S}_k(x_k)\| \\
&\leq \|x_k - y_m\| + \|\tilde{S}_k(y_m) - \tilde{S}(y_m)\| + \lambda_k\|F\tilde{S}_k(x_k)\|.
\end{aligned}$$

Since as $k \to \infty$, $\tilde{S}_k(y_m) \to \tilde{S}(y_m)$ for each y_m, $\lambda_k \to 0$ and $\{\|F\tilde{S}_k(x_k)\|\}_{k \geq 1}$ is bounded, by using the properties of any Banach limit and the last inequality, we obtain that

$$\mu_k\|x_k - \tilde{S}y_m\|^2 = \mu_k\|x_{k+1} - \tilde{S}y_m\|^2 \leq \mu_k\|x_k - y_m\|^2. \tag{3.3}$$

From (3.1) with T replaced by $\tilde{S}$, we can write that

$$\begin{aligned}
x_k - y_m &= x_k - (I - \lambda_m F)\tilde{S}(y_m) \\
&= (1 - \lambda_m)\big[x_k - \tilde{S}(y_m)\big] + \lambda_m\big[x_k - \tilde{S}(y_m) + F\tilde{S}(y_m)\big],
\end{aligned}$$

and hence,

$$(1 - \lambda_m)\big[x_k - \tilde{S}(y_m)\big] = x_k - y_m - \lambda_m\big[x_k - \tilde{S}(y_m) + F\tilde{S}(y_m)\big].$$

Therefore, by virtue of the first inequality in Lemma 2.2,

$$(1 - \lambda_m)^2 \|x_k - \tilde{S}(y_m)\|^2$$
$$\geq \|x_k - y_m\|^2 - 2\lambda_m \langle x_k - \tilde{S}(y_m) + F\tilde{S}(y_m), j(x_k - y_m)\rangle$$
$$= \|x_k - y_m\|^2 - 2\lambda_m \langle x_k - y_m + y_m - \tilde{S}(y_m) + F\tilde{S}(y_m), j(x_k - y_m)\rangle$$
$$= (1 - 2\lambda_m)\|x_k - y_m\|^2 - 2\lambda_m \langle y_m - \tilde{S}(y_m) + F\tilde{S}(y_m), j(x_k - y_m)\rangle,$$

which together with (3.3) and $\lambda_m \in (0,1)$ implies that

$$(1 - \lambda_m)^2 \mu_k \|x_k - y_m\|^2$$
$$\geq (1 - \lambda_m)^2 \mu_k \|x_k - \tilde{S}(y_m)\|^2$$
$$\geq (1 - 2\lambda_m)\mu_k \|x_k - y_m\|^2$$
$$+ 2\lambda_m \mu_k \langle -y_m + \tilde{S}(y_m) - F\tilde{S}(y_m), j(x_k - y_m)\rangle.$$

Consequently,

$$\frac{\lambda_m}{2}\mu_k \|x_k - y_m\|^2 \geq \mu_k \langle -y_m + \tilde{S}(y_m) - F\tilde{S}(y_m), j(x_k - y_m)\rangle, \quad \forall\, m \geq 1.$$

Letting $m \to \infty$ in the last inequality and basing on that $\lambda_m \to 0$ and $y_m \to p^*$, we obtain that

$$0 \geq \mu_k \langle -F\tilde{S}(p^*), j(x_k - p^*)\rangle.$$

It means

$$\mu_k \langle F(p^*), j(x_k - p^*)\rangle \geq 0.$$

Thus,

$$\limsup_{k \to \infty} \langle F(p^*), j(x_k - p^*)\rangle \geq 0,$$

that implies (3.2). The lemma is proved.   □

Now, we are in position to prove our main result.

Theorem 3.3. *Let E, F be as in Lemma 3.1 and let $\{T_i\}_{i \geq 1}$ be an infinite family of nonexpansive mappings on E such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Assume that $\alpha_i \in (0,1)$ for all $i \geq 1$, $\lambda_k \in (0,1)$ satisfy the conditions (L1)-(L2), and $\{s_i\}_{i \geq 0}$ is a strictly decreasing to zero sequence of real positive numbers. Then, the sequence $\{x_k\}_{k \geq 1}$ converges strongly to p^*, as $k \to \infty$, defined by (1.12)-(1.13) solving (1.1).*

Proof. Clearly, form (1.13) and Lemma 3 in [3], it follows, that $\tilde{S}_k$ is a non-expansive mapping on E and $\mathrm{Fix}(T^i) = \mathrm{Fix}(T_i)$, respectively. Consequently,

$\tilde{S}_k p = p$ for any point $p \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \subseteq \bigcap_{i=1}^{k} \text{Fix}(T_i)$ for any $k \geq 1$. Therefore, using Lemma 2.1, we have the following estimate:

$$
\begin{aligned}
\|x_{k+1} - p\| &= \|(I - \lambda_k F)\tilde{S}_k(x_k) - p\| \\
&\leq (1 - \lambda_k \tau)\|x_k - p\| + \|(I - \lambda_k F)\tilde{S}_k(p) - \tilde{S}_k(p)\| \\
&\leq (1 - \lambda_k \tau)\|x_k - p\| + \lambda_k \|F(p)\| \\
&= (1 - \lambda_k \tau)\|x_k - p\| + \lambda_k \tau \frac{1}{\tau}\|F(p)\| \\
&\leq \max\left\{ \|x_1 - p\|, \frac{1}{\tau}\|F(p)\| \right\}.
\end{aligned}
$$

This imply that the sequence $\{x_k\}_{k\geq 1}$ is bounded. Hence, the sequences $\{\tilde{S}_k(x_k)\}_{k\geq 1}$, $\{\tilde{S}_{k+1}(x_k)\}_{k\geq 1}$, $\{F\tilde{S}_k(x_k)\}_{k\geq 1}$, $\{T_i(x_k)\}_{k\geq 1}$, $\{T^i(x_k)\}_{k\geq 1}$ for all $i \geq 1$, $\{\tilde{S}_{k,-1}(x_k)\}_{k\geq 1}$ and $\{\tilde{S}_{k+1,-1}(x_k)\}_{k\geq 1}$ are also bounded, where

$$
\tilde{S}_{k,-1}(x) = \frac{1}{s_1 - s_k} \sum_{i=2}^{k} (s_{i-1} - s_i)T^i(x).
$$

Without loss of generality, we can assume that they are bounded by a positive constant M_1. Further, from (1.12) and (1.13), we can write that

$$
\begin{aligned}
x_{k+1} &= \lambda_k(I - F)\tilde{S}_k(x_k) + (1 - \lambda_k)\tilde{S}_k(x_k) \\
&= \lambda_k(I - F)\tilde{S}_k(x_k) + (1 - \lambda_k)\left[\frac{(s_0 - s_1)(1 - \alpha_1)}{s_0 - s_k}x_k \right. \\
&\quad \left. + \frac{(s_0 - s_1)\alpha_1}{s_0 - s_k}T_1(x_k) + \frac{s_1 - s_k}{s_0 - s_k}\tilde{S}_{k,-1}(x_k) \right] \\
&= h_k x_k + (1 - h_k)z_k
\end{aligned}
\tag{3.4}
$$

with

$$
\begin{aligned}
h_k &= \frac{(1 - \lambda_k)(s_0 - s_1)(1 - \alpha_1)}{s_0 - s_k}, \\
z_k &= \frac{\lambda_k(I - F)\tilde{S}_k(x_k)}{1 - h_k} + \frac{(1 - \lambda_k)(s_0 - s_1)\alpha_1}{(1 - h_k)(s_0 - s_k)}T_1(x_k) \\
&\quad + \frac{(1 - \lambda_k)(s_1 - s_k)}{(1 - h_k)(s_0 - s_k)}\tilde{S}_{k,-1}(x_k).
\end{aligned}
$$

Since

$$C_1 := \frac{\lambda_{k+1}(I-F)\tilde{S}_{k+1}(x_{k+1})}{1-h_{k+1}} - \frac{\lambda_k(I-F)\tilde{S}_k(x_k)}{1-h_k}$$

$$= \frac{\lambda_{k+1}}{1-h_{k+1}}\big[(I-F)\tilde{S}_{k+1}(x_{k+1}) - (I-F)\tilde{S}_{k+1}(x_k)\big]$$

$$+ \frac{\lambda_{k+1}}{1-h_{k+1}}\big[(I-F)\tilde{S}_{k+1}(x_k) - (I-F)\tilde{S}_k(x_k)\big]$$

$$+ \left[\frac{\lambda_{k+1}}{1-h_{k+1}} - \frac{\lambda_k}{1-h_k}\right](I-F)\tilde{S}_k(x_k),$$

$$C_2 := \frac{1-\lambda_{k+1}}{(1-h_{k+1})(s_0-s_{k+1})}T_1(x_{k+1}) - \frac{1-\lambda_k}{(s_0-s_k)(1-h_k)}T_1(x_k)$$

$$= \frac{1-\lambda_{k+1}}{(1-h_{k+1})(s_0-s_{k+1})}\big[T_1(x_{k+1}) - T_1(x_k)\big]$$

$$+ \left[\frac{(1-\lambda_{k+1})}{(1-h_{k+1})(s_0-s_{k+1})} - \frac{(1-\lambda_k)}{(1-h_k)(s_0-s_k)}\right]T_1(x_k),$$

$$C_3 := \frac{(1-\lambda_{k+1})(s_1-s_{k+1})}{(1-h_{k+1})(s_0-s_{k+1})}\tilde{S}_{k+1,-1}(x_{k+1}) - \frac{(1-\lambda_k)(s_1-s_k)}{(1-h_k)(s_0-s_k)}\tilde{S}_{k,-1}(x_k)$$

$$= \frac{(1-\lambda_{k+1})(s_1-s_{k+1})}{(1-h_{k+1})(s_0-s_{k+1})}\big[\tilde{S}_{k+1,-1}(x_{k+1}) - \tilde{S}_{k+1,-1}(x_k)\big]$$

$$+ \frac{(1-\lambda_{k+1})(s_1-s_{k+1})}{(1-h_{k+1})(s_0-s_{k+1})}\big[\tilde{S}_{k+1,-1}(x_k) - \tilde{S}_{k,-1}(x_k)\big]$$

$$+ \left[\frac{(1-\lambda_{k+1})(s_1-s_{k+1})}{(1-h_{k+1})(s_0-s_{k+1})} - \frac{(1-\lambda_k)(s_1-s_k)}{(1-h_k)(s_0-s_k)}\right]\tilde{S}_{k,-1}(x_k),$$

$\tilde{S}_{k+1,-1}(x)$ is nonexpansive on E, and

$$\|\tilde{S}_{k+1,-1}(x_k) - \tilde{S}_{k,-1}(x_k)\|$$

$$= \left\|\frac{1}{s_1-s_{k+1}}\sum_{k=2}^{k+1}(s_{i-1}-s_i)T^i(x_k) - \frac{1}{s_1-s_k}\sum_{k=2}^{k}(s_{i-1}-s_i)T^i(x_k)\right\|$$

$$= \left\|\left(\frac{1}{s_1-s_{k+1}} - \frac{1}{s_1-s_k}\right)\sum_{k=2}^{k}(s_{i-1}-s_i)T^i(x_k) + \frac{s_k-s_{k+1}}{s_1-s_{k+1}}T^{k+1}(x_k)\right\|$$

$$\leq \left|\frac{-s_k+s_{k+1}}{(s_1-s_{k+1})(s_1-s_k)}\right|(s_1-s_k)M_1 + \frac{s_k-s_{k+1}}{s_1-s_{k+1}}M_1$$

$$= \frac{2M_1(s_k-s_{k+1})}{(s_1-s_{k+1})},$$

we have

$$\|z_{k+1} - z_k\| \le \|C_1\| + (s_0 - s_1)\alpha_1\|C_2\| + \|C_3\|$$

$$\le \frac{\lambda_{k+1}}{1 - h_{k+1}}(1 - \tau_1)\left[\|x_{k+1} - x_k\| + \frac{4M_1}{1 - \tau_1}\right]$$

$$+ \left|\frac{\lambda_{k+1}}{1 - h_{k+1}} - \frac{\lambda_k}{1 - h_k}\right|2M_1$$

$$+ \frac{(1 - \lambda_{k+1})(s_0 - s_1)\alpha_1}{(1 - h_{k+1})(s_0 - s_{k+1})}\|x_{k+1} - x_k\|$$

$$+ (s_0 - s_1)\alpha_1\left|\frac{(1 - \lambda_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})} - \frac{(1 - \lambda_k)}{(1 - h_k)(s_0 - s_k)}\right|M_1$$

$$+ \frac{(1 - \lambda_{k+1})(s_1 - s_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})}\|x_{k+1} - x_k\|$$

$$+ \frac{(1 - \lambda_{k+1})(s_1 - s_{k+1})2M_1(s_k - s_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})(s_1 - s_{k+1})}$$

$$+ \left|\frac{(1 - \lambda_{k+1})(s_1 - s_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})} - \frac{(1 - \lambda_k)(s_1 - s_k)}{(1 - h_k)(s_0 - s_k)}\right|M_1.$$

Therefore,

$$\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|$$

$$\le \left[\frac{\lambda_{k+1}(1 - \tau_1)}{1 - h_{k+1}} + \frac{(1 - \lambda_{k+1})(s_0 - s_1)\alpha_1}{(1 - h_{k+1})(s_0 - s_{k+1})}\right. \tag{3.5}$$

$$\left. + \frac{(1 - \lambda_{k+1})(s_1 - s_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})} - 1\right]\|x_{k+1} - x_k\| + c_k$$

that is the sum of the remain terms in the right hand-side of the before-last inequality. Clearly, $1 - h_k \to [s_0 - (s_0 - s_1)(1 - \alpha_1)]/s_0$ as $k \to \infty$ and hence, $c_k \to 0$ as $k \to \infty$. Moreover,

$$\lim_{k \to \infty} \frac{(1 - \lambda_{k+1})(s_0 - s_1)\alpha_1}{(1 - h_{k+1})(s_0 - s_{k+1})} + \frac{(1 - \lambda_{k+1})(s_1 - s_{k+1})}{(1 - h_{k+1})(s_0 - s_{k+1})} - 1$$

$$= \frac{(s_0 - s_1)\alpha_1 + s_1}{s_0 - (s_0 - s_1)(1 - \alpha_1)} - 1 = 0.$$

So, from (3.5) it follows that

$$\limsup_{k \to \infty}\left(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|\right) \le 0.$$

By virtue of Lemma 2.4,

$$\lim_{k \to \infty}\|x_k - z_k\| = 0,$$

which together with (3.4) implies that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\|$$

$$= \lim_{k\to\infty} \left[1 - \frac{(1-\lambda_k)(s_0-s_1)(1-\alpha_1)}{s_0-s_k} \right] \|x_k - z_k\| = 0. \qquad (3.6)$$

Now, we estimate the value of $\|x_{k+1} - p^*\|^2$ as follows:

$$\begin{aligned}
\|x_{k+1} - p^*\|^2 &= \|(I - \lambda_k F)\tilde{S}_k(x_k) - p^*\|^2 \\
&= \|(I - \lambda_k F)\tilde{S}_k(x_k) - (I - \lambda_k F)\tilde{S}_k(p^*) - \lambda_k F(p^*)\|^2 \\
&\leq (1 - \lambda_k \tau)\|x_k - p^*\|^2 - 2\lambda_k \langle F(p^*), j(x_{k+1} - p^*) \rangle \\
&= (1 - \lambda_k \tau)\|x_k - p^*\|^2 + 2\lambda_k [\langle F(p^*), j(p^* - x_k) \rangle \\
&\quad + \langle F(p^*), j(p^* - x_{k+1}) - j(p^* - x_k) \rangle] \\
&= (1 - b_k)\|x_k - p^*\|^2 + b_k c_k,
\end{aligned}$$

where

$$b_k = \lambda_k \tau,$$
$$c_k = 2[\langle F(p^*), j(p^* - x_k) \rangle + \langle F(p^*), j(p^* - x_{k+1}) - j(p^* - x_k) \rangle]/\tau.$$

Since $\sum_{k=0}^{\infty} \lambda_k = \infty$, $\sum_{k=0}^{\infty} b_k = \infty$. So, from Lemma 2.3 with $a_k = \|x_k - p^*\|^2$, (3.2) and the norm to weak star continuity of j with (3.6), it follows that $\lim_{k\to\infty} \|x_k - p^*\|^2 = 0$. The theorem is proved. $\qquad \square$

4. APPLICATION AND NUMERICAL EXAMPLE

Obviously, the result above is applicable to solve the following optimization problem: find a point $p^* \in C$ such that

$$\varphi(p^*) = \min_{x \in C} \varphi(x), \qquad (4.1)$$

where $\varphi(x)$ is a Gâteaux differentiable functional whose derivative $\varphi'(x)$ is strongly monotone and Lipschitz continuous on the Euclidian space $\mathbb{R}^n$ and

$$C = \bigcap_{i=1}^{\infty} C_i, \qquad (4.2)$$
$$C_i = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i x_n \geq b_i\},$$

where a_j^i and b_i are the real numbers for all $i \geq 1$ and $1 \leq j \leq n$.

For computation, we consider the case that $n = 2$ with

$$\varphi(x) = x^T A x + b^T x + c,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad \text{and} \quad c = 13,$$

and $a_1^i = 1$, $a_2^i = i$ and $b_i = 2$ for all $i \geq 1$. In this case, $p^* = (2.0; 3.0)$ is the unique solution of (4.1)-(4.2).

We use algorithm (1.12)-(1.13) with $F(x) = (2x_1 - 4; 2x_2 - 6)$, the derivative of φ at $x \in \mathbb{R}^2$, $T_i = P_{C_i}$, the started point of iteration is $x^1 = (-3.0; -3.0)$, $\lambda_k = 1/(k+2)$ and $s_i = \alpha_i = 1/(i(i+1))$. Clearly, F is 1-strongly monotone and 2-Lipschtiz continuous and P_{C_i} is nonexpansive on Euclidian space $\mathbb{R}^2$.

The following table of numerical results indicates that the method works well in practice.

k	x_1^k	x_2^k	k	x_1^k	x_2^k
1	-3.0000000000	-3.0000000000	40	1.9850772357	2.981455284
4	0.7763333333	1.4793333333	50	1.9904026143	2.9880732026
6	1.4173015873	2.2758730158	60	1.9933132969	2.9916903460
8	1.6600925925	2.5775925925	70	1.9950757880	2.9938806170
10	1.7775151515	2.7235151515	100	1.9975768976	2.9969887788
20	1.9417301587	2.9275873015	500	1.9999023020	2.9998785894
30	1.9736845878	2.9672974910	1000	1.9999755511	2.9999696170

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